Fernando Ferreira · Reinhard Kahle ·
Giovanni Sommaruga *Eds.*

Axiomatic Thinking II

Springer

Axiomatic Thinking II

Photography of the participants of the annual meeting of the Swiss Mathematical Society 1917, taken at the back side of the *Landesmuseum* in Zurich. Hilbert is standing in the center of the first row. Source: Bildarchiv of the ETH Zurich. https://doi.org/10.3932/ethz-a-000046430

People which were identified by George Pólya: 1 Fehr; 2 Carathéodory; 3 Grossmann; 4 Hilbert; 5 Gaiser; 6 Mrs. Weyl; 7 Weyl; 8 Bernays; 9 Plancherel; 10 Gonseth; 11 Speiser; 12 Spiess; 13 Hecke; 14 DuPasquier. The drawing it taken from: George Pólya. *The Pólya picture album: encounters of a mathematician*, Birkhäuser, 1987, p. 14

Fernando Ferreira · Reinhard Kahle ·
Giovanni Sommaruga
Editors

Axiomatic Thinking II

Springer

Editors
Fernando Ferreira
Departamento de Matemática
Faculdade de Ciências
Universidade de Lisboa
Lisboa, Portugal

Reinhard Kahle
Carl Friedrich von Weizsäcker-Zentrum
Universität Tübingen
Tübingen, Germany

Giovanni Sommaruga
Department of Mathematics
ETH Zurich
Zurich, Switzerland

ISBN 978-3-030-77798-2 ISBN 978-3-030-77799-9 (eBook)
https://doi.org/10.1007/978-3-030-77799-9

Mathematics Subject Classification: 03-06, 03-03, 03A05, 03A10

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This book originates with two events commemorating the centenary of David Hilbert's seminal talk on *Axiomatic Thinking (Axiomatisches Denken)* which he delivered on September 11, 1917, at Zurich University for the Swiss Mathematical Society. This talk marks arguably the birth of proof theory as it was conceived by David Hilbert in the 1920s. It makes clear that the formalistic endeavor which one may find in the development of mathematical logic by the Hilbert school is, at best, a technical ingredient of a much larger enterprise which attempts to base every science deserving this predicate on a transparent *framework of concepts* (*Fachwerk von Begriffen*), developed and investigated by the axiomatic method.

On September 14–15, 2017, a joint meeting of the Swiss Mathematical Society and the Swiss Society for Logic and Philosophy of Science on *Axiomatic Thinking* took place at the University of Zurich, Switzerland, the place where Hilbert had spoken 100 years ago. It was followed, on October 11–14, 2017, by a conference on the same topic at the *Academia das Ciências de Lisboa* and *Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa* in Lisbon which was also the annual meeting of the *Académie Internationale de Philosophie des Sciences*. This meeting included a *Panel Discussion on the Foundations of Mathematics* with Peter Koellner, Michael Rathjen, and Mark van Atten as invited panelists.

The current volumes contain contributions of speakers of both meetings and also papers by other researchers in the field. In accordance with the broad range of topics addressed by Hilbert, the articles in Vol. I focus on reflections on the *History and Philosophy* of Axiomatic Thinking; Vol. II provides in Part I examples of developments of axiomatic thinking in *Logic*, especially in Proof Theory, inspired by Hilbert's ideas; Part II is concerned with applications of the axiomatic method in *Mathematics*; and Part III addresses the use of the axiomatic method in *other sciences*, namely Computer Science, Physics, and Theology.

Our dear friend Thomas Strahm, an inspired logician, followed the development of this book closely. But sadly he is not here to see its publication. Thomas Strahm died at the end of April, 2021. We dedicate this book to him—an excellent logician, and even more a kind, sensitive, humorous and wonderful friend.

Lisboa, Portugal — Fernando Ferreira
Tübingen, Germany — Reinhard Kahle
Zurich, Switzerland — Giovanni Sommaruga
March 2021

Acknowledgements

The editors are grateful to all speakers in Zurich and Lisbon who contributed to the success of the two meetings, as well as to the meetings' co-organizers, Thomas Kappeler and Viktor Schroeder in Zurich, and Gerhard Heinzmann, João Cordovil, João Enes, Mirko Engler, António Fernandes, Gilda Ferreira, Emanuele Frittaion, Nuno Jerónimo, Isabel Oitavem, Cecília Perdigão, and Gabriele Pulcini in Lisbon. The meeting in Zurich was supported by the Swiss Mathematical Society SMS, the Swiss Society for Logic and Philosophy of Science SSLPS, the Swiss Academy of Sciences SCNAT, and the Institute of Mathematics, University of Zurich, which is gratefully acknowledged. Equally, many thanks for the support of the conference in Lisbon to the Académie Internationale de Philosophie des Sciences AIPS; the Academia das Ciências de Lisboa; the Centro Internacional de Matemática, CIM; and the following research centres: Centro de Matemática, Aplicações Fundamentais e Investigação Operacional, CMAF-CIO; Centro de Matemática e Aplicações, CMA; Centre for Philosophy of Sciences of the University of Lisbon, CFCUL. The Portuguese Science Foundation (Fundação para a Ciência e a Tecnologia, FCT) supported the conference through the projects, UID/FIL/00678/2013, UID/MAT/00297/2013, UID/MAT/04561/2013, and PTDC/MHC-FIL/2583/2014 (*Hilbert's 24th Problem*). For the preparation of the current volumes, the first editor acknowledges the support by the Portuguese Science Foundation (UIDB/04561/2020) and the second editor was also supported by the Udo Keller Foundation.

Pro domo, we'd like to acknowledge Reinhard's enormous efforts and drive without which this book might still be a mere project. (Fernando and Giovanni).

Contents Overview of Vol. 1

Contents

Editors and Contributors

About the Editors

Fernando Ferreira is Professor of Mathematics at Universidade de Lisboa. He received his Ph.D. at Pennsylvania State University in 1988 under the direction of Stephen Simpson. He was a Fulbright Scholar at Harvard University (Spring 2004) and Tinker Visiting Professor at Stanford University (Fall 2009). He has written papers in weak systems of arithmetic and analysis, proof theory (especially functional interpretations) and philosophy and foundations of mathematics. He also wrote two papers on the problem of truth in Parmenides and Plato. He is a member of Academia das Ciências de Lisboa.

