Springer Proceedings in Mathematics & Statistics

Alexey N. Karapetyants
Vladislav V. Kravchenko
Elijah Liflyand
Helmuth R. Malonek *Editors*

Operator Theory and Harmonic Analysis

OTHA 2020, Part I – New General Trends and Advances of the Theory

Springer

Springer Proceedings in Mathematics & Statistics

Volume 357

This book series features volumes composed of selected contributions from workshops and conferences in all areas of current research in mathematics and statistics, including operation research and optimization. In addition to an overall evaluation of the interest, scientific quality, and timeliness of each proposal at the hands of the publisher, individual contributions are all refereed to the high quality standards of leading journals in the field. Thus, this series provides the research community with well-edited, authoritative reports on developments in the most exciting areas of mathematical and statistical research today.

More information about this series at http://www.springer.com/series/10533

Alexey N. Karapetyants •
Vladislav V. Kravchenko • Elijah Liflyand •
Helmuth R. Malonek
Editors

Operator Theory and Harmonic Analysis

OTHA 2020, Part I – New General Trends and Advances of the Theory

Springer

Editors
Alexey N. Karapetyants
Institute of Mathematics, Mechanic and Computer Sciences and Regional Mathematical Center
Southern Federal University
Rostov-on-Don, Russia

Vladislav V. Kravchenko
Department of Mathematics
Cinvestav
Santiago de Queretaro, Querétaro, Mexico

Elijah Liflyand
Department of Mathematics
Bar-Ilan University
Ramat-Gan, Israel

Helmuth R. Malonek
Department of Mathematics
University of Aveiro
Aveiro, Portugal

ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-3-030-77492-9 ISBN 978-3-030-77493-6 (eBook)
https://doi.org/10.1007/978-3-030-77493-6

Mathematics Subject Classification: 26-02, 30-02, 32-02, 35-02, 42-02, 45-02, 46-02, 47-02, 60-02, 92-02

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This is the first volume of the two-volume series entitled

Operator Theory and Harmonic Analysis.

Vol. 1: New General Trends and Advances of the Theory

and

Vol. 2: Probability-Analytical Models, Methods, and Applications

Volume 1 is devoted to harmonic analysis and its applications in general, while Volume 2 is focused on probabilistic and mathematical (statistical) methods in applied sciences, but still in the context of general harmonic analysis and its numerous applications.

The volumes' readership is the pool of researchers interested in various aspects of harmonic analysis and operator theory: real and complex variable methods, applications to PDE's, mathematical modeling based on applied harmonic analysis and probability-analytical methods, and exploration of new themes and trends.

The contributions to both volumes are based on the matter supposed to be presented at the Annual International Scientific Conference on Modern Methods and Problems of Operator Theory and Harmonic Analysis and Their Applications (OTHA-2020, http://otha.sfedu.ru/), canceled due to Covid19 restrictions.

The Editors are very grateful to all the authors for their valuable contributions and for a strong willingness to support mathematical activities and communications in the hope of the soonest resumption of regular conferences and safe mutual visits. The Editors express an immense sorrow on the occasion of the recent loss of remarkable scientists and brilliant persons, Hrachik Hayrapetyan (Armenia), who is one of the authors of the first volume, Vladimir Pilidi (Russia), who was an active member of Program Committees of OTHA conferences, and Vladimir Nogin (Russia), who was a colleague and a teacher of quite a few participants of OTHA.

The first volume contains words in memoriam of our dear friends Hrachik Hayrapetyan, Vladimir Pilidi, and Vladimir Nogin.

Rostov-on-Don, Russia A. Karapetyants
Santiago de Queretaro, Mexico V. Kravchenko
Ramat-Gan, Israel E. Liflyand
Aveiro, Portugal H. Malonek

In Memory of Prof. Hrachik M. Hayrapetyan (25.10.1946–06.11.2020)

On November 6, 2020, the world mathematical community lost a brilliant mathematician and a wonderful personality Hrachik Hayrapetyan.

Professor Hayrapetyan was born in 1946 in Dilijan (Armenia). His mathematical talents were noticed since his adolescence. His mathematical inclinations were influenced by contacts with his first teacher A. Sahakyan. In 1964, he started his studies at the Faculty of Mathematics and Mechanics of the Yerevan State University from which he graduated in 1969. After two years of service in the Soviet Army, he entered the Institute of Mathematics of the National Academy of Sciences of Armenia as a junior scientific researcher. Since then, his collaboration with the academician Mkhitar Jrbashyan started, who proposed to him the study of free interpolation and basis properties of rational fractions. Hrachik Hayrapetyan succeeded in discovering a series of essential results in this research field. In particular, he proved that if the multiplicities in the interpolation problem are not bounded, then this problem may have no solution and the rational fractions may fail to be a basis in the closure of their linear span. In 1975, he completed his PhD thesis. In 1979, he entered the National Polytechnic University of Armenia as an associate professor of the Chair of Applied Mathematics. At that time, the scientific group of Prof. N.E. Tovmasyan was developing the theory of boundary value problems for partial differential equations. H.M. Hayrapetyan was actively involved in this research. As a specialist in the theory of complex variable functions, he was interested in theoretical-functional approach to these problems. He succeeded in obtaining a series of important results. Particularly, it is worth mentioning the new formulation of the classical Riemann boundary value problem, which allowed to solve this problem first in the classes of integrable functions and afterwards in the class of essentially bounded functions. Later, applying proposed method, Prof. Hayrapetyan with his students investigated boundary value problems in various functional spaces. The obtained results not only extended the theory of boundary value problems but also permitted to develop the theory of elliptic partial differential equations. Hrachik Hayrapetyan defended his Doctor of Science thesis "Riemann–Hilbert boundary value problem in the sense of mean convergence and applications in the theory of elliptic partial differential equations" in the M.V. Lomonosov Moscow State University. Specialists evaluated his results as a great success in the theory of boundary value problems. Developing his theory during the last decade, he studied boundary value problems in weighted classes of functions. He succeeded to describe the classes of functions, where the Dirichlet and Riemann–Hilbert boundary value problems in the classes of polyanalytic and polyharmonic functions are normally solvable in both bounded and unbounded domains. These results are highly evaluated by specialists in Armenia as well as abroad.

