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General Relativity for Planetary Navigation

INTERNATIONAL SPACE UNIVERSITY ISU

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Contents

Chapter 1
Einstein Field Equations

The General Theory of Relativity, as it relates to navigation of spacecraft, can be separated into two parts. The first part involves derivation of a set of differential field equations that can be solved for the metric tensor. The second part involves inserting the metric tensor into the equation of geodesics to obtain equations of motion, which can be solved for formulae describing the precession of Mercury's orbit, the bending of light, radar time delay, gravitational red shift, and the time measured by clocks. In this chapter, the solution for the metric tensor is obtained from equations that provide a statement of the theory's fundamental assumptions. The assumptions are simply that the speed of light is constant, matter or energy curves space and the universe have some symmetrical properties. These assumptions are observed and cannot be proven. Two methods are used to solve for the metric tensor. The first is a computer solution that involves parameterizing the metric tensor and solving for the parameters using an orbit determination filter. The second is an analytic solution developed by Einstein by defining a covariant derivative and differentiating to obtain the Riemann tensor, Ricci tensor, and Einstein's field equations, which can be solved for the metric tensor.

1.1 Introduction

The Einstein field equations have been solved exactly for the case of spherical symmetry by Schwarzschild. This solution and Einstein's solution have spawned a number of formulae describing the precession of Mercury's orbit, the bending of light, radar time delay, gravitational red shift, and several more that relate to special relativity. The Schwarzschild solution has been transformed to a form such that the equations of motion look like Newton's equations of motion with a small relativistic perturbation. For orbit determination, these equations have been programmed into software used for navigation. One might question whether this is really necessary,

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since the perturbations due to general relativity are so small. The justification is that the orbit solution used for prediction of a spacecraft orbit is obtained after analysis of data residuals, the difference between the real world and the world computed by a mathematical model. Since the data is very high precision, a very small modeling error will show up as a signature in the data residuals. Without relativity modeling, a serious modeling error in another variable could be masked. A navigation analyst might initially conclude that the signature is caused by relativity or some other error source such as a clock failing to keep the right time. Eventually the signature will grow in magnitude and the alarm bells will ring indicating a problem. The earlier the problem is detected, the more likely a solution can be found before the spacecraft crashes into something. The problem of an inaccurate gravity harmonic caused an exponential rise in the Doppler signature on the Near Earth Asteroid Rendezvous (NEAR) mission which was detected early and corrected before anything catastrophic happened. For this reason, general and special relativity are programmed into the navigation operational software.

In the 1960s, general relativity was programmed into the Orbit Determination Program (ODP) at the Jet Propulsion Laboratory (JPL). At the time, those outside of navigation thought this was not needed. Since that time, many orbits have been determined using the ODP and little attention has been given to general relativity. The ODP is treated as a black box. With the advent of comet and asteroid missions a new orbit determination program was needed. This effort required implementing general relativity. Finding and understanding the equations presented a major difficulty. After consulting many sources including relativity experts at JPL, equations were programmed into the software used for the NEAR mission. We can assume the equations are correct because the spacecraft completed its mission successfully.

The derivation of the relativity equations of motion was initiated from the metric tensor which was assumed to be correct. The goal of deriving the equations from Einstein's original assumptions that the speed of light is constant and matter curves space has been difficult to achieve. The equations of motion were worked out long before Einstein's death. His theory written in books published up to that time were close to his original 1916 paper [1]. After his death, cosmologists got hold of the theory, and engineers had difficulty understanding the mathematics. The main source of confusion was the normalization of coordinates removing c, the speed of light, and G the gravitational constant from the equations. Einstein did this to make his theory look more profound and mathematical. In this chapter, the part of Einstein's theory pertaining to navigation of spacecraft in the solar system has been extracted from Einstein's original paper [1], Eddington's book [2] written in 1923, Harry Lass's book [3] on tensors written in 1950, and Sokolnikoff's book [4] written in 1951. What goes on inside the sun, earth or black holes is not relevant to navigation of spacecraft in the solar system. While Einstein's paper is difficult to understand, all the essential equations are there. Einstein's audience was other mathematicians and physicists. Eddington, who was a mathematician, explained some of the theory in a clear way that is comparatively easy to understand. His audience was much wider than Einstein's. Sokolnikoff shows how the Riemann tensor is put together and Schwarzschild's solution is obtained. Harry Lass described the properties of tensors.