Reinhard Kahle is Carl Friedrich von Weizsäcker Professor for Philosophy and History of Science at the University of Tübingen. Before he holds professorships in Mathematics at the University of Coimbra and at the Universidade Nova in Lisbon, at the end as full professor for Mathematical Logic. He is a fellow of the *Académie Internationale de Philosophie des Sciences*. His main research interests are proof theory and the history and philosophy of modern mathematical logic, in particular, of the Hilbert School. He has (co-)edited more than ten books and special issues as, for instance, Gentzen's Centenary: The quest for consistency (Springer, 2015) and The Legacy of Kurt Schütte (Springer, 2020), both together with Michael Rathjen.

Giovanni Sommaruga did his studies in philosophy and philosophical and mathematical logic at the University of Freiburg (Switzerland), Stanford University, and the University of Siena. In 1996 he became assistant professor in logic and philosophy of science at the Albert Ludwig University Freiburg (Germany), and since 2008 he has been senior scientist in philosophy of the formal sciences (logic, mathematics, theoretical computer science) at ETH Zurich. His main research interests are in philosophy and foundations of mathematics, in the history of mathematical logic, and more recently in the history and philosophical issues concerning computability and information in theoretical computer science.

Contributors

John L. Bell University of Western Ontario, London, ON, Canada

Udi Boker School of Computer Science, The Interdisciplinary Center, Herzliya, Israel

Nachum Dershowitz School of Computer Science, Tel Aviv University, Ramat Aviv, Israel

Domenico Giulini Leibniz University of Hannover, Institute for Theoretical Physics, Hannover, Germany

Lorenz Halbeisen Department of Mathematics, ETH Zentrum, Zürich, Switzerland

Gerhard Jäger Institut für Informatik, Universität Bern, Bern, Switzerland

Colin McLarty Department of Philosophy, Case Western Reserve University, Cleveland, OH, USA

Victor Pambuccian School of Mathematical and Natural Sciences, Arizona State University - West Campus, Phoenix, AZ, USA

Jean Petitot Centre d'Analyse et de Mathématique Sociales, EHESS, Paris, France

Wolfram Pohlers Institut für Mathematische Logik und Grundlagenforschung, Münster, Germany

Michael Rathjen Department of Pure Mathematics, University of Leeds, Leeds, UK

Paul Weingartner Innsbruck, Tirol, Austria

Part I
Logic I(2)

Chapter 1
A Framework for Metamathematics

Lorenz Halbeisen

Abstract First, we consider Hilbert's program, focusing on the three different aspect of mathematics called *actual mathematics*, *formal mathematics*, and *metamathematics*. Then, we investigate the relationship between *metamathematics* and *actual mathematics*, describe what shall be achieved with *metamathematics*, and propose a framework for *metamathematics*.

1.1 Hilbert's Program Revisited

Motivated by prior work of Frege and Russell, Hilbert describes in [3] what he calls *axiomatic thinking*. He concludes his article with the following words:

> Ich glaube: Alles, was Gegenstand des wissenschaftlichen Denkens überhaupt sein kann, verfällt, sobald es zur Bildung einer Theorie reif ist, der axiomatischen Methode und damit mittelbar der Mathematik. Durch Vordringen zu immer tieferliegenden Schichten von Axiomen [...] gewinnen wir auch in das Wesen des wissenschaftlichen Denkens selbst immer tiefere Einblicke und werden uns der Einheit unseres Wissens immer mehr bewußt. In dem Zeichen der axiomatischen Methode erscheint die Mathematik berufen zu einer führenden Rolle in der Wissenschaft überhaupt.[1]

At this early stage of Hilbert's program, the focus is on the "objects of scientific thought" which become dependent on the axiomatic method. When these "objects of

[1] I believe: anything at all that can be the object of scientific thought becomes dependent on the axiomatic method, and thereby indirectly on mathematics, as soon as it is ripe for the formation of a theory. By pushing ahead to ever deeper layers of axioms [...] we also win ever-deeper insights into the essence of scientific thought itself, and we become ever more conscious of the unity of our knowledge. In the sign of the axiomatic method, mathematics is summoned to a leading role in science. (*Translation taken from* [2].)

L. Halbeisen (✉)
Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland
e-mail: lorenz.halbeisen@math.ethz.ch

F. Ferreira et al. (eds.), *Axiomatic Thinking II*,
https://doi.org/10.1007/978-3-030-77799-9_1

scientific thought" are mathematical objects, one can think of these objects as being part of the *real mathematical world*.

Later in 1922, Hilbert went a step further. The focus is now not on the "objects of scientific thought" which shall be axiomatized, but on the consistency of the axiomatic systems. In [5, p. 174], Hilbert summarizes his program as follows:

> Erstens: Alles, was bisher die eigentliche Mathematik ausmacht, wird nunmehr streng formalisiert, so daß die *eigentliche Mathematik* oder die Mathematik im engeren Sinne zu einem Bestande an beweisbaren Formeln wird. [. . .]
>
> Zweitens: Zu dieser eigentlichen Mathematik kommt eine gewissermaßen neue Mathematik, eine *Metamathematik*, hinzu, die zur Sicherung jener dient, indem sie sie vor dem Terror der unnötigen Verbote sowie der Not der Paradoxien schützt. In dieser Metamathematik kommt—im Gegensatz zu den rein formalen Schlußweisen der eigentlichen Mathematik—das inhaltliche Schließen zur Anwendung, und zwar zum Nachweis der Widerspruchsfreiheit der Axiome.
>
> Die Entwicklung der mathematischen Wissenschaft geschieht hiernach beständig wechselnd auf zweierlei Art: durch Gewinnung neuer "beweisbarer" Formeln aus den Axiomen mittels formalen Schließens und durch Hinzufügung neuer Axiome nebst dem Nachweis ihrer Widerspruchsfreiheit mittels inhaltlichen Schließens.[2]

What we see here is the beginning of a paradigm shift: In classical mathematics, axioms were a statement that was taken to be true. Axioms served as premises or starting points for further reasoning and arguments. Therefore, it would have been absurd to consider different contradicting axiom systems, since at most one of these systems can be true in an absolute sense, and all the others systems must be false or meaningless. Now, focusing on the consistency of axiom systems rather than on their inherent truth, we do not need to restrict ourselves to axiom systems which are relevant for *actual mathematics* (i.e. for the investigation of objects in the, to some extend, *real mathematical world*), but could investigate any consistent axiomatic system, no matter whether it is relevant for *actual mathematics* or not.