Hrachik Hayrapetyan was one of the members of the first elected Council of the Armenian Mathematical Union created in 1991 following Armenia's independence from the Soviet Union.

Prof. Hayrapetyan was an active organizer. He served two terms as the President of Mathematical Association of Armenia and he was the Head of specialized mathematical education chair in National Polytechnic University of Armenia and the Head of Mathematical Analysis and Function Theory chair in the Yerevan State University. His devotion to the science and excellent empathy skills helped him to interest young students in mathematical research. He was a scientific advisor of 15 PhD theses; his students continue the work on ideas of their teacher and mentor, Hrachik Hayrapetyan, in various universities and research institutions of Armenia.

The fond memory of our friend will forever rest in our hearts.

Doctor of Science, Professor Armenak H. Babayan
Doctor of Science, Professor Vanya A. Mirzoyan
Doctor of Science, Professor Levon Z. Gevorgyan
Chair of Specialized Mathematical Education of National Polytechnic University of Armenia

In Memory of Prof. Vladimir Pilidi (07.11.1946–19.01.2021)

Vladimir Pilidi became a student at the Rostov State University (now Southern Federal University) in 1964. All his scientific career from a talented student (diploma with honors) to distinguished Chair was related to this university during 57 years.

In 1972, under the guidance of Professor I.B. Simonenko, he defended his Ph.D. thesis "Local method for the study of linear operator equations of the type of bisingular integral equations," and in 1990 at the Dissertation Council of the Tbilisi Institute of Mathematics named after I. A. Razmadze of the Georgian Academy of Sciences, he defended doctoral dissertation (second degree) "Bisingular operators and classes of operators close to them", which became a significant scientific achievement that enriched the general theory of operators of local type. Seven students of Vladimir Pilidi became candidates of science (PhD's).

Professor Pilidi was a highly qualified expert in the field of mathematics and its applications, reviewer of scientific articles, member and chairman of the Dissertation Council, scientific consultant of several research institutions, chairman of the State Examination Commissions of universities. After he became the head of the Chair of Informatics and Computational Experiment in 2000, Vladimir Pilidi expanded his area of scientific interests towards the application of mathematical methods in cryptography, the theory of pattern recognition, and graphic information processing.

Professor Pilidi is known as the author of the bilocal method, an analogue of the classical local method of Simonenko, and in his research he successfully applied it to the study of bisingular and related operators, as well as algebras of such operators. Thanks to these achievements, the name of Vladimir Pilidi will forever remain among the names of outstanding researchers in analysis and operator theory.

Professor Pilidi actively participated in the scientific life of the Mathematics Department, in the organization of scientific seminars, conferences, and schools. He was one of the main organizers of the OTHA conference series, a regular participant, and a member of the Program Committee of these conferences. Professor Pilidi and his students made a valuable contribution to the development of this series of conferences and to the development of publication activity following the conferences.

Vladimir Pilidi is known as a brilliant lecturer of various courses in mathematics and computer science. He had remarkable achievements as a teacher in the lecture course in algebra and geometry for students of applied and mechanical engineering, which he taught for about 20 years. His textbook “Linear Algebra” (Vuzovskaya Kniga, Moscow, 2005), co-authored with A.V. Kozak, is standard for other authors. Other textbooks by Professor Pilidi are: “Mathematical Analysis” (Phoenix, 2009), “Mathematical Foundations of Information Security” (Southern Federal University, 2019), and Analytic Geometry (Southern Federal University, 2020). Vladimir Pilidi developed a deep modern course on mathematical methods of cryptography, which he taught to students of the Department of Fundamental Informatics and Information Technology and students of the Department of Applied Mathematics, specialized in the field of mathematical methods of information security.

His distinguished features were not only erudition and professionalism but also modesty, discreetness, and goodwill in relations with colleagues and students. Vladimir Pilidi was a wonderful head of his mathematical family. His wife, daughter, and son-in-law devoted themselves to mathematics, and his grandchildren are preparing to become mathematicians as well.

The bright memory of Vladimir Pilidi—of a mathematician, a teacher, and a brilliant person, will remain in our hearts.

I. M. Erusalimskiy
A. N. Karapetyants
V. S. Rabinovich
S. G. Samko

In Memory of Vladimir Nogin (20.12.1955–31.05.2021)

Dr. Nogin Vladimir Alexandrovich was born on December 20, 1955. Vladimir Nogin graduated from the Faculty of Mechanics and Mathematics of the Rostov State University (now it is Southern Federal University) in 1979, defended his Ph.D. thesis in 1982 and worked 35 years as an assistant professor, senior teacher and then associate professor of the Department (Chair) of Differential and Integral Equations at the same University. During his work at the university, V.A. Nogin taught courses in mathematical analysis, higher mathematics, and mathematical physics. He also developed and taught more than five special courses for undergraduate and postgraduate math students, which included contemporary results in the field of functional analysis and mathematical physics.

Dr. Nogin's scientific interests were in the classical area of analysis related to the study of operators of mathematical physics, the construction and study of fractional powers of these operators, their inversion, and the description of the image of such operators in the framework of Lebesgue spaces. At the same time, he dealt with questions of functional analysis—the description of function spaces that arise in analysis in the context of the above-mentioned theory of operators. He and his students obtained profound results in this theory; he successfully developed the so-called method of approximate inverse operators. He has published about 70 scientific papers and a significant number of textbooks.

