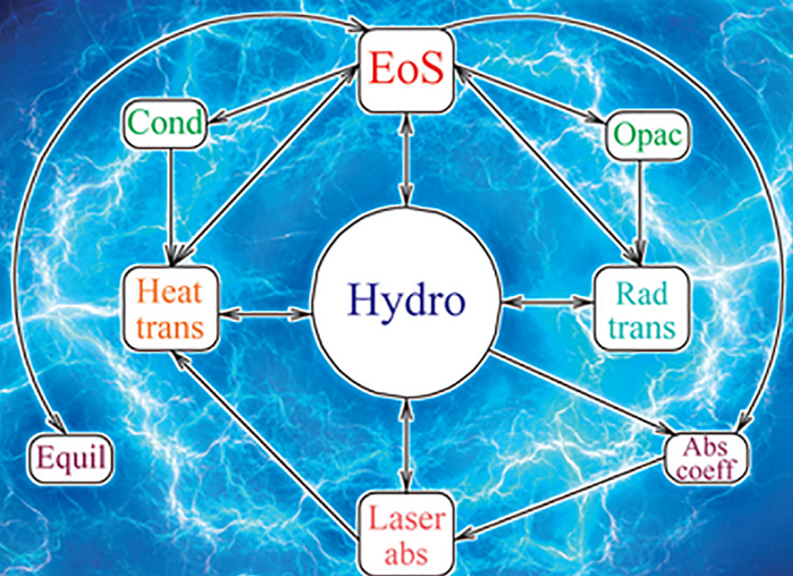
Geoffrey J. Pert

Foundations of Plasma Physics for Physicists and Mathematicians



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This book is dedicated with much appreciation to the mentors of my early career.
Peter Smy, Stuart Ramsden and Michael Woolfson
who all showed great faith in an unknown physicist
and whose guidance shaped my career as a professional scientist.
Also to
Malcolm Haines
who forgot more plasma physics than I ever knew.
Finally, but not least, to my longstanding friend and colleague
Greg Tallents
without his support, help, and encouragement this book
would have never seen the light of day.

Contents

Preface

Plasma often called 'the fourth state of matter' is the most abundant form of observable matter in the universe. Nonetheless plasma physics is a relatively new discipline, unknown before 1900. Ionisation studies originated in the experimental study of gas breakdown in strong electric fields by Paschen (1889). The inception of low temperature plasma physics may be considered to have been the result of the discovery by Thomson (1897) of the negatively charged electron in gas discharges in 1897, although plasma physics as we know it today did not evolve until the mid-1920s. Positively charged ions, essentially gas molecules which had lost an electron, were found soon after. In the early years of the twentieth century, gas discharges and arcs were empirically investigated. The related topics of breakdown of gases and ionisation and recombination dominated this field known as *ionised gases*. This activity becoming progressively more realistic during the early years of the twentieth century as the mobile role of molecules and particles became better understood; but still only in media of low ionisation, where the number of charged particles is small compared to that of the background gas. von Engel (1965) gives an interesting history of the study of ionised gases up to about to 1900. The early studies of breakdown, glow discharges and arcs are covered in detail by von Engel (1965), Cobine (1958), and Loeb (1955). By 1930 an essential understanding of the low temperature plasma had been developed exemplified by the two volume text (von Engel and Steenbeck 1932). At that time quantum physics was not well developed and essentially relatively simple classical models had to used. These yielded basic formal structures with functional relationships but with coefficients to be determined by experiment – an approach which has worked well and is still used today, exemplified by volumes such as Brown (1967). The field, now known as *low temperature plasma physics* has become increasingly developed (Raizer, 1997; Lieberman and Lichtenberg, 2005; Smirnov, 2015) as the earlier models have been progressively refined and improved. Although the basic theory has remained unchanged since the 1930s the detailed methodology has been refined and agreement between experiment and theory improved. In recent years a wide range of direct uses for both direct current and microwave discharges has been found such as gas laser pumping. Important commercial applications now include plasma processing, coating, etching, lighting, and lightning (Lieberman and Lichtenberg, 2005).

In the 1920s, the subject broadened after it became realised that the interior of stars for example must be at such high temperatures that they could contain only fully stripped bare atomic nuclei and electrons (Eddington, 1959). Astrophysical plasmas became an important subject area today embracing a much wider field than simply stellar interiors. Nowadays plasma physics embraces the whole gamut of astronomical bodies from aurora, the magnetosphere to the edge of black holes and led to many conceptual advances in subject (Alfvén, 1950). This area remains an active and important discipline in its own right. At this stage, the end of 1920s, we see the identification of the essential criteria defining plasma through two seminal works. Both involve the nature of the interactive force between charged particle in the plasma.