However, since the ultimate goal of Hilbert's program was to give a firm (i.e. provably consistent) foundation of *actual mathematics*, there are still some axiomatic systems which are more relevant for mathematics, and some which are less relevant for mathematics. This situation is similar to geometry, where one could argue that the only geometric system which is relevant, is the one which describes the space

---

[2] First: everything that hitherto made up [actual mathematics] is now to be strictly formalized, so that ***actual mathematics***, or mathematics in the strict sense, becomes a stock of provable formulae. [. . .]
Secondly: in addition to actual mathematics, there appears a mathematics that is to some extent new, a ***metamathematics*** which serves to safeguard it by protecting it from the terror of unnecessary prohibitions as well as from the difficulty of paradoxes. In this metamathematics—in contrast to the purely formal modes of inference in actual mathematics—we apply contentual inference; in particular, to the proof of the consistency of the axioms.
The development of mathematical science accordingly takes place in two ways that constantly alternate: the derivation of new "provable" formulae from the axioms by means of formal inference; and the adjunctionn of new axioms together with a proof of their consistency by means of contentual inference. (*Translation taken from* [2], *except that we translated "eigentliche Mathematik" as "actual mathematics" and not as "mathematics proper"*.)

in which we live. Even though from a physical point of view this argument makes sense, from a mathematical point of view it is immaterial. For example, it is very unlikely that our space satisfies the axioms of projective geometry, but nevertheless, projective geometry is the key tool in the investigation of conic sections in Euclidian geometry.

To sum up, we can say that Hilbert's program was the beginning of a paradigm shift from "axioms as obviously true statements" towards "axioms as mutually non-contradictory statements". However, since there is still a presupposed *actual mathematics*, this paradigm shift was not carried out thoroughly. For example, let us consider the axiomatic system ZFC, which is Zermelo-Fraenkel Set Theory ZF with the Axiom of Choice AC. One of the earliest problems in set theory was the question whether the Continuum Hypothesis CH holds (which is the first of the twenty-three problems Hilbert presented at the *ICM 1900 in Paris*). On the one hand, it is known that CH is independent from ZFC (i.e. within ZFC we can neither prove nor disprove CH), and on the other hand, ZFC serves as a foundation of mathematics. Now, if one believes in a unique *actual mathematics*, then CH should be either true or false, which implies that ZFC is not strong enough to serve as the foundation of *actual mathematics*. So, we have to extend ZFC by adding new axioms in such a way that the extended systems decide CH. However, by *Gödel's Second Incompleteness Theorem*, this does not really help, since no matter how we extend ZFC, we always obtain a sentence which is undecidable within the extended system. In other words, having *Gödel's Second Incompleteness Theorem* in mind, it is not possible to axiomatize *actual mathematics* in such a way that the axiomatic system obtained fully represents *actual mathematics*, which also shows that Hilbert's program must fail.

Let us turn back to the paradigm shift from "axioms as true statements" towards "axioms as mutually non-contradictory statements", which was initiated by Hilbert's program: The above explanations show that in order to make the paradigm shift complete, we have to give up the idea of *actual mathematics* as the unique *real mathematical world*, since strictly formalizing *actual mathematics* in 1st order logic yields a formal axiomatic system of which *actual mathematics* is just one of numerous models. However, we can conceive *actual mathematics* as the collection of all models of axiomatic systems which form a foundation for mathematics. Such systems are consistent extensions of ZF, whose models are proper models for mathematics, i.e. models in which we can carry out essentially all mathematics.

In order to see how and where we build these models, we have to combine *Gödel's Completeness Theorem* for 1st order logic with Hilbert's metamathematics: *Gödel's Completeness Theorem* together with the *Soundness Theorem* states that a sentence ϕ is provable from an axiomatic system S, denoted $\mathsf{S} \vdash \phi$, if and only if ϕ is valid in each model of S. In particular, we obtain that an axiomatic system S has a model if and only if S is consistent. So, for any axiomatic system S, Hilbert's metamathematics has the task to decide whether S is consistent, or equivalently, to decide whether S has a model. By *Gödel's Incompleteness Theorems* we know that this task cannot be carried out in a formal system. In other words, Hilbert's metamathematics can not be formalized, and therefore, does not belong to actual mathematics—which is indicated by the prefix "meta", which means "behind" (i.e. metamathematics is a kind of "background-mathematics"). Moreover, even in the case when we know that some axiomatic system S is consistent, and therefore has a model, in general, the construction of a model of S cannot be carried out in a formalized system, i.e. the construction of a model must in general be carried out in metamathematics.

Since metamathematics plays an important role in the investigation of axiomatic systems and in the construction of models, and since metamathematics cannot be formalized, it is natural to ask what kind of principles we have in metamathematics. An answer to this question is given in the next section.

1.2 Non-constructive Principles of Metamathematics

The previous section can be summarized as follows: In mathematics we investigate formal axiomatic systems. In particular, we investigate which sentences can be derived from a given axiomatic system S, which sentences are consistent with S, and which sentences are independent of S, where the investigations themselves are based on the construction of various models of axiomatic systems. In particular, we have to construct models of variations of ZF (i.e. models of extensions of ZF). The construction of models is carried out in a moderate constructive way, which we are going to circumscribe now.

1.2.1 What We Need

The construction of a model for an axiomatic system is carried out by following Henkin's proof of *Gödel's Completeness Theorem* for 1st order logic. Now, beside the constructive parts of Henkin's proof, which are described explicitly or by algorithms, there are also some non-constructive parts using principles which are usually tacitly assumed. The goal is now to make these principles explicit.

The most important principle we need in metamathematics is the notion of FINITENESS. Hilbert writes in [4, p. 154]:

> Die beweisbaren Formeln [...] haben sämtlich den Charakter des Finiten, d.h. die Gedanken, deren Abbilder sie sind, können [...] mittels Betrachtung endlicher Gesamtheiten erhalten werden.[3]

The notion of FINITENESS plays a crucial role not only in the investigation of provable formulae, but also in the proof of *Gödel's Incompleteness Theorems*. In fact, if the notion of FINITENESS could be formalized (i.e. if FINITENESS were a notion of

[3] The provable formulae [...] all have the character of the finite; that is, the thoughts whose images they are can also be obtained [...] from the examination of finite totalities. (*Translation taken from* [2].)

formal mathematics), then *Gödel's Incompleteness Theorems* would disappear and Hilbert's program would succeed.

What we also need to construct models is the notion of a POTENTIALLY INFINITE SET, like the natural numbers 0, 1, 2, ... Notice that we do not require to have the entire set $\mathbb{N}$ of natural numbers, which would be an actual infinite set. In fact, a closer look at Henkin's proof of *Gödel's Completeness Theorem* shows that in order to construct non-finite models (e.g. models of Peano Arithmetic PA or models of ZFC), a POTENTIALLY INFINITE SET is sufficient—but also necessary.