1.2 Summary of General Relativity Fundamental Assumptions

The universe assumed for navigation consists of the solar system and massless stars that are infinitely far away and emit light. The center is the solar system barycenter or center of gravity. The goal is to define the equations of motion in curved space. A Euclidean coordinate system is defined far away from the sun but not as far as the stars. If we move this coordinate system so it is centered at the solar system barycenter, we can define a curved space coordinate system as a covariant mapping from Euclidean coordinates. Sometimes we are interested in curved space coordinates and other times we are interested in Euclidean coordinates. For the equations of motion, we are interested in curved space coordinates. To define the Einstein tensor, we consider volume elements that have the same size and we use Euclidean coordinates. In Euclidean coordinates the volume elements are equal and cubical. For constant density the mass of every volume element is the same. When we map to curved space, the volume elements vary in size and shape. Since a one to one mapping exists, the curved space volume elements would have to be assigned different densities to have the same mass. Einstein defines the density in Euclidean space as scalar invariant density and this is mapped to curved space to keep the total mass the same as defined in Euclidean space.

The fundamental assumptions of General Relativity are stated in equations without proof. The first assumption is that the speed of light is constant defined by c and the observed speed of light defined by the path length ds is also constant and equal to c. It is also necessary to define a measurement (Z) which is the projection of the observed acceleration of a point mass or any vector that can be observed in curved space on the trajectory of a curved line in the gravity field defined by the equation of geodesics.

$$Z = A_u \frac{dx_u}{ds}$$

This scalar measurement gives us one equation, but there are 10 independent elements in the metric tensor. To determine them we need nine more equations. For an analytic solution, we can differentiate this measurement with respect to the assumed coordinate system to obtain four more equations that can be measured. Differentiating again gives four more equations that define the curvature and can also be observed. We need one more equation to solve for the metric tensor. It is obtained by assuming the scale or curvature is proportional to mass. The assumption that the curvature of space is proportional to mass is satisfied by placing a boundary condition on the solution to the Einstein field equations or solving the Einstein tensor by equating it to the stress–energy tensor.

There are other assumptions associated with mathematics that are difficult to state in simple equations. These include symmetry, linearity, and continuity. Not only the trajectory of a particle but all the higher order derivatives must be continuous. They trace a smooth curve when drawn on graph paper and they have slopes and areas under the curve. Once the above fundamental equations are defined, the work of the scientist is complete. For a solution, we turn the problem over to mathematicians. Einstein was the essential bridge between the two camps. His main contribution besides special relativity was the Einstein tensor which is a purely mathematical result but required considerable physical insight to derive.

1.3 Geodesic Equation

The shortest distance between two points on a curved surface is called a geodesic. When an airplane flies over the North pole on its way to Europe, it is following a geodesic or great circle arc. The metric tensor (g_{uv}) defines the arc length due to curvature of space where u and v are indices in four-space corresponding to coordinates. For example, coordinates may be x, y, z, ct as u and v vary from integer values 1 to 4. The metric tensor defines a differential line element (ds). The elements of g_{uv} are functions of space and time that define g_{uv} at some point in space. The integral of the line element (ds) gives the distance between two points or the length of the curve connecting them. Consider two points A and B. A coordinate system can be used to locate the two points relative to one another. Since the reference coordinate system is arbitrary, the coordinates of the points are of little use. The only useful physical reality is the distance between the two points. The metric tensor can be integrated to determine the length of this line. Next, we consider a line between the two points that is the shortest distance. The variation of the path length with respect to the coordinates must be zero since only one path is the shortest. Thus we have

$$ds^2 = g_{uv} dx_u dx_v$$

$$2ds\delta(ds) = dx_u dx_v \delta g_{uv} + g_{uv} dx_u \delta(dx_v) + g_{uv} dx_v \delta(dx_u)$$

and

$$2ds\ \delta(ds) = dx_u dx_v \frac{\partial g_{uv}}{\partial x_\sigma} \delta x_\sigma + g_{uv} dx_u\ d(\delta x_v) + g_{uv} dx_v\ d(\delta x_u)$$

and the stationary condition is

$$\int \delta(ds) = 0$$