The first important conceptual step forward towards distinctive plasma physics was taken in 1923 by the publication of the paper by Debye and Hückel (1923) relating to the behaviour of electrolyte ions in solution. Considering the behaviour of positively and negatively charged particles with equal average charge density, Debye and Hückel argued that each ion of one charge is surrounded more closely by ions of the opposite charge thereby decreasing the field of the primary ion below that associated with Coulomb force law over distances in excess of a characteristic length, known as the Debye length (Section 1.3), which tends to zero as the particle temperature tends to zero.

The second key step was the identification of collective waves by Langmuir (1928) (and more fully Tonks and Langmuir 1929). It was proposed that cold plasma moved as a 'block', the individual particles oscillating in the field of their neighbours through the Coulomb force in a *plasma wave* Section 1.2. Again the medium is acting collectively through the Coulombic interparticle field, and *not* through the individual particle fields (see Section 6.2).

Realising that they were observing behaviour not described by conventional nomenclature Langmuir coined the name *plasma* in Langmuir (1928). More details on the origin of the name are given by Tonks (1967) and Mott-Smith (1971) who state that it arose from similarities with blood plasma as a transport fluid. In fact the word plasma stems from the Greek πλάσμα meaning a 'maleable substance' which is appropriate in view of the uncanny ability of the positive column of a glow discharge to fill the space available to it. The name was slow to become adopted but by the 1950s was widely used, particularly in connection with controlled fusion.

The following decade was characterised by the development of plasma kinetic theory.

Firstly Landau (1936) developed a form of Fokker–Planck equation Section 7.3 to account for the fact that most particle interactions are long range and therefore weak involving many particles simultaneously. In contrast to earlier work analogous to gases where the interaction was assumed short range and the Chapman–Enskog method, Section 4.11, used for fluids, had been incorrectly used to determine the hydrodynamic behaviour in plasma. This approach was followed by an improved more formal stochastic picture by Chandrasekhar (1943) (see Section 7.2) in an astrophysical context. The method was completed for application in plasma by Rosenbluth, MacDonald and Judd (1957), the final element being the introduction of the Coulomb logarithm by Cohen, Spitzer, and Routly (1950) following several early workers. The development of *magneto-hydrodynamics*, the plasma equivalent of fluid theory which treats ensembles of large numbers of particles averaged over velocity, was now complete.

The second development of kinetic theory was in the area of the microscopic distribution in which the particle motions are treated individually before a final averaging. Recognising the relative importance of collective behaviour over collisional, Vlasov (1938) introduced what is essentially a collisionless Boltzmann equation Section 6.2. This kinetic equation has proved very successful in dealing with the large number of plasma modes of oscillation. An important development of the theory due to Landau (1946) took account of the velocity resonance at the wave phase speed in warm plasma to develop damping Appendix 12.A. The approach has been widely used to identify micro-instability and damping.

These two approaches complete the underlying theoretical structure of plasma physics, individually appropriate for different problems. By about 1960 the formal structure of plasma physics was well established. Further development involved the development of theoretical pictures to investigate specific problems.

The successful achievement of man-made thermo-nuclear reactions in 1952 led to a marked increase in activity on the possibility of controlled reactions suitable for the generation of electrical power in the USA, UK, and USSR, which was mostly hidden behind security barriers. One of the first, and probably best known, was by John Lawson at Harwell, initially issued as a secret report (Lawson, 1955), but later a revised form in the open literature (Lawson, 1957), introducing the familiar Lawson criterion giving a necessary condition for useful thermo-nuclear power. His final conclusion (Lawson, 1955) remains as true today as it was 60 years ago.

> Even with the most optimistic possible assumptions, it is evident that the conditions for the operation of a useful thermonuclear reactor are very severe.

Nonetheless as we shall see in the course of this book, a major part of the road to controlled nuclear fusion has now been achieved.

Despite these problems, the possibility of such a device and its anticipated rewards led to surge in activity in plasma physics as it is was clear that the lowest temperatures require the working medium to be plasma. The general interest in the possibility of achieving fusion power using plasma as the medium has led to an enormous increase in the resources allocated to these problems and a wide variety of experimental device (Glasstone and Lovberg, 1960). As a consequence, the subject as a whole has moved rapidly forward with particular emphasis on conditions required for stable confined plasma, particularly toroidal geometries (tokamaks). Since 1970, laser generated plasma has offered an alternative route to fusion through inertial confinement.