Vladimir Alexandrovich always found enough time for his students, and scientific work was his main passion in life. 8 PhD theses defended under his supervisorship. One of his students, Mikhail Gurov, became the teacher of the year in the Russian Federation in 2020.

Vladimir Alexandrovich was distinguished by his modesty and delicacy in relation to colleagues. The bright memory of Dr. V.A. Nogin will always be in the hearts of his colleagues and students.

On behalf of the colleagues and students,

O. G. Avsyankin, A. P. Chegolin, M. N. Gurov, A. N. Karapetyants, D. N. Karasev, B. G. Vakulov

Contents

Weighted Hadamard–Bergman Convolution Operators 1
Smbat A. Aghekyan and Alexey N. Karapetyants

On Initial Extensions of Mappings 21
A. B. Antonevich and C. Dolicanin

On Multidimensional Integral Operators with Homogeneous Kernels in Classes with Asymptotics 39
O. G. Avsyankin

A Dirichlet Problem for Non-elliptic Equations and Chebyshev Polynomials 55
A. H. Babayan

Martingale Hardy-Amalgam Spaces: Atomic Decompositions and Duality 73
Justice Sam Bansah and Benoît F. Sehba

Robust Estimation of European and Asian Options 101
G. I. Beliavsky, N. V. Danilova, and A. D. Logunov

On Reflexivity and Other Geometric Properties of Morrey Spaces 119
Evgenii I. Berezhnoi

Justification of the Averaging Method for a System with Multipoint Boundary Value Conditions 137
D. Bigirindavyi and V. B. Levenshtam

Spanne-Guliyev Type Characterization for Fractional Integral Operator and Its Commutators in Generalized Orlicz–Morrey Spaces on Spaces of Homogeneous Type 143
Fatih Deringoz

Spanne Type Characterization of Parabolic Fractional Maximal Function and Its Commutators in Parabolic Generalized Orlicz–Morrey Spaces 161
V. S. Guliyev, A. Eroglu, and G.A. Abasova

Eigenvalues of Tridiagonal Hermitian Toeplitz Matrices with Perturbations in the Off-diagonal Corners 179
Sergei M. Grudsky, Egor A. Maximenko, and Alejandro Soto-González

Some Properties of the Kernel of a Transmutation Operator 203
T. N. Harutyunyan and A. M. Tonoyan

On a Riemann Boundary Value Problem with Infinite Index in the Half-plane 221
Hrachik M. Hayrapetyan, Smbat A. Aghekyan, and Artavazd D. Ohanyan

Repeated Distances and Dot Products in Finite Fields 243
Alex Iosevich and Charles Wolf

Banach Spaces of Functions Delta-Subharmonic in the Unit Disc 259
Armen Jerbashian and Jesus Pejendino

Banach Spaces of Functions Delta-Subharmonic in the Half-Plane 279
Armen Jerbashian and Daniel Vargas

Weighted Estimates Containing Quasilinear Operators 297
A. Kalybay and R. Oinarov

A Neumann Series of Bessel Functions Representation for Solutions of the Radial Dirac System 319
Vladislav V. Kravchenko, Elina L. Shishkina, and Sergii M. Torba

Some Algebraic Aspects of Boolean Valued Analysis 333
Anatoly G. Kusraev

Some Questions Related to Extended Eigenvalues and Extended Eigenoperators 345
M. Lacruz, F. León-Saavedra, and M. P. Romero de la Rosa

Summability to the Hilbert Transform 361
E. Liflyand

Applying Wavelet Transforms as a Solution for Convolution Type Equations 369
Vladimir A. Lukianenko

Boundary Behaviour of Partial Derivatives for Solutions to Certain Laplacian-Gradient Inequalities and Spatial QC Maps 393
Miodrag Mateljević

Is Every Nontrivial Hausdorff Operator on Lebesgue Space a Non-Riesz Operator? 419
A. R. Mirotin

Homogenization, Drift, Stabilization and Destabilization for the Patlak-Keller-Segel Systems Driven by the Indirect Taxis and the Short-Wavelength External Signals 429
Andrey Morgulis and Konstantin Ilin

Nontangential Summability of Fourier Series 447
A. D. Nakhman and B. P. Osilenker

On Some Fast Implementations of Fourier Interpolation 463
Anry Nersessian

Boundary Value Problems for 3D-Dirac Operators and MIT Bag Model 479
Vladimir Rabinovich

A Refinement of the Adams Theorem on the Riesz Potential 497
Yoshihiro Sawano

Inversion of the Weighted Spherical Mean 507
Elina Shishkina

On Subadditivity for Subspace-Valued Maps 521
Ekaterina Shulman

Generalized Cauchy–Riemann Equations with Power-law Singularities in Coefficients of Lower Order 535
A. P. Soldatov and A. B. Rasulov

Norms of Linear Functionals, Summability of Trigonometric Fourier Series and Wiener Algebras 549
R. M. Trigub

Unilateral Ball Potentials in Grand Lebesgue Spaces 569
S. M. Umarkhadzhiev and M. U. Yakhshiboev

Lebesgue Means of Trigonometric Integrals and Their Rate of Convergence 577
S. S. Volosivets

Weighted Hadamard–Bergman Convolution Operators

Smbat A. Aghekyan and Alexey N. Karapetyants

Abstract Following the ideas of the recent paper by Karapetiants and Samko (Hadamard–Bergman convolution operators. Complex analysis operator theory) we extend the introduced in the mentioned paper notion of Hadamard–Bergman convolution operators to a weighted settings. We treat operators of fractional integration and differentiation as important examples of operators in the above mentioned class, and study mapping properties of certain generalized potentials in generalized Hölder spaces.