Finally, we need a kind of LAW OF EXCLUDED MIDDLE. This law is crucial in the completion of axiomatic systems S, since in each step of the completion of S, for some ϕ we have to decide whether or not ϕ is consistent with the extension of S we already have constructed. In other words, for every axiomatic systems S and each sentence ϕ, we must be able to decide whether ϕ is provable from S (i.e. $\mathsf{S} \vdash \phi$), and since a formal proof is just a special FINITE sequence of formulae, either there is such a sequence or there is no such sequence. The difficulty is, that we probably cannot decide in FINITELY many steps, whether or not $\mathsf{S} \vdash \phi$. Now, this non-constructive part in the proof of *Gödel's Completeness Theorem* is handled by the LAW OF EXCLUDED MIDDLE. If we would formalize this law, we would obtain what is known as the WEAK KÖNIG'S LEMMA, which is just KÖNIG'S LEMMA for infinite, binary 0-1-trees.

1.2.2 What We Obtain

In the framework described above, we can construct models of all kind of axiomatic systems. For example, we can construct models of ZFC, or models of ZF in which the Axiom of Choice fails, and we can carry out Forcing constructions in order to obtain models of ZFC (or of ZF) in which certain statements become valid. In particular, we can construct models of ZFC in which CH holds or in which CH fails. This way, we obtain different models of the standard real numbers. On the other hand, we can also construct non-standard models of the real numbers, for example, the hyperreal numbers or the surreal numbers, which give us also non-standard models of Peano Arithmetic. In fact, even non-standard approaches to mathematics, like intuitionism, can be modelled. There is a lot of freedom we have, and it might be this freedom, which Cantor meant when he writes ([1, p. 564])

> ... das Wesen der Mathematik liegt [...] in ihrer Freiheit.[4]

[4] ... the essence of mathematics lies [...] in its freedom.

1.3 Conclusion

The view of mathematics we proposed can be described as follows:

- In mathematics we investigate formal axiomatic systems. In particular, we investigate which sentences we can derive from a given axiomatic system S, which sentences are consistent with S, and which sentences are independent of S.
- The investigations are based on the construction of various models of axiomatic systems, in particular, on the construction of models of variations of ZFC.
- The construction of models is carried out in metamathematics, where metamathematics consists of all we can describe explicitly or by algorithms, together with the notions of FINITENESS and POTENTIALLY INFINITE SET, and the LAW OF EXCLUDED MIDDLE.

On the one hand, this view of mathematics is quite formal in the sense that there is no unique *real mathematical world* anymore, but on the other hand, we have a realm of models of various axiomatic systems, which distinguishes this view from pure formalism. Moreover, one of the features of this view is that we do not have any kind of "ideology" like constructivism, platonism, or intuitionism, which would lead us to the "right" mathematical world: No matter which approach we take, with Hilbert's axiomatic thinking—enriched by Gödel's work—we are able to create various mathematical worlds. With respect to this kind of mathematics, we would like to say:

> From the realm of mathematics, which Hilbert and Gödel created for us, no-one shall expel us.

References

1. Georg Cantor. Ueber unendliche, lineare Punktmannichfaltigkeiten. *Mathematische Annalen*, 21 (4): 545–591, 1883.
2. William Bragg Ewald. *From Kant to Hilbert: a source book in the foundations of mathematics II.* Oxford University Press, New York, 1996.
3. David Hilbert. Axiomatisches denken. *Mathematische Annalen*, 78 (1): 405–415, 1918.
4. David Hilbert. Die logischen Grundlagen der Mathematik. *Mathematische Annalen*, 88 (1–2): 151–165, 1922a.
5. David Hilbert. Neubegründung der Mathematik, Erste Mitteilung. *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, 1:157–177, 1922b.

Lorenz Halbeisen studied mathematics, focused on set theory and logic, at the universities of Basel and Zürich, graduated in 1990 at the University of Zürich and received his Ph.D. in 1994 from the ETH Zürich. After post-doctoral positions in Caen (France), Barcelona (Spain) and Berkeley (U.S.A.), he has been lecturer at Queen's University Belfast (Northern Ireland) and since 2014 he is senior scientist at the ETH Zürich.

Chapter 2
Simplified Cut Elimination for Kripke-Platek Set Theory

Gerhard Jäger

Abstract The purpose of this article is to present a new and simplified cut elimination procedure for KP. We start off from the basic language of set theory and add constants for all elements of the constructible hierarchy up to the Bachmann-Howard ordinal $\psi(\varepsilon_{\Omega+1})$. This enriched language is then used to set up an infinitary proof system IP whose ordinal-theoretic part is based on a specific notation system $C(\varepsilon_{\Omega+1}, 0)$ due to Buchholz and his idea of operator controlled derivations. KP is embedded into IP and complete cut elimination for IP is proved.

2.1 Introduction

Kripke-Platek set theory KP (with infinity) is a remarkable subsystems of Zermelo-Fraenkel set theory and had an enormous impact on the interaction between various fields of mathematical logic; see, for example, Barwise [1]. Its proof-theoretic analysis has been carried through in Jäger [11], and it is known that the proof-theoretic ordinal of KP is the Bachmann-Howard ordinal $\psi(\varepsilon_{\Omega+1})$ and that KP proves the same arithmetical sentences as the theory ID_1 of positive inductive definitions (cf. Feferman [8] and Buchholz, Feferman, Pohlers, and Sieg [6]). Functional interpretations of KP have been studied by Burr [7] and Ferreira [9].

The purpose of this article is to present a new and simplified cut elimination procedure for KP. We start off from the basic language of set theory and add constants for all elements of the constructible hierarchy up to the Bachmann-Howard ordinal $\psi(\varepsilon_{\Omega+1})$. This enriched language is then used to set up an infinitary proof system IP whose ordinal-theoretic part is based on a specific notation system $C(\varepsilon_{\Omega+1}, 0)$ due to Buchholz (see, for example, Buchholz [3]) and his idea of operator controlled derivations. KP is embedded into IP and complete cut elimination for IP is proved.

In the older proof-theoretic treatments of theories for admissible sets infinitary systems of ramified set theory play a central role. The build up of the set terms in

G. Jäger (✉)
Institut für Informatik, Universität Bern, Neubrückstrasse 10, 3012 Bern, Switzerland
e-mail: gerhard.jaeger@inf.unibe.ch

F. Ferreira et al. (eds.), *Axiomatic Thinking II*,
https://doi.org/10.1007/978-3-030-77799-9_2

these systems has always been complicated—requiring a lot of technical intermediate steps to deal with, for example, extensionality and equality—and is now for free.