It is a characteristic of any new field, that its development is likely to be reflected in the way plasma physics has developed over the years, namely in fits and starts. 1900–1940 the era of gas discharges, and collisional effects, 1940–1950 the start of what we might call the modern era where collective behaviour was being understood. 1950–1970 the age of major theoretical and experimental development when the basic understanding of waves and instabilities was developed. 1970–present the construction of large toroidal machines and detailed theory. plus laser-plasma interactions. This is reflected in the text books used. Several of the best date from 1960–1980, but have been updated. We have tried to reflect this historical growth in the structure of the book and included as many of the classic papers and books as possible. Consequently many of the papers cited may appear ‘old-hat’.

The historical development is interesting as case study in itself as plasma the fourth state of matter was only recognised from about 1925 and its development as a discipline has been rapid. The period 1950–1970 was particularly important as due to security issues relating to fusion much of the work carried out in laboratories in the US, USSR, and UK was classified. A major event in 1956 was the lecture given by Kurchatov (1956) at Harwell revealing the progress made by Soviet workers and opening the security blanket.

Major declassification occurred as a result of the second UN conference on the Peaceful Uses of Atomic Energy at Geneva when fusion was included and lead to a sharing of results on Plasma Physics. Since then the subject aside from matters relating to H-bombs has lain outside the security blanket.

A key assessment by Lawson (1957) of the conditions necessary for controlled fusion to take place defined *ignition* as the state when more energy is released in the fusion reaction that is put into heating the fuel.[1]

1 Allowing for the conversion efficiency η from plasma energy to output, we equate the fusion energy release to the sum of initial energy of the particles $3kT$ and the bremsstrahlung heat loss $\alpha T^{1/2}\tau$ to the fusion energy release

$$\frac{\eta}{4(1-\eta)}\,\langle\sigma v\rangle\,\Delta E\tau = 3kT + \alpha T^{1/2}\tau \tag{1}$$

It turned out that on both sides of the ‘iron curtain’ the major experimental efforts had been on toroidal linear pinches, loosely based on the steady Bennett pinch (problem 7). Unfortunately, such plasmas are inherently unstable as we show in Section 16.2.2. As we shall see work in the USSR had gone farthest to mitigate these effects, but none were successful. None-the-less in the west and the USSR work continued to proceed relatively independently. In 1960, the US programme contained many different elements aside from pinches, which were conveniently surveyed in Glasstone and Lovberg (1960); the major effort being directed towards steady plasma in the *stellarator*. In the United Kingdom, the study of toroidal Z-pinches had been actively pursued since 1945 (Hendry and Lawson, 1993) ultimately leading to the construction of *ZETA* a large toroidal pinch.[2] USSR led the development of toroidal discharges initially proposed by Tamm (1990) and Shafranov (2001) which metamorphosed into *tokamaks*. By 1969, the Russians were demonstrating very promising results with their T3 Tokamak which had been distrusted in the United States on the basis of their unconventional diagnostics. Despite the cold war, a group from Culham, United Kingdom led by Derek Robinson and Nic Peacock were invited to Russia (Forrest, 2011) to independently check the ion temperatures. The Culham group were probably the world leaders on laser scattering at that time, a method recognised as being unequivocal. These experiments confirmed the Russian claims and a result tokamaks have become recognised as the most probable route to magnetically confined fusion with large machines being built in Europe (JET), the United states (PLT), and Japan (JT60). At present an international collaboration is constructing the ITER device to demonstrate break-even.[3]

Following de-classification in 1972, a second possible route to controlled thermo-nuclear fusion was opened with laser compression and *inertial confinement fusion* with the publication by Nuckolls, Wood, Thiessen and Zimmerman (1972), and shortly later by Clark, Fisher and Mason (1973). In this scheme, fusion takes place as a small pellet or shell of thermo-nuclear fuel is compressed by the ablation pressure and then blows apart due to its high temperature having being heated to ‘burn’ temperatures by a high power laser pulse. Since then the conditions necessary to achieve thermo-nuclear reactions have been progressively refined from these simple geometries as technical problems have arisen until today’s designs involve complex targets and either direct heating by the laser or indirect by X-rays inside a small cavity (*hohlraum*) (Lindl, 1993). Large-scale laser

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where n is the particle density, τ the containment time, and T the temperature; η is the efficiency parameter, the fusion reaction rate averaged over the particle velocity distribution, and α the bremsstrahlung emission coefficient. This occurred when the *Lawson criterion* is satisfied

$$n\,\tau > \frac{3 k T}{\frac{\eta}{4(1-\eta)}\,\langle\sigma v\rangle\;\Delta E - \alpha T^{1/2}} \tag{2}$$

Taking representative values for η, α; and noting that the DT reaction requires the lowest temperature for ignition with minimum $kT \approx 20$ keV (Lawson 1957) identified the key criterion as $n\tau > 10^{20}\ \mathrm{m}^{-3}$ s which with only the following slight modification has remained unchanged today (see Wesson, 2011). Later studies have indicated that triple product $n\ \tau\ T$ obtained under a constant pressure condition giving a minimum temperature for the DT reaction of 14 keV and triple product $nT\tau \approx 3 \times 10^{21}\ \mathrm{keV\ s\ m}^{-3}$.