Keywords Hadamard–Bergman convolutions · Holomorphic fractional integrals and derivatives · Hölder space

1 Introduction

In the recent paper by A. Karapetyants and S. Samko [16] there was introduced the notion of Hadamard–Bergman convolution. Here we extend the results of the mentioned paper to the weighted case, i.e., we study convolutions

$$g \times f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad z \in \mathbb{D},$$

where $\mathrm{dA}_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda \mathrm{dA}(z)$, and $d\mathrm{A}(z) = \frac{1}{\pi} dxdy$, $z = x+iy \in \mathbb{D}$. Here either f and g are both in $H(\mathbb{D})$, or $g \in L^1_\lambda(\mathbb{D})$ and $f \in H(\mathbb{D})$, see Sect. 3 for definitions.

S. A. Aghekyan · A. N. Karapetyants (✉)
Institute of Mathematics, Mechanic and Computer Sciences & Regional Mathematical Center, Southern Federal University, Rostov-on-Don, Russia

A. N. Karapetyants et al. (eds.), *Operator Theory and Harmonic Analysis*, Springer Proceedings in Mathematics & Statistics 357,
https://doi.org/10.1007/978-3-030-77493-6_1

Introducing the weight parameter is not just a technical issue; it seems more natural to consider such weighted convolutions and corresponding weighted Hadamard–Bergman operators in view of the important examples such as the weighted operators of fractional integration and differentiation (see [26] and also Sect. 5), which are particular cases of the Hadamard–Bergman convolution operators.

Certainly the proofs for the weighted case in most occasions are similar to the unweighted situation, however the corresponding formulas and conclusions can provide some tricky issues in view of the weight parameter. Therefore we prefer to provide a reader with a proof or at least with a sketch of the proof.

The idea can be developed further in the direction of Bergman type operators which appear in the study of generalized holomorphic functions related to Vekua and some other equations. Such study attracts now attention of many authors, see e.g. [5, 6, 14] and references therein. We plan such investigation in another work.

The paper is organized as follows. Section 2 collects necessary preliminaries. In Sect. 3 we give definitions and discuss some properties of weighted Hadamard–Bergman convolutions, and in Sect. 4 we proceed with the corresponding operators. The important examples, the operators of weighted fractional integrodifferentiation, are discussed in Sect. 5. Some mapping properties in weighted Lebesgue spaces are discussed in Sects. 6, and 7 presents mappings by weighted fractional operators in generalized Hölder spaces. This section serves as an example of application of our results.

2 Preliminaries

Let $\mathrm{dA}(z) = \frac{1}{\pi} dxdy$ be the normalized Lebesgue measure on the unit disc $\mathbb{D}$. Let $-1 < \lambda < +\infty$, $\mathrm{dA}_\lambda(z) = (\lambda+1)(1-|z|^2)^\lambda \mathrm{dA}(z)$. We equip weighted Lebesgue spaces $L^p_\lambda(\mathbb{D}) = L^p(\mathbb{D}, \mathrm{dA}_\lambda)$ with the norm

$$\|f\|_{p,\lambda} = \left(\int_{\mathbb{D}} |f(z)|^p \mathrm{dA}_\lambda(z)\right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty, \tag{1}$$

and we treat the case $p = \infty$ as usual: $\|f\|_\infty = \mathrm{esssup}_{z\in\mathbb{D}}|f(z)|$. By $\mathcal{A}^p_\lambda(\mathbb{D})$, as usual, we denote the subspace in $L^p_\lambda(\mathbb{D})$ consisting of holomorphic in $\mathbb{D}$ functions (see [3, 4, 9, 10]) and also [26, 27]. Let $H(\mathbb{D})$ be the set of functions f, holomorphic in $\mathbb{D}$, equipped with the topology defined by the countable set of norms

$$\|f\|_n = \sup_{|z|<1-\frac{1}{n+1}} |f(z)| = \sup_{|z|<1-\frac{1}{n+1}} \left|\sum_{m=0}^{\infty} f_m z^m\right|, \quad n = 1, 2, \ldots, \tag{2}$$

where f_m are the Taylor coefficients of f :

$$f_m = \frac{f^{(m)}(0)}{m!} = \frac{1}{2\pi i}\int_{|\tau|=r}\frac{f(\tau)}{\tau^{m+1}}d\tau \tag{3}$$
$$= \frac{1}{(\lambda+1)B(r^2; m+1, \lambda+1)}\int_{|w|<r} f(w)\overline{w}^m \mathrm{dA}_\lambda(w),$$

where

$$B(\tau; a, b) = \int_0^\tau t^{a-1}(1-t)^{b-1}dt, \quad a > 0, \quad b > 0, \quad \tau > 0$$

is the incomplete Beta-function, see [7, page 910]. The space $H(\mathbb{D})$ may be identified with the set of series

$$\sum_{n=0}^{\infty} a_n z^n \quad \text{such that} \quad \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leqslant 1.$$

It can be viewed as a subspace of the space S of all formal power series $f = \sum_{n=0}^{\infty} f_n z^n$ or, which is the same, with the space of sequences $\{f_n\}_{n=0}^{\infty}$. By $\mathcal{L}(H(\mathbb{D}))$ we denote the set of linear operators on $H(\mathbb{D})$. We refer to the book [25], Chapter 1.3.1, for more details.

The following lemma is intuitive.

Lemma 2.1 *Let* $f \in \mathcal{A}^p_\lambda(\mathbb{D})$, $1 \leqslant p < \infty$, *and* $0 \leqslant \alpha < 2$, *then*

$$\int_{\mathbb{D}} \frac{|f(w)|^p}{|w|^\alpha}\mathrm{dA}_\lambda(w) \leqslant C\|f\|^p_{p,\lambda}, \quad 1 \leqslant p < \infty,$$

where $C > 0$ *does not depend on* f.

The following lemma follows from part (a) of Lemma 3.10 in [27].

Lemma 2.2 *Let* $\alpha + \beta < 2 + \lambda$, $\lambda < \beta < \lambda + 1$. *Then* $\frac{1}{(1-|z|)^\alpha(1-z)^\beta} \in L^1_\lambda(\mathbb{D})$.