This article is organized as follows: We begin with a very compact presentation of Kripke-Platek set theory KP (with infinity) and its Tait-style variant KP^T. Then we discuss the ordinal notation system and the derivation operators needed for our analysis. Here we can confine ourselves to a "slimmed down" version of Buchholz [3]. It follows the presentation of the new infinitary system IP with its very simple term structure. After some partial soundness and completeness results for IP we show how to embed KP^T into IP. The last two sections are dedicated to cut elimination: predicative cut elimination and collapsing. The Hauptsatz then tells us that the Bachmann-Howard ordinal is an upper bound for the cut-free embedding of the Σ fragment of KP into IP; also, the constructible hierarchy up to the Bachmann-Howard ordinal is a model of the Π_2 fragment of KP.

2.2 Kripke-Platek Set Theory

Let $\mathcal{L}$ be the standard first-order language of set theory with $\in$ as the only relation symbol, countably many set variables, and the usual connectives and quantifiers of first-order logic. With regard to the later proof-theoretic analysis we want all formulas of $\mathcal{L}$ to be in negation normal form. Thus, the *atomic formulas* of $\mathcal{L}$ are all expressions $(u \in v)$ and $(u \notin v)$. The formulas of $\mathcal{L}$ are built up from these atomic formulas by means of $\vee$, $\wedge$, $\exists$, $\forall$ as usual. We use as metavariables (possibly with subscripts):

- u, v, w, x, y, z for set-theoretic variables,
- A, B, C, D for formulas.

As you can see, we have no connective for negation. However, the negation $\neg A$ of A is defined via de Morgan's laws and the law of double negation. In addition, we work with the following abbreviations:

$$\begin{aligned}
(A \to B) &:\equiv (\neg A \vee B),\\
(A \leftrightarrow B) &:\equiv ((A \to B) \wedge (B \to A)),\\
(\exists x \in u)A[x] &:\equiv \exists x(x \in u \wedge A[x]),\\
(\forall x \in u)A[x] &:\equiv \forall x(x \in u \to A[x]),\\
(u = v) &:\equiv (\forall x \in u)(x \in v) \wedge (\forall x \in v)(x \in u).
\end{aligned}$$

To simplify the notation we often omit parentheses if there is no danger of confusion. Moreover, we shall employ the common set-theoretic terminology and the standard notational conventions. For example, A^u results from A by restricting all unbounded quantifiers to u. The Δ_0, Σ, Π, Σ_n, and Π_n formulas of $\mathcal{L}$ are defined as usual.

The logic of Kripke-Platek set theory is classical first-order logic. The *set-theoretic axioms of* KP consist of

(Equality) $\quad u \in w \wedge u = v \;\to\; v \in w$,
(Pair) $\quad \exists z(u \in z \wedge v \in z)$,

(Union) $\exists z(\forall y \in u)(\forall x \in y)(x \in z)$,
(Infinity) $\exists z(z \neq \varnothing \wedge (\forall x \in z)(x \cup \{x\} \in z))$,
(Δ_0-Sep) $\exists z(z = \{x \in u : D[x]\})$,
(Δ_0-Col) $(\forall x \in u)\exists y D[x, y] \rightarrow \exists z(\forall x \in u)(\exists y \in z)D[x, y]$,
($\in$-Ind) $\forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$.

The formulas D in the schemas (Δ_0-Sep) and (Δ_0-Col) are Δ_0 whereas the formula A in the schema ($\in$-Ind) ranges over arbitrary formulas of $\mathcal{L}$.

2.3 A Tait-Style Reformulation of KP

For the later embedding into the infinitary system IP it is technically convenient to work with a Tait-style variant KP^T of KP. In KP^T we derive finite sets of $\mathcal{L}$ formulas rather than individual formulas. In the following the Greek letters Γ, Θ, Λ (possibly with subscripts) act as metavariables for finite sets of $\mathcal{L}$ formulas. Also, we write (for example) $\Gamma, A_1, \ldots, A_n$ for $\Gamma \cup \{A_1, \ldots, A_n\}$; similarly for expressions such as Γ, Θ, A. Finite sets of formulas are to be interpreted disjunctively.

Axioms of KP^T.

(Tnd) $\Gamma, A, \neg A$ for all $\mathcal{L}$ formulas A.
(Equality) $\Gamma, u \in w \wedge u = v \rightarrow v \in w$.
(Pair) $\Gamma, \exists z(u \in z \wedge v \in z)$.
(Union) $\Gamma, \exists z(\forall y \in u)(\forall x \in y)(x \in z)$.
(Infinity) $\Gamma, \exists z(\varnothing \neq z \wedge (\forall x \in z)(x \cup \{x\} \in z)$.
(Δ_0-Sep) $\Gamma, \exists z(z = \{x \in u : D[x]\})$.
(Δ_0-Col) $\Gamma, (\forall x \in u)\exists y D[x, y] \rightarrow \exists z(\forall x \in u)(\exists y \in z)D[x, y]$.
($\in$-Ind) $\Gamma, \forall x((\forall y \in x)A[y] \rightarrow A[x]) \rightarrow \forall x A[x]$.

The formulas A in the Tertium-non-datur axioms (Tnd) and $\in$-induction axioms ($\in$-Ind) range over arbitrary $\mathcal{L}$ formulas whereas the formulas D in (Δ_0-Sep) and (Δ_0-Col) are supposed to be Δ_0.

Rules of inference of KP^T.

$$(or)\ \frac{\Gamma, A_i \quad \text{for } i \in \{0,1\}}{\Gamma, A_0 \vee A_1} \qquad (and)\ \frac{\Gamma, A_0 \qquad \Gamma, A_1}{\Gamma, A_0 \wedge A_1}$$

$$(b\text{-}ex)\ \frac{\Gamma, u \in v \wedge A[u]}{\Gamma, (\exists x \in v)A[x]} \qquad (b\text{-}all)\ \frac{\Gamma, u \in v \rightarrow A[u]}{\Gamma, (\forall x \in v)A[x]}$$

$$(ex)\ \frac{\Gamma, A[u]}{\Gamma, \exists x A[x]} \qquad (all)\ \frac{\Gamma, A[u]}{\Gamma, \forall x A[x]}$$

$$(cut)\ \frac{\Gamma, A \qquad \Gamma, \neg A}{\Gamma}.$$

In the rules (*b-all*) and (*all*) the eigenvariable u of these rules must not occur in their conclusion.