2 Unfortunately ZETA was initially stigmatised by an unfortunate premature press release claiming the observation of controlled thermonuclear reactions, whereas in fact the measured neutrons were generated in the electric fields produced by instabilities. Despite this ZETA later carried out much novel and valuable work, including the serendipitous discovery of self-reversal in *reversed field pinches* (Bodin and Newton, 1980) and improved stability.

3 Author’s note: I was fortunate to be attending the APS Plasma Division meeting when these results were announced and remember well the excitement caused. In retrospect looking back over 50 years it was possibly the most important result in that time, both scientifically and geopolitically, reconciling the Russian and American fusion programmes across cold war divisions. It also saw the death knell of the pathologically damaging anomalous Bohm type diffusion in fusion studies (see p. 35 in chapter 2).

facilities are required for successful implementation of this scheme and have therefore been constructed. The core of the *inertial confinement* scheme is essentially hydro-dynamic and is treated in some detail in our previous work (Pert, 2013, Ch.14). The interaction of the laser with plasma surface on the other hand has many features which lie within the ambit of this work and are treated here in Chapters 11 and 14.

One of the characteristics of plasma is the large number of degrees of freedom it possesses especially in the presence of a background magnetic field. As a result, there are a wide range of differing wave motions, both stable and growing, possible. Consequently, much of this discussion is concerned with waves of different types, both free and driven.

This text in two parts, reflecting this development of plasma physics. The first part, Chapters 1–8 guide the reader through the underlying theoretical structure of the subject. The remaining Chapters 9–14 deal with specific problems, principally associated with fusion

Chapter 1. Characteristic Collective Behaviour of Plasma

Chapter 2. Classical Behaviour for Plasma with Simple Collisions

Chapter 3. Single Particle Model – Motion of Charged Particles in Static Fields Neglecting Particle Interactions

Chapter 4. Fluid Theory and Hydrodynamic Equations Obtained by Integrating Basic Kinetic Theory Over the Particle Distribution with Only Short Range Collisional Interactions

Chapter 5. Review of the Properties of Waves in Dispersive and Anisotropic Media Including Geometrical Optics

Chapter 6. Development of Kinetic Effects When Collective Effects Dominate, Therefore Neglecting any Short Range Collisions – Vlasov Equation

Chapter 7. Plasma Theory and magnetohydrodynamic (MHD) Equations Obtained by Integrating Basic Kinetic Theory Over the Particle Distribution Contrasting the Difference Between Short Range Collisions (Gases) and Long Range Coulombic Interactions (Plasma)

Chapter 8. MHD Hydrodynamics for Plasma Where the Coulomb Force Is Strong Inducing Co-operative Behaviour Modifying Fluid Dynamics to Include the Role of the Magnetic Field on Charged Particles

Chapter 9. Simplification to MHD Flow Introduced by Infinite Conductivity – Absence of Collisional Effects

Chapter 10. Spectrum of Hydro-magnetic Acoustic Waves Found in Ideal MHD Plasma Including Shock Waves

Chapter 11. Spectrum of Waves in Cold (Zero Temperature) Plasma in Magnetic Fields

Chapter 12. Dielectric Properties of Waves in Warm Plasma Mainly in the Absence of a Magnetic Field

Chapter 13. Generalisation of the Classical Theory of Electro-magnetic Waves to Plasma and the Non-linear Interactions at High Beam Intensity

Chapter 14. Brief Introduction to Interaction Between a Laser Beam and Plasma Within the Classical Quasi-linear Regime, Where the Absorption/Thermal Conduction Are All Within the Constraints of 'Linear' Theory. Non-linear Regimes Needing Extensive Numerical Calculation Are Appropriate to Later Research Texts Such as Atzeni and ter Vehn (2004)

Chapter 15. MHD Theory of Stable Plasma Configurations, Particularly Toroidal Configurations

Chapter 16. MHD Theory of Instability in Plasma Configurations

Supplementary sections on the basic models of low temperature plasma, and the theory of complex variables and Laplace transforms are included to provide, if necessary, background material and revision.