We will need the following asymptotic of the ratio of Gamma functions:

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\sum_{m=0}^{N}\frac{C_m}{z^m} + z^{a-b}O\left(z^{-N-1}\right), \quad |z| \to \infty, \tag{4}$$

where $|\arg(z+a)| < \pi$, $C_0 = 1$, and $C_m = \frac{(-1)^m\Gamma(b-a+m)}{m!\Gamma(b-a)}B_m^{a-b+1}(a)$, $m = 1, 2, \ldots, N$, are expressed in terms of generalized Bernoulli polynomials, see [19] and [22, page 17].

3 Weighted Hadamard–Bergman Convolution of Functions

3.1 Convolutions of Holomorphic Functions

Keeping a study of weighted holomorphic spaces in mind, we define the Hadamard product type composition of holomorphic functions in $H(\mathbb{D})$:

$$f(z) = \sum_{m=0}^{\infty} f_m z^m \quad \text{and} \quad g(z) = \sum_{m=0}^{\infty} g_m z^m,$$

as follows

$$f * g(z) = (\lambda + 1) \sum_{k=0}^{\infty} B(k+1, \lambda+1) f_k g_k z^k, \quad z \in \mathbb{D}.$$

See, e.g. [8, 22, 25]) for Hadamard product composition (Hadamard fractional integrodifferentiation) theory. Here the right hand side is well defined for all $f, g \in H(\mathbb{D})$, since $H(\mathbb{D})$ is identified with the set of series $\sum_{n=0}^{\infty} a_n z^n$ with $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \leqslant 1$.

From the above formula at least formally we have for $f \in \mathcal{A}^2_\lambda(\mathbb{D})$:

$$f * \overline{f}(z) = \int_{\mathbb{D}} |f(z)|^2 \mathrm{dA}_\lambda(z) = (\lambda + 1) \sum_{m=0}^{\infty} B(m+1, \lambda+1) |f_m|^2.$$

See Theorem 3.1 below for the justification of this formula.

We extend the definition of Hadamard–Bergman convolution given in [16] for the weighted case, though in most places in this paper we skip the word "weighted".

Definition 1 Let $f, g \in H(\mathbb{D})$. The construction

$$g \times f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad z \in \mathbb{D}, \tag{5}$$

is called the (weighted) Hadamard–Bergman convolution of functions $g, f \in H(\mathbb{D})$. Here and everywhere in the sequel, the Hadamard–Bergman convolution of holomorphic functions is treated in the improper sense as justified in Theorem 3.1 below.

Theorem 3.1 *Let $f, g \in H(\mathbb{D})$. Then the following statements are valid*

1. For any $0 < r < 1$, $z \in \mathbb{D}$

$$\int_{|w|<r} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w) = \int_{|w|<r} f(z\overline{w})g(w)\mathrm{dA}_\lambda(w) \tag{6}$$

$$= (\lambda+1)\sum_{k=0}^{\infty} B(r^2; k+1, \lambda+1) f_k g_k z^k.$$

2. The integral

$$\int_{\mathbb{D}} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w), \quad z \in \mathbb{D}, \tag{7}$$

exists as improper integral:

$$\int_{\mathbb{D}} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w) = \lim_{r\to 1}\int_{|w|<r} f(w)g(z\overline{w})\mathrm{dA}_\lambda(w)$$

$$= (\lambda+1)\sum_{k=0}^{\infty} B(k+1, \lambda+1) f_k g_k z^k.$$

Proof We have

$$\int_{|w|<r} f(z\overline{w})g(w)\mathrm{dA}_\lambda(w) = \sum_{k=0}^{\infty} f_k z^k \int_{|w|<r} g(w)\overline{w}^k \mathrm{dA}_\lambda(w).$$

It remains to refer to (3); the first equality of (6) immediately follows. For the proof of the second statement we note that from (3) there follows that the integral

$$\int_{\mathbb{D}} f(w)\overline{w}^m \mathrm{dA}_\lambda(w) = \lim_{r\to 1}\int_{|w|<r} f(w)\overline{w}^m \mathrm{dA}_\lambda(w)$$

exists as improper integral for all $f \in H(\mathbb{D})$. Now it suffices to use (6). □

It is clear from the above, but it worth to underline separately that the convolution formula (5) is symmetric in the sense that:

$$g \times f(z) = \int_{\mathbb{D}} g(w)f(z\overline{w})\mathrm{dA}_\lambda(w), \tag{8}$$

$$= \int_{\mathbb{D}} g(z\overline{w})f(w)\mathrm{dA}_\lambda(w), \quad z \in \mathbb{D}.$$

This formula holds in the more general "integration by parts" form:

$$\int_{\mathbb{D}} (\mathcal{D}^m f)(z\overline{w}) g(w) \mathrm{dA}_\lambda(w) = \int_{\mathbb{D}} f(w)(\mathcal{D}^m g)(z\overline{w}) \mathrm{dA}_\lambda(w), \tag{9}$$

where $\mathcal{D}^m f(z) = z^m \left(\frac{d}{dz}\right)^m f(z)$.

3.2 Convolutions of Holomorphic and non Holomorphic Functions

Convolution formula (5) may be in general considered for non necessarily holomorphic functions. In the formula (5) the function g maybe taken non holomorphic, while $f \in H(\mathbb{D})$.

However, in this case we are not able to define the convolution as an improper integral and have to assume $g \in L^1_\lambda(\mathbb{D})$. Also, the symmetry (8) does not hold, in general.