The notions of *principal formula* and *minor formula(s)* of an inference and that of *cut formula(s)* of a cut are as usual. We say that Γ is provable in KP^T iff there exists a finite sequence of finite sets of $\mathcal{L}$ formulas

$$\Theta_0, \ldots, \Theta_n$$

such that Θ_n is the set Γ and for any $i = 0, \ldots, n$ one of the following two conditions is satisfied:

- Θ_i is an axiom of KP^T;
- Θ_i is the conclusion of an inference of KP^T whose premise(s) are among $\Theta_0, \ldots, \Theta_{i-1}$.

In this case we write $\mathsf{KP}^T \vdash \Gamma$. It is an easy exercise to show that a formula A is provable in one of the usual Hilbert-style formalizations of KP iff $\mathsf{KP}^T \vdash A$. We leave all details to the reader.

2.4 An Ordinal System for the Bachmann-Howard Ordinal

Buchholz has developed several ordinal notation systems based on so called collapsing functions; see, for example Buchholz [2–4]. In the following we work with a reduced version, which is sufficient for our purposes. For that we need the following ingredients:

(1) Let *On* be the collection of all ordinals and let Ω be a sufficiently large ordinal. To simplify matters we set $\Omega := \aleph_1$, but also ω_1^{ck} or even somewhat smaller ordinals could do the job.
(2) The basic ordinal operations $\lambda\eta, \xi.(\eta + \xi)$ and $\lambda\xi.\omega^\xi$.
(3) The binary Veblen function φ, where φ_α is defined by transfinite recursion on α as the ordering function of the class

$$\{\omega^\beta : \beta \in On \ \&\ (\forall \xi < \alpha)(\varphi_\xi(\omega^\beta) = \omega^\beta\}.$$

In the following we write $\varphi\alpha\beta$ for $\varphi_\alpha(\beta)$.
(4) An ordinal α is called *strongly critical* iff $\alpha = \varphi\alpha 0$.
(5) Every ordinal α has a *normal form*

$$\alpha =_{NF} \varphi\alpha_1\beta_1 + \ldots + \varphi\alpha_n\beta_n$$

with $\beta_i < \varphi\alpha_i\beta_i$ for $i = 1, \ldots, n$ and $\varphi\alpha_1\beta_1 \geq \ldots \geq \varphi\alpha_n\beta_n$.
(6) The collection $SC(\alpha)$ of *strongly critical components* of an ordinal α is defined by

$$SC(\alpha) := \begin{cases} \varnothing & \text{if } \alpha = 0, \\ \{\alpha\} & \text{if } \alpha \text{ is strongly critical,} \\ SC(\beta) \cup SC(\gamma) & \text{if } \alpha = \varphi\beta\gamma \text{ and } \beta, \gamma < \alpha, \\ \bigcup_{i=1}^{n} SC(\alpha_i) & \text{if } \alpha =_{NF} \alpha_1 + \ldots + \alpha_n \text{ and } n > 1. \end{cases}$$

Based on that we can now introduce, for all α and β, the ordinals $\psi(\alpha)$ and the sets of ordinals $C(\alpha, \beta)$.

Definition 2.4.1 By recursion of α we simultaneously define:

(1) $\psi(\alpha) := \min(\beta : C(\alpha, \beta) \cap \Omega = \beta)$.
(2) $C(\alpha, \beta)$ is the closure of $\beta \cup \{0, \Omega\}$ under $+$, φ, and $(\xi \mapsto \psi(\xi))_{(\xi<\alpha)}$.

Since $C(\alpha, \beta)$ is countable, it is clear that $\psi(\alpha)$ is always defined and less than Ω in case that Ω is interpreted as $\aleph_1$. If Ω is interpreted as ω^{ck} (or a smaller ordinal), then additional considerations are required.

Now we list a series of properties of the sets $C(\alpha, \beta)$. Their proofs are either standard or follow from the results in the articles of Buchholz mentioned above.

Lemma 2.4.2 *We have for all ordinals α and β:*

(1) $C(\alpha, 0) = C(\alpha, \psi(\alpha))$.
(2) $C(\alpha, \psi(\alpha)) \cap \Omega = \psi(\alpha)$.
(3) $C(\alpha, \beta) \cap \Omega$ *is an ordinal.*

Every set $C(\alpha, 0)$ is well-ordered by the usual less relation $<$ on the ordinals but is not an ordinal itself. For example,

$$\Omega \in C(\alpha, 0) \quad \text{and} \quad (\forall \xi < \Omega)(\psi(\alpha) \leq \xi \rightarrow \xi \notin C(\alpha, 0)).$$

If we write $ot(\alpha, \xi)$ for the order-type of an element ξ of $C(\alpha, 0)$ with respect to $C(\alpha, 0)$, then

- $ot(\alpha, \xi) = \xi$ for all $\xi \leq \psi(\alpha)$,
- $ot(\alpha, \xi) < \xi$ for all elements of $C(\alpha, 0)$ greater than $\psi(\alpha)$.

In particular, we have $ot(\alpha, \Omega) = \psi(\alpha)$ and the order types $ot(\alpha, \xi)$ of all elements ξ of $C(\alpha, 0)$ are countable.

We write $\varepsilon_{\Omega+1}$ for the least ordinal $\alpha > \Omega$ such that $\omega^\alpha = \alpha$. Its collapse $\boldsymbol{\eta} := \psi(\varepsilon_{\Omega+1})$ is called the *Bachmann-Howard ordinal*. This number gained importance in proof theory since it is the proof-theoretic ordinal of the theory ID_1 of one positive inductive definition and of Kripke-Platek set theory KP; see, for example, Buchholz and Pohlers [5], Jäger [11], and Pohlers [13].

2.5 Derivation Operators

The general theory of derivation operators and operator controlled derivations has been introduced in Buchholz [3]. His main motivation was to provide a conceptually clear and flexible approach to infinitary proof theory that allows to put the finger on that part of the ordinal analysis of a sufficiently strong formal theory where the uniformity of proofs and a collapsing technique play the central role.

In this article we confine ourselves to that part of the general theory that goes along with the notation system $C(\varepsilon_{\Omega+1}, 0)$ described in the previous section.