The book was initially planned to be aimed at final year undergraduate and postgraduate students. However as work progressed, it became clear that the extended and complex nature of the subject introduced many new concepts and techniques from other areas which would be unfamiliar to the targeted students. The objective of the book therefore changed to provide the background material in both basic plasma physics and other related topics which would allow an easy access to complex research. The book is not intended to provide a review of the current state of plasma research, but to provide a ready source to the methodology and techniques used in that research, a sort of compendium in the basic physics underlying plasmas. Although the book is now aimed at postgraduate students, it remains very suitable for use as a basic final year undergraduate course in the methodology of plasma physics. It is the result of many years working in plasma physics research, especially laser-plasma interactions. Much of the material is based on different lecture courses given at the Universities of Alberta, Hull and York, over the years. The choice of topics included has ranged over different fusion orientated syllabi. In the interests of conciseness, much important (and interesting to the author) material in atomic and ionisation physics, astrophysics, magnetospheric physics, and low temperature industrially orientated plasma that would have been included in a full survey of the field has of necessity been omitted. However, the basic material contained in the book should have wide application.

July 2020

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1

Fundamental Plasma Parameters – Collective Behaviour

1.1 Introduction

Plasma is generally defined as *an electrically neutral, conducting gas*. The designation was introduced by Langmuir (1928) to describe the behaviour of the medium in an electrical discharge distant from the electrodes.[1] In this region, the gas contains a balanced number of positively and negatively electrically charged ions and electrons so that the total charge density is zero. Thus, if the ion density is n_i and charge is Ze, the electron density is $n_e = Z\, n_i$. Collisions are relatively infrequent, and the conductivity correspondingly high.

The behaviour of the plasma is determined by the motion of individual particles subject to electric and magnetic forces either externally applied or resulting from their charge. In fact, this is a complex problem as the particles themselves generate space charges and currents, and in consequence, electric and magnetic fields. Thus, the particles mutually interact through these self-generated fields.

This mutual interaction can take place in two ways due to the long-range nature of the fields. First, over short ranges, the particles act independently, and the interaction resembles the form of the conventional two particle collision found in gases. However, over longer distances, the particles behave collectively leading to group interactions such as a wave oscillations.

In all conventional plasmas, the ions and electrons can be treated as classical particles. The basic equations governing plasma are therefore Maxwell's equations for the fields combined with Newton's law of motion for the dynamics. Relativistic plasmas can be formed under exceptional conditions, but will not be treated here.

An immediate consequence of this mutual interaction is that a new form of longitudinal oscillation becomes possible. In contrast to sound waves in gases, where the forces are due to pressure and communicated by short range collisions, the force is due to the long-range nature of electrostatic forces. This oscillation introduces a characteristic frequency, the electron plasma frequency Π_e, at which the plasma can respond to temporal changes imposed upon it. The plasma cannot follow changes occurring over times $\sim \Pi_e^{-1}$. As a consequence, it may be anticipated that there is also a limiting distance over which spatial change can occur determined by the thermal velocity and the plasma frequency $\lambda_D \approx \overline{v}_e/\Pi_e$, where $\overline{v}_e = \sqrt{k T_e/m_e}$, m_e being the electron mass, k Boltzmann's constant, and T_e the electron temperature.

1 See Tonks (1967) for an account of the origin of the name.

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In this chapter and the next, we introduce the essential characteristics of dilute plasmas by considering some basic problems, which illustrate the collective behaviour of the constituent particles resulting from their electric charges. The electrostatic Coulomb force has a potential which varies as $1/r$, where r is the distance from the charge. The force is therefore effective at long ranges; as distances increase, the cumulative effect of the larger number particles balances the fall-off of the individual fields. This leads to a number of characteristic effects, of which we examine some of the simplest in this chapter: the space charge shielding of electric fields and longitudinal waves. The collision of particles by many charges simultaneously and their effects are deferred until the following Chapter 2.

A final important characteristic of plasma concerns the relative strength of the electrostatic interaction energy compared to thermal. If this ratio is small, the particles move freely and the correlation between particles is small; the plasma is *dilute*. The particle distribution is therefore approximately Maxwellian. In contrast if this ratio is large, particles are 'tied' to each other and the distribution highly correlated. The ultimate limit of this *strongly coupled plasma* behaviour can be seen in an ionic crystal lattice.

1.2 Cold Plasma Waves

Consider a cold (zero temperature) quasi-neutral plasma comprising ions of charge $Z\,e$ and electrons of charge $-e$ so that

$$Zn_{i0} = n_{e0}$$

The plasma is considered to be collision-free. Therefore, if the electrons are disturbed, the ions remaining stationary, a collective motion of the electrons is established by the internally generated fields. This contrast with a gas where collisions transmit the motion from element to element.