We define the Fourier type transform of a function $g \in L^1_\lambda(\mathbb{D})$:

$$g \in L^1_\lambda(\mathbb{D}) \longrightarrow \mu_{m,\lambda}(g) = \int_{\mathbb{D}} g(w)\overline{w}^m \mathrm{dA}_\lambda(w), \quad m = 0, 1, \ldots. \tag{10}$$

In the case $g \in L^1_\lambda(\mathbb{D})$ and $f \in H(\mathbb{D})$ the convolution (5) also reduces to multiplier form, i.e.

$$g \times f(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(g) f_m z^m, \quad z \in \mathbb{D}. \tag{11}$$

Therefore, it is clear, that the Hadamard–Bergman convolution (5) of $g \in L^1_\lambda(\mathbb{D})$ and $f \in H(\mathbb{D})$ may be represented as a convolution with a certain holomorphic kernel g_{hol}:

$$g \times f(z) = \int_{\mathbb{D}} g_{hol}(w) f(z\overline{w}) \mathrm{dA}_\lambda(w) \tag{12}$$

The relation between the holomorphic kernel g_{hol} and the initially given non holomorphic kernel g is as follows:

$$g_{hol}(z) = B^\lambda_{\mathbb{D}} g(z) = \int_{\mathbb{D}} \frac{g(w)}{(1 - z\overline{w})^{2+\lambda}} \mathrm{dA}_\lambda(w), \quad z \in \mathbb{D},$$

where $B^\lambda_{\mathbb{D}}$ is the weighted Bergman projection. Indeed, consider the holomorphic function $g_{hol}(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(g) z^m$ in $\mathbb{D}$, where $\mu_{m,\lambda}(g)$ are given in (10). We have

$$g_{hol}(z) = \sum_{m=0}^{\infty} z^m \frac{1}{(\lambda+1)B(m+1,\lambda+1)} \int_{\mathbb{D}} g(w)\overline{w}^m \mathrm{dA}_\lambda(w)$$
$$= \int_{\mathbb{D}} g(w) \left(\sum_{m=0}^{\infty} \frac{1}{(\lambda+1)B(m+1,\lambda+1)} z^m \overline{w}^m \right) \mathrm{dA}_\lambda(w)$$
$$= \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\lambda}} \mathrm{dA}_\lambda(w) = B_{\mathbb{D}}^\lambda g(z), \quad z \in \mathbb{D}.$$

Here we used the known expansion formula for the weighed Bergman kernel for the unit disc:

$$K_\lambda(z,w) := \frac{1}{(1-z\overline{w})^{2+\lambda}} = \sum_{m=0}^{\infty} \frac{1}{(\lambda+1)B(m+1,\lambda+1)} z^m \overline{w}^m, \quad z, w \in \mathbb{D}.$$

The extension of the notion of Hadamard–Bergman convolution for non holomorphic function g is very important for further analysis. An immediate example of the Hadamard–Bergman convolution is the construction given by the weighted Bergman projection $B_{\mathbb{D}}^\lambda$ defined on $g \in L_\lambda^1(\mathbb{D})$ as

$$B_{\mathbb{D}}^\lambda g(z) = \int_{\mathbb{D}} K_\lambda(z,w) g(w) \mathrm{dA}_\lambda(w) = \int_{\mathbb{D}} \frac{g(w)}{(1-z\overline{w})^{2+\lambda}} \mathrm{dA}_\lambda(w),\ z \in \mathbb{D}.$$

Further development concerns the theory of Toeplitz operators; we plan to threat this issue in another paper.

4 Weighted Hadamard–Bergman Convolution Operators

Fix a function $g \in H(\mathbb{D})$ and define the the Hadamard–Bergman convolution operator as an operator

$$\mathbb{K}_g f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w)$$
$$= \lim_{r\to 1} \int_{|w|<r} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad f \in H(\mathbb{D}).$$

We note that for the Hadamard–Bergman convolution operator $\mathbb{K}_g$ with holomorphic kernel $g(z) = \sum_{m=0}^{\infty} g_m z^m$ we have

$$\mathbb{K}_g f(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(\mathbb{K}_g) f_m z^m, \quad \text{for} \quad f(z) = \sum_{m=0}^{\infty} f_m z^m \in H(\mathbb{D}),$$

where

$$\mu_{m,\lambda}(\mathbb{K}_g) \equiv \mu_{m,\lambda}(g) = (\lambda+1)B(m+1,\lambda+1)g_m, \quad m = 0, 1, \ldots. \tag{13}$$

The converse is also true: every operator $K \in \mathcal{L}(H(\mathbb{D}))$ of the form

$$Kf(z) = \sum_{m=0}^{\infty} \mu_m f_m z^m, \quad z \in \mathbb{D}, \tag{14}$$

(such operator is called a coefficient multiplier and it is automatically continuous, see e.g., [21]) is represented as Hadamard–Bergman convolution with the kernel

$$g(z) = \frac{1}{(\lambda+1)} \sum_{m=0}^{\infty} \frac{\mu_m z^m}{B(m+1,\lambda+1)} = K\left(\frac{1}{(1-w)^{2+\lambda}}\right)(z), \quad z \in \mathbb{D}. \tag{15}$$

Indeed, we have

$$\begin{aligned} Kf(z) &= K\left(\sum_{m=0}^{\infty} f_m z^m\right) = \sum_{m=0}^{\infty} f_m \mu_m z^m \\ &= (\lambda+1)\sum_{m=0}^{\infty} B(m+1,\lambda+1) g_m f_m z^m, \end{aligned} \tag{16}$$

where $g_m = \frac{\mu_m}{(\lambda+1)B(m+1,\lambda+1)}$, and we arrive at the Hadamard–Bergman convolution operator with the kernel

$$\begin{aligned} g(z) &= \frac{1}{(\lambda+1)} \sum_{m=0}^{\infty} \frac{\mu_m z^m}{B(m+1,\lambda+1)} \\ &= K\left(\frac{1}{(\lambda+1)} \sum_{m=0}^{\infty} \frac{z^m}{B(m+1,\lambda+1)}\right) = K\frac{1}{(1-z)^{2+\lambda}}. \end{aligned}$$

The above results can be also obtained from the results of [24] on the general form of Hadamard or coefficient type multipliers (see also [20]).