Definition 2.5.1 Let $Pow(On)$ denote the collection of all sets of ordinals. A class function

$$\mathcal{H} : Pow(On) \to Pow(On)$$

is called a *derivation operator (d-operator for short)* iff it satisfies the following conditions for all $X, Y \in Pow(On)$:

(i) $X \subseteq \mathcal{H}(X)$.
(ii) $Y \subseteq \mathcal{H}(X) \implies \mathcal{H}(Y) \subseteq \mathcal{H}(X)$.
(iii) $\{0, \Omega\} \subseteq \mathcal{H}(X)$.
(iv) For all α,

$$\alpha \in \mathcal{H}(X) \iff SC(\alpha) \subseteq \mathcal{H}(X).$$

Hence every d-operator $\mathcal{H}$ is monotone, inclusive, and idempotent. Every $\mathcal{H}(X)$ is closed under $+$ and the binary Veblen function φ, and the decomposition of its members into their strongly critical components.

Let $\mathcal{H}$ be a d-operator. Then we define for all finite sets of ordinals $\mathfrak{m}$ the operators

$$\mathcal{H}[\mathfrak{m}] : Pow(On) \to Pow(On)$$

by setting for all $X \subseteq On$:

$$\mathcal{H}[\mathfrak{m}](X) := \mathcal{H}(\mathfrak{m} \cup X).$$

If $\mathcal{H}$ and $\mathcal{K}$ are d-operators, then we set

$$\mathcal{H} \subseteq \mathcal{K} := (\forall X \subseteq On)(\mathcal{H}(X) \subseteq \mathcal{K}(X)).$$

In this case $\mathcal{K}$ is called an *extension* of $\mathcal{H}$. The following observation is immediate from this definition.

Lemma 2.5.2 *If $\mathcal{H}$ is a d-operator, then we have for all finite sets of ordinals $\mathfrak{m}$, $\mathfrak{n}$:*

(1) $\mathcal{H}[\mathfrak{m}]$ *is a d-operator and an extensions of* $\mathcal{H}$.
(2) If $\mathfrak{m} \subseteq \mathcal{H}(\varnothing)$, *then* $\mathcal{H}[\mathfrak{m}] = \mathcal{H}$.
(3) $\mathfrak{n} \subseteq \mathcal{H}[\mathfrak{m}](\varnothing) \implies \mathcal{H}[\mathfrak{n}] \subseteq \mathcal{H}[\mathfrak{m}]$.

Now we turn to specific operators $\mathcal{H}_\sigma$. They will play a crucial role in connection with the embedding of KP into the infinitary proof system IP—to be introduced in the next section—and the collapsing procedure for IP.

Definition 2.5.3 We define, for all ordinals σ, the operators

$$\mathcal{H}_\sigma : Pow(On) \to Pow(On)$$

by setting for all $X \subseteq On$:

$$\mathcal{H}_\sigma(X) := \bigcap \{C(\alpha, \beta) : X \subseteq C(\alpha, \beta) \text{ and } \sigma < \alpha\}.$$

The following lemmas summarize those properties of these operators that will be needed later. For their proof we refer to [3], in particular Lemma 4.6 and Lemma 4.7. Assertion (5) is a consequence of Lemma 2.4.2(3).

Lemma 2.5.4 *We have for all ordinals σ, τ and all $X \subseteq On$:*

(1) $\mathcal{H}_\sigma$ *is a derivation operator.*
(2) $\mathcal{H}_\sigma(\varnothing) = C(\sigma + 1, 0))$.
(3) $\tau \leq \sigma$ *and* $\tau \in \mathcal{H}_\sigma(X) \implies \psi(\tau) \in \mathcal{H}_\sigma(X)$.
(4) $\sigma < \tau \implies \mathcal{H}_\sigma \subseteq \mathcal{H}_\tau$.
(5) $\mathcal{H}_\sigma(X) \cap \Omega$ *is an ordinal.*

Lemma 2.5.5 *Let $\mathfrak{m}$ a finite set of ordinals and σ an ordinal such that the following conditions are satisfied:*

$$\mathfrak{m} \subseteq C(\sigma + 1, \psi(\sigma + 1)) \cap \Omega \quad \textit{and} \quad \sigma \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing).$$

Then we have for $\widehat{\alpha} := \sigma + \omega^{\Omega+\alpha}$ and $\widehat{\beta} := \sigma + \omega^{\Omega+\beta}$:

(1) $\alpha \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing) \implies \widehat{\alpha} \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing)$ *and* $\psi(\widehat{\alpha}) \in \mathcal{H}_{\widehat{\alpha}}[\mathfrak{m}](\varnothing)$.
(2) $\alpha \in \mathcal{H}_\sigma[\mathfrak{m}](\varnothing)$ *and* $\alpha < \beta \implies \psi(\widehat{\alpha}) < \psi(\widehat{\beta})$.
(3) $\mathcal{H}_\sigma[\mathfrak{m}](\varnothing) \cap \Omega \subseteq \psi(\sigma + 1)$.

From now on the letter $\mathcal{H}$ will be used as a metavariable that ranges over d-operators.

2.6 The Infinitary Proof System IP

Henceforth, all ordinals used on the metalevel range over the set $C(\varepsilon_{\Omega+1}, 0)$ if not stated otherwise. In this section we introduce an infinitary proof system whose terms are constants for the elements of the initial segment of the constructible hierarchy $\boldsymbol{L}_\eta$ and whose proofs are controlled by derivation operators. Later we show that KP can be embedded into IP and that IP permits cut elimination and collapsing.

Definition 2.6.1 The language of IP is the following extension $\mathcal{L}[\boldsymbol{\eta}]$ of $\mathcal{L}$:

(1) For each element a of $\boldsymbol{L}_\eta$ we fix a fresh constant $\bar{a}$. These constants are the *terms of* IP. The letters r, s, t (possibly with subscripts) act as metavariables for the terms of IP.
(2) The *level* $|\bar{a}|$ *of* $\bar{a}$ is the least ξ such that $a \in \boldsymbol{L}_{\xi+1}$.
(3) The *formulas of* IP are now easily obtained from the formulas of $\mathcal{L}$ by simply replacing all their free variables by terms of IP; i.e. the formulas of IP are the sentences of $\mathcal{L}[\boldsymbol{\eta}]$.

Accordingly, the Δ_0, Σ, Π Σ_n, and Π_n formulas of IP are the Δ_0, Σ, Π Σ_n, and Π_n sentences of $\mathcal{L}[\boldsymbol{\eta}]$, respectively.