In one dimension, the motion of the particles takes place as planar slabs moving normal to their surface. Consider a block of electrons initially at x_0 of thickness δx_0 displaced at a distance X, the ions remaining stationary. As a result, there is an increase in the number of electrons at one end and a corresponding increase of an equal number of unbalanced positive charges at the other end of the displaced block of width X, which have been uncovered by the moving slab. Applying Gauss' theorem across the block, it is trivial to show that a uniform field

$$E_x = n_{e0} e X/\epsilon_0 \tag{1.1}$$

is generated on the plasma within the block. This field will accelerate the electrons back toward the ions

$$\frac{\mathrm{d}^2 X}{\mathrm{d}t^2} = -\frac{eE_x}{m_e} = -\frac{n_{e0}\,e^2}{\epsilon_0\,m_e} X = -\Pi_e{}^2 X \tag{1.2}$$

The electrons therefore oscillate about the ions with a frequency

$$\Pi_e = \sqrt{n_{e0} e^2/\epsilon_0\,m_e} \tag{1.3}$$

known as the *electron plasma frequency* or more commonly succinctly the *plasma frequency*.

Clearly, we may build up a more general bulk oscillation by summing over a series of slabs to allow for general forms of the initial displacement, i.e. as a wave with varying amplitude, provided the linear assumption is made that the ordering of the electrons remains unchanged. These *cold plasma waves* are non-propagating and are standing waves with zero group velocity.

The characteristic plasma frequency essentially represents the fastest speed at which electrons can respond. Therefore, we can propagate waves with frequency $\omega > \Pi_e$, but lower frequencies may be damped; in waves where $\omega < \Pi_e$, the electrons can move sufficiently rapidly to prevent the propagation of the wave.

1.2.1 Wave Breaking

Consider the slab whose ambient position is x_0 and is displaced by X and compare it with the neighbouring slab from $x_0 + \delta x_0$ which is displaced by $X + \delta X$. The spacing between the slabs

$$\begin{aligned}\delta x(t) &= [x(t) + \delta x(t)] - x(t) = [x_0 + \delta x_0 + X(t) + \delta X(t)] - [x_0 - X(t)] \\ &= \delta x_0 + \delta X(t) \end{aligned} \tag{1.4}$$

Clearly, the ordering of the slabs is maintained provided $\delta x(t) > 0$ if $X(t) > 0$ and *vice versa*, i.e.

$$\frac{\partial X(t)}{\partial x_0} > -1 \tag{1.5}$$

If this condition is not upheld, the electron sheets lose their integrity, the phases are mixed, and the wave damped. The wave energy is converted into disordered electron kinetic energy.

For a cold plasma wave differentiating Eq. (1.2) with respect to x_0, multiplying the result by $\partial X/\partial x_0$ and integrating yields

$$\Pi_e{}^2\left(\frac{\partial X}{\partial x_0}\right)^2 + \left(\frac{\partial \dot{X}}{\partial x_0}\right)^2 = \text{const} = W \tag{1.6}$$

so that if $W < \Pi_e{}^2$ at any time, the wave does not break.

The spatial position of the element x_0 is given by $x = X + x_0$, and the displacement $X(t)$ at time t by Eq. (1.2). Hence, the instantaneous electric field at point x at time t, $E(x, t)$ is easily obtained from Eq. (1.1). The behaviour of the wave at breaking is easily demonstrated by a simple example introduced by Dawson (1959). Consider a sinusoidal wave such that

$$X(0) = A\ \sin(k\,x_0) \quad \text{and} \quad \dot{X}(0) = 0$$

The electric field at time $t = 0$ is, therefore,

$$E(x, 0) = \frac{e n_{e0}}{\epsilon_0} A \sin(k\,x_0) = E_{\max}\ \sin(k\,x_0) \tag{1.7}$$

where $x = x_0 + A\sin(k\,x_0)$. It is easily established from Eq. (1.5) that this wave breaks if $A \geq 1/k$. The structure of the electric field as the amplitude of the wave varies is shown in Figure 1.1. It is clearly seen that wave breaking occurs if the amplitude of the wave $A \geq 1/k$ as predicted. Once the wave has broken the calculated field is multi-valued, which is, of course, unphysical. Since the maximum field intensity $E_{\max} = e n_{e0} A/\epsilon_0$, the amplitude of the critical field leading to wave breaking is

$$E_{\text{break}} = \frac{e\,n_{e0}}{\epsilon_0\,k} \tag{1.8}$$

which may be written in terms of the 'bounce frequency' $\omega_b = \sqrt{ekE_0/m}$ when E_0 is the intensity amplitude of the wave. At the wave breaking condition, $\Pi_e = \omega_b$.