Fix now a function $g \in L^1_\lambda(\mathbb{D})$. The Hadamard–Bergman convolution operator $\mathbb{K}_g$ is well defined as

$$\mathbb{K}_g f(z) = \int_{\mathbb{D}} g(w) f(z\overline{w}) \mathrm{dA}_\lambda(w), \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}. \tag{17}$$

It possesses the multiplier realization

$$\mathbb{K}_g f(z) = \sum_{m=0}^{\infty} \mu_{m,\lambda}(\mathbb{K}_g) f_m z^m, \quad \text{for} \quad f(z) = \sum_{m=0}^{\infty} f_m z^m \in H(\mathbb{D}),$$

where $\mu_{m,\lambda}(\mathbb{K}_g) \equiv \mu_{m,\lambda}(g)$, and $\mu_{m,\lambda}(g)$ is given in (10).

We conclude this section with the following important remark. By a change of variables the convolution operator $\mathbb{K}_g$ with the kernel $g \in L^1_\lambda(\mathbb{D})$ can be written in the form

$$\mathbb{K}_g f(z) = \frac{\lambda+1}{|z|^2} \int_{|w|<|z|} g\left(\frac{\overline{w}}{\overline{z}}\right)\left(1 - \left|\frac{\overline{w}}{\overline{z}}\right|^2\right)^{\lambda} f(w) \mathrm{dA}(w), \quad z \in \mathbb{D}. \tag{18}$$

In this form the operator $\mathbb{K}_g$ is well defined in the setting of measurable functions for which the integral converges. It is represented as the integral operator with homogeneous of degree (-2) kernel invariant with respect to rotations. Such operators of the form (18) belong to the class of operators with homogeneous kernels well studied in analysis (see books [11, 12], and review paper [13]). The operators (18) may be also considered as generalized Hardy operators. The algebra of operators with homogeneous kernels is well studied (see [1, 2] for recent development in this direction); these results may be used for the study of the algebra of Hadamard–Bergman convolution operators in the framework of holomorphic functions. We plan to study such questions in another paper. The multidimensional case is of a special interest as well. Hoverer in such a case we most likely need to deal with a general Banach lattices and orthogonally theory (see e.g., [17, 18]) using homogeneous complex polynomials which are substitution of spherical harmonics techniques proved to work very well in the case of operators with homogeneous kernel in real analysis.

5 Operators of Fractional Integrodifferentiation

The operator of fractional integration

$$\mathbb{I}^{\alpha}_{\lambda} f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2+\lambda-\alpha}} \mathrm{dA}_\lambda(w), \quad \alpha > 0, \quad \lambda > -1. \tag{19}$$

is an example of Hadamard–Bergman convolution. Direct calculation provides

$$\mathbb{I}^{\alpha}_{\lambda} f(z) = \frac{\Gamma(2+\lambda)}{\Gamma(2+\lambda-\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(m+2+\lambda-\alpha)}{\Gamma(2+m+\lambda)} f_m z^m, \tag{20}$$

$$\alpha > 0, \lambda \neq -2, -3, \ldots .$$

The formula (20) defines $\mathbb{I}_\lambda^\alpha$ for a wider range of parameter λ, i.e. $\lambda \in \mathbb{R}, \lambda \neq -2, -3, \ldots$.

We will also need the operator of fractional differentiation

$$\mathbb{D}_\lambda^\alpha f(z) = \frac{\Gamma(2+\lambda-\alpha)}{\Gamma(2+\lambda)} \sum_{m=0}^{\infty} \frac{\Gamma(2+m+\lambda)}{\Gamma(m+2+\lambda-\alpha)} f_m z^m, \tag{21}$$

$$\alpha > 0, \lambda - \alpha \neq -2, -3, \ldots,$$

so that

$$\mathbb{D}_\lambda^\alpha \mathbb{I}_\lambda^\alpha f(z) = \mathbb{I}_\lambda^\alpha \mathbb{D}_\lambda^\alpha f(z) = f(z), \quad z \in \mathbb{D}.$$

In order to distinguish between integration and differentiation we prefer to consider $\mathbb{I}_\lambda^\alpha$ and $\mathbb{D}_\lambda^\alpha$ for positive $\alpha > 0$. Clearly the corresponding constructions (20) and (21) make sense for $\alpha \in \mathbb{R}$, and, in particular,

$$\mathbb{D}_\lambda^\alpha = \mathbb{I}_{\lambda-\alpha}^{-\alpha}, \quad \lambda - \alpha \neq -2, -3, \ldots. \tag{22}$$

The operator $\mathbb{I}_\lambda^\alpha$ does not satisfy the semigroup property, however we can indicate the following rule for the composition:

$$\mathbb{I}_\lambda^\alpha \mathbb{I}_{\lambda-\alpha}^\beta = \mathbb{I}_\lambda^{\alpha+\beta}, \tag{23}$$

where $\alpha > 0, \beta > 0$ and neither λ nor $\lambda - \alpha$ equal to $-2, -3, \ldots$.