Definition 2.6.2 Every IP formula is an expression of the form $F[\bar{a}_1, \dots, \bar{a}_n]$ where $F[u_1, \dots, u_n]$ is a formula of $\mathcal{L}$ with the free variables indicated and $a_1, \dots, a_n$ are elements of $\boldsymbol{L}_\eta$. The set

$$par(F[\bar{a}_1, \dots, \bar{a}_n]) \ := \ \{|\bar{a}_1|, \dots, |\bar{a}_n|\}$$

is called the *parameter set* of this formula.

Observe that each Δ_0 sentence of $\mathcal{L}[\boldsymbol{\eta}]$ has a non-empty parameter set. Below it will be necessary to measure the complexities of the cut formulas appearing in a derivation. To this end we assign a rank to each $\mathcal{L}[\boldsymbol{\eta}]$ sentence.

Definition 2.6.3 The *rank* $rk(F)$ of an $\mathcal{L}[\boldsymbol{\eta}]$ sentence F is defined by induction on the number of symbols occurring in F as follows.

(1) $rk(\bar{a} \in \bar{b}) \ := \ rk(\bar{a} \notin \bar{b}) \ := \ \omega\cdot \max(|\bar{a}|, |\bar{b}|)$.
(2) $rk(F \vee G) \ := \ rk(F \wedge G) \ := \ \max(rk(F), rk(G)) + 1$.
(3) $rk((\exists x \in \bar{a})F[x]) \ := \ rk((\forall x \in \bar{a})F[x]) \ := \ \max(\omega\cdot|\bar{a}|, rk(F[\bar{\varnothing}]) + 1)$.
(4) $rk(\exists x F[x]) \ := \ rk(\forall x F[x]) \ := \ \max(\Omega, rk(F[\bar{\varnothing}]) + 1)$.

Finally, we define the *level* $lev(F)$ of an IP formula F by

$$lev(F) \ := \ \begin{cases} \max(par(F)) & \text{if } rk(F) < \Omega, \\ \Omega & \text{if } \Omega \le rk(F)). \end{cases}$$

Some important properties of the ranks of $\mathcal{L}[\boldsymbol{\eta}]$ sentences are summarized in the following lemma. Its proof is straightforward and left to the reader.

Lemma 2.6.4 *We have for all* IP *formulas* F, G *and all* $a, b \in \boldsymbol{L}_\eta$*;*

(1) $rk(F) = rk(\neg F)$.
(2) $rk(F) < \omega\cdot lev(F) + \omega$.
(3) $rk(F), rk(G) < rk(F \vee G)$.
(4) $|\bar{b}| < lev(F[\bar{\varnothing}]) \quad \Longrightarrow \quad rk(F[\bar{b}]) = rk(F[\bar{\varnothing}])$.
(5) $b \in a \quad \Longrightarrow \quad rk(F[\bar{b}]) < rk((\exists x \in \bar{a})F[x])$.

(6) $rk(F[\bar{b}]) < rk(\exists x F[x])$.
(7) $rk(F) \in \mathcal{H}[par(F)](\varnothing)$.
(8) $\alpha \in par(F) \implies \alpha \leq rk(F)$.

The proof system for IP will be Tait-style. From now on we let the Greek letters Γ, Θ, Λ (possibly with subscripts) also range over finite sets of $\mathcal{L}[\boldsymbol{\eta}]$ sentences.

For a finite set $\mathfrak{S} = \{F, \ldots, F_m, r_1, \ldots, r_n\}$ of formulas and terms of IP we set

$$par(\mathfrak{S}) \;:=\; par(F_1) \cup \ldots \cup par(F_m) \cup \{|r_1|, \ldots, |r_n|\}.$$

Accordingly,

$$\mathcal{H}[\Gamma, F_1, \ldots, F_m, r_1, \ldots, r_n] \;:=\; \mathcal{H}[par(\Gamma) \cup par(\{F_1, \ldots, F_m, r_1, \ldots, r_n\})].$$

Variants of this notation may also be used. However, it will always be clear from the context what is meant.

Axioms of IP. The axioms of IP are all finite sets

$$\Gamma,\ (\bar{a}_1 \in \bar{b}_1) \qquad \text{and} \qquad \Gamma,\ (\bar{a}_2 \notin \bar{b}_2)$$

with $a_1, a_2, b_1, b_2 \in \boldsymbol{L}_\eta$, $a_1 \in b_1$, and $a_2 \notin b_2$.

So the axioms of IP are the finite sets that contain true atomic sentences of $\mathcal{L}[\boldsymbol{\eta}]$. The next definition introduces the derivability relation, controlled by derivation operators.

Definition 2.6.5 $\mathcal{H} \overset{\alpha}{\underset{\rho}{\vdash}} \Gamma$ iff $par(\Gamma) \cup \{\alpha\} \subseteq \mathcal{H}(\varnothing)$ and one of the following cases holds:

(Ax)	Γ is an axiom				
($\vee$)	$F_0 \vee F_1 \in \Gamma$	&	$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F_i$	&	$\alpha_0 < \alpha,\ i \in \{0, 1\}$
($\wedge$)	$F_0 \wedge F_1 \in \Gamma$	&	$\mathcal{H} \overset{\alpha_i}{\underset{\rho}{\vdash}} \Gamma, F_i$	&	$\alpha_i < \alpha$ for $i = 0, 1$
($b\exists$)	$(\exists x \in \bar{a}) F[x] \in \Gamma$	&	$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F[\bar{b}]$	&	$\alpha_0, \lvert\bar{b}\rvert < \alpha,\ b \in a$
($b\forall$)	$(\forall x \in \bar{a}) F[x] \in \Gamma$	&	$\mathcal{H}[\bar{b}] \overset{\alpha_b}{\underset{\rho}{\vdash}} \Gamma, F[\bar{b}]$	&	$\alpha_b < \alpha$ for all $b \in a$
($\exists$)	$\exists x F[x] \in \Gamma$	&	$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F[\bar{b}]$	&	$\alpha_0, \lvert\bar{b}\rvert < \alpha,$
($\forall$)	$\forall x F[x] \in \Gamma$	&	$\mathcal{H}[\bar{b}] \overset{\alpha_b}{\underset{\rho}{\vdash}} \Gamma, F[\bar{b}]$	&	$\alpha_b < \alpha$ for all $b \in \boldsymbol{L}_\eta$
(Ref)	$\exists x F^x \in \Gamma$	&	$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F$	&	$\alpha_0, \Omega < \alpha,\ F \in \Sigma$
(Cut)	$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, F$	&	$\mathcal{H} \overset{\alpha_0}{\underset{\rho}{\vdash}} \Gamma, \neg F$	&	$rk(F) < \rho,\ \alpha_0 < \alpha$