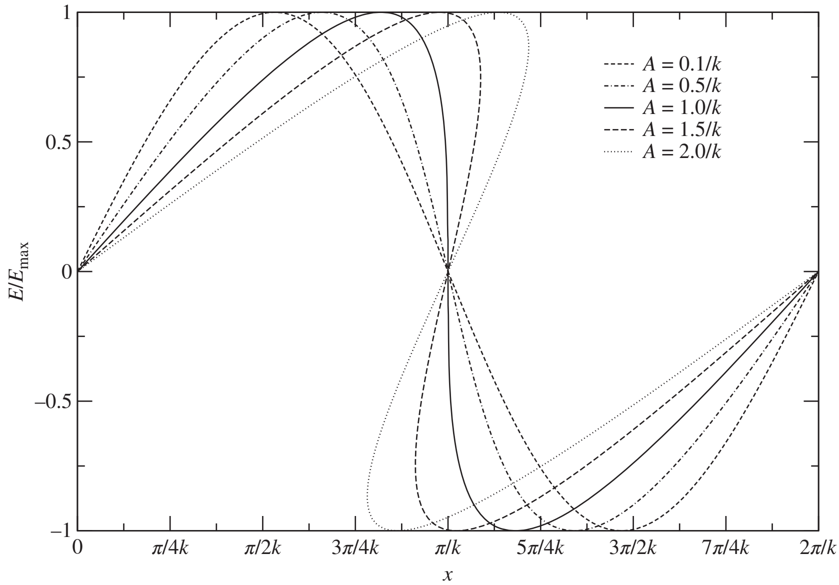

Figure 1.1 Plots of the electric field distribution as fractions of the peak field for the model plasma wave. For $A < 1/k$, it can be seen the field is well behaved, but for $A > 1/k$, it is multi-valued characteristic of breaking.

1.3 Debye Shielding

In a plasma, both ions and electrons are charged. As a consequence of the statistical distribution of the particles, there is a rapidly varying space charge field, the *micro-field*. The spatial and temporal fluctuations of this micro-field are determined by the random motion of the charged particles. In the presence of an applied electric field, the particles become polarised as in a dielectric, setting up a field which neutralises the applied one. The distance over which this space charge field is established is known as the *Debye length*. It is important to note that in contrast to a dielectric, the polarisation is dynamic, particles moving rapidly within the space charge, and the shielding is established as a statistical average over temporal fluctuations.

In particular, as a consequence of their negative charges, electrons tend to 'clump' around the positive charges of the ions. Provided the temperature is sufficiently high, the electrons are not bound to the ions but move freely, with slightly increased density in the neighbourhood of the ion. Provided the spatial range of the perturbation is extensive, a large number of particles are weakly slightly perturbed. Individual particles are averaged, and the space charge forms a continuum, established by the perturbed particle distribution.

Consider a test ion inserted into a plasma, which contains on average n_i^0 ions of charge Ze per unit volume and n_e^0 electrons with charge $-e$ per unit volume. The ions have an equilibrium thermal distribution with temperature T_i and the electrons T_e. Within the plasma, there are local variations of the average electric potential ϕ within which will be corresponding variations of the ion and electron densities. Since each particle is in thermal equilibrium with its own temperature, the local densities are given by the Boltzmann distribution

$$\begin{aligned} n_e &= n_e^0 \ \exp[e\,\phi/kT_e] \\ n_i &= n_i^0 \ \exp[-Ze\,\phi/kT_i] \end{aligned} \tag{1.9}$$

although overall charge neutrality holds for the average densities

$$Z\, n_i^0 = n_e^0 \tag{1.10}$$

Consider the field in the neighbourhood of the test ion with charge Ze at $r = 0$. Its positive charge will attract electrons and repel ions giving rise to a net charge imbalance locally. If this region of charge, separation is large compared to the inter-particle separation distance $n_i^{-1/3}$, the densities will show the net increase or decrease predicted by the Boltzmann equation (1.9) when time averaged. Over large distances and times, the inherent 'graininess' of the particles is averaged out to give a quasi-continuum distribution. The statistical distribution of the electrons will give a slowly varying potential, ϕ, described by Poisson's equation:

$$\nabla^2\phi = -\frac{1}{\epsilon_0}[Zn_i e - n_e e] \tag{1.11}$$

Near the ion, the field is spherically symmetric

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = -\frac{1}{\epsilon_0}n_e^0\left[\exp\left[\frac{-Ze\phi}{kT_i}\right] - \exp\left[\frac{e\phi}{kT_e}\right]\right] \tag{1.12}$$

and subject to the boundary conditions:

1. Near the ion, the field tends to that of the ion alone: $\phi = Ze/4\pi\epsilon_0 r$ as $r \to 0$.
2. Far from the ion, the field must vanish: $\phi \to 0$ as $r \to \infty$.

Since the field is small except close to the ion, we may expand the exponentials retaining only the leading terms

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\phi}{\mathrm{d}r}\right) = \frac{n_e^0 e^2}{\epsilon_0}\left\{\frac{1}{kT_e} + \frac{Z}{kT_i}\right\}\phi = \frac{\phi}{\lambda_D^2} \tag{1.13}$$

where λ_D is the Debye length

$$\lambda_D^2 = \frac{\epsilon_0}{n_e e^2}\Big/\left\{\frac{1}{kT_e} + \frac{Z}{kT_i}\right\} \tag{1.14}$$

To solve this differential equation, substitute: $\phi = f/x$ and $x = r/\lambda_D$ to obtain

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = f \tag{1.15}$$

subject to $f(0) = 1$, and $f(x) \to 0$ as $x \to \infty$ whose solution is

$$f(x) \sim \exp(-x) \tag{1.16}$$

Hence, we see that the field at large distances from the ion is

$$\phi = \frac{Ze}{4\pi\epsilon_0 r}\exp\left(-\frac{r}{\lambda_D}\right) \tag{1.17}$$

and falls off more rapidly for distances $r \gtrsim \lambda_D$ than Coulombic due to additional electrons and depleted ions. The electrons therefore on a statistical basis screen the ion. At this statistical level, there is locally a higher-than average electron density reflecting weak electron correlation over distances $\sim \lambda_D$. Since the field falls off rapidly as $r \gtrsim \lambda_D$, the Coulomb field is screened, and its range reduced from ∞ to $\sim \lambda_D$.

It is often convenient to separate the Debye length into separate electron and ion components: λ_e and λ_i represent the electron and ion correlation lengths, respectively.

$$\left.\begin{aligned} \lambda_e^2 &= \frac{\epsilon_0 \kappa T_e}{n_e e^2} \\ \lambda_i^2 &= \frac{\epsilon_0 \kappa T_i}{Z^2 n_i e^2} \end{aligned}\right\} \quad \frac{1}{\lambda_D^2} = \frac{1}{\lambda_e^2} + \frac{1}{\lambda_i^2}$$

In many cases, screening is due to the electrons alone. The large ion mass ensures that they move much more slowly. As a result, the ions are nearly static and, as a consequence, tend to be more grainy in their averaged distribution; in particular, they cannot respond to rapidly oscillating fields. They generally play little role in screening. Consequently, the electron Debye length is generally used as a measure of the screening distance $\lambda_D \approx \lambda_e$. Note that the effect of a uniform background ion space charge is to neutralise some of the electron charge and therefore increase the Debye length.

The Debye length measures the penetration of electric fields into the plasma. However, as we have pointed out, the model leading to the Debye length is only valid if the number of particles in the Debye sphere ($n_i\ {\lambda_D}^3$) is large, or alternatively that the Debye length is much larger than the inter-particle separation. At distances $\gtrsim \lambda_D$, the plasma particles act collectively to smooth out electric field fluctuations, although as we shall show well organised coherent collective motions are possible. For distance less than λ_D, thermal fluctuations are insufficient to dominate space charge disturbances on that scale, and electric fields can be found.

The screening effect of the Debye shielding over lengths $> \lambda_D$ clearly marks a difference in the form of the interactions between particles. When the particle separation is small $d \gtrsim \lambda_D$, the particles clearly interact on a 'short range', i.e. on an individual basis as between gas molecules – *collisional interactions*. On the other hand, when the separation is large $d \gtrsim \lambda_D$, the particle no longer 'sees' individual scatterers but only a 'smoothed out' continuum of interacting elements – *coherent or collective interactions*.

Clearly, the plasma frequency is related to the Debye length as both represent terms over which fields may be damped by the plasma. In fact

$$\lambda_D\,\Pi_e = \sqrt{\frac{\kappa T_e}{m_e}} = \bar{v}_e \tag{1.18}$$

the electron thermal speed.

1.3.1 Weakly and Strongly Coupled Plasmas

The analysis (with its imperfection) which leads to Debye shielding is only valid if the number of electrons and ions in the Debye sphere is sufficiently large that the 'graininess' of the particle distribution is lost in the averaging. In this case, the plasma is weakly coupled, and the thermal energies are large compared to electrostatic energies. Such a plasma is called *dilute*, when

$$n\ {\lambda_D}^3 \gg 1 \tag{1.19}$$

If this condition is not obeyed, individual particles in the neighbourhood of an ion are strongly influenced by the field. In this case, the averaging we have used is no longer applicable, and individual particles are strongly influenced by their neighbours, i.e. strongly correlated and the plasma is dense: $\lambda_D \lesssim n_i^{-\frac{1}{3}}$. We can no longer express the distribution in terms of the one particle (Boltzmann)