In the following theorem we find a representation of the operator (21) in convolution terms with the kernel expressed in the term of elementary function. Denote

$$\mathcal{D} f(z) = z \frac{d}{dz} f(z).$$

Theorem 5.2 *Let* $0 < \alpha < 1$ *and* $\lambda - \alpha \neq -2, -3, \ldots$*, then*

$$\mathbb{D}_\lambda^\alpha f(z) = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right) \mathbb{I}_{\lambda-\alpha}^{1-\alpha} f(z), \quad z \in \mathbb{D},$$

where $Ef = f$ *is the identity operator. Hence, for* $\lambda - \alpha > -1$ *we have*

$$\mathbb{D}_\lambda^\alpha f(z) = \left(E + \frac{1}{1+\lambda}\mathcal{D}\right) \int_{\mathbb{D}} \frac{f(w)}{(1-z\overline{w})^{1+\lambda}} \mathrm{dA}_{\lambda-\alpha}(w), \quad z \in \mathbb{D}.$$

Proof We have $\left(E+\frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^1_\lambda = E$, hence

$$\mathbb{D}^\alpha_\lambda = \mathbb{I}^{-\alpha}_{\lambda-\alpha} = E\mathbb{I}^{-\alpha}_{\lambda-\alpha} = \left(E+\frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^1_\lambda\mathbb{I}^{-\alpha}_{\lambda-\alpha}$$
$$= \left(E+\frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^{-\alpha}_{\lambda-\alpha}\mathbb{I}^1_\lambda = \left(E+\frac{1}{1+\lambda}\mathcal{D}\right)\mathbb{I}^{1-\alpha}_{\lambda-\alpha},$$

according to (23). □

Lemma 5.3 *Let* $\alpha > 0$, $\beta \in \mathbb{R}$. *Then*

$$\mathbb{I}^\alpha_\lambda \frac{1}{(1-z)^\beta} = \frac{A(z)}{(1-z)^{\beta-\alpha}}, \quad \lambda \neq -2, -3, \ldots, \quad z \in \mathbb{D}, \tag{24}$$
$$\mathbb{D}^\alpha_\lambda \frac{1}{(1-z)^\beta} = \frac{B(z)}{(1-z)^{\beta+\alpha}}, \quad \lambda-\alpha \neq -2, -3, \ldots, \quad z \in \mathbb{D}, \tag{25}$$

where $A, B \in H(\mathbb{D})$, *and* $A \in C(\overline{\mathbb{D}})$, $A(1) = 1$ *when* $\beta - \alpha < 1$, *and* $B \in C(\overline{\mathbb{D}})$, $B(1) = 1$, *when* $\beta + \alpha < 1$.

Besides this for $m = 0, 1, 2, \ldots$

$$\mathbb{I}^\alpha_\lambda \frac{z^m}{(1-z)^{2+\lambda+m}} = \frac{z^m}{(1-z)^{2+\lambda-\alpha+m}}, \quad \lambda \neq -2, -3, \ldots, \quad z \in \mathbb{D}, \tag{26}$$

$$\mathbb{D}^\alpha_\lambda \frac{z^m}{(1-z)^{2+\lambda+m}} = \frac{z^m}{(1-z)^{2+\lambda+\alpha+m}}, \quad \lambda-\alpha \neq -2, -3, \ldots, \quad z \in \mathbb{D}. \tag{27}$$

Proof To prove (24) we calculate Taylor expansion of

$$R(z) \equiv \mathbb{I}^\alpha_\lambda \frac{1}{(1-z)^\beta} - \frac{\Gamma(\beta-\alpha)\Gamma(2+\lambda)}{\Gamma(\beta)\Gamma(2+\lambda-\alpha)} \frac{1}{(1-z)^{\beta-\alpha}}$$

and then use the asymptotic behaviour of Gamma function (4):

$$R(z) = \frac{\Gamma(2+\lambda)}{\Gamma(\beta)\Gamma(2+\lambda-\alpha)} \sum_{m=0}^\infty c_{n,\alpha,\beta,\lambda} z^n,$$
$$c_{n,\alpha,\beta,\lambda} = \frac{\Gamma(n+\beta)}{\Gamma(n+1)} \frac{\Gamma(n+2+\lambda-\alpha)}{\Gamma(n+2+\lambda)} - \frac{\Gamma(n+\beta-\alpha)}{\Gamma(n+1)}$$
$$= O\left(\frac{1}{n^{2+\alpha-\beta}}\right), \quad n \to \infty.$$

Now (24) is clear. The formula (25) follows by (22).

To prove the second statement observe that for $\lambda > -1$ we have

$$\begin{aligned}\mathbb{I}_\lambda^\alpha \frac{1}{(1-z)^{2+\lambda}} &= \int_{\mathbb{D}} \frac{\mathrm{dA}_\lambda(w)}{(1-z\overline{w})^{2+\lambda-\alpha}(1-w)^{2+\lambda}}\\ &= \int_{\mathbb{D}} \frac{\mathrm{dA}_\lambda(w)}{(1-z\overline{w})^{2+\lambda}(1-w)^{2+\lambda-\alpha}}\\ &= B_{\mathbb{D}}^\lambda \frac{1}{(1-z)^{2+\lambda-\alpha}} = \frac{1}{(1-z)^{2+\lambda-\alpha}},\end{aligned}$$

for $z \in \mathbb{D}$. Therefore,

$$\mathcal{D}^m \mathbb{I}_\lambda^\alpha \frac{1}{(1-z)^{2+\lambda}} = \frac{\Gamma(2+\lambda-\alpha+m)}{\Gamma(2+\lambda-\alpha)} \frac{z^m}{(1-z)^{2+\lambda-\alpha+m}}, \quad z \in \mathbb{D}.$$

From the other side, differentiating under the integral sign we have

$$\begin{aligned}&\mathcal{D}^m \mathbb{I}_\lambda^\alpha \frac{1}{(1-z)^{2+\lambda}} = \mathcal{D}^m \int_{\mathbb{D}} \frac{\mathrm{dA}_\lambda(w)}{(1-w)^{2+\lambda}(1-z\overline{w})^{2+\lambda-\alpha}}\\ &= z^m \frac{\Gamma(2+\lambda-\alpha+m)}{\Gamma(2+\lambda-\alpha)} \int_{\mathbb{D}} \frac{\overline{w}^m \mathrm{dA}_\lambda(w)}{(1-w)^{2+\lambda}(1-z\overline{w})^{2+\lambda+m-\alpha}}\\ &= \frac{\Gamma(2+\lambda-\alpha+m)}{\Gamma(2+\lambda-\alpha)} \int_{\mathbb{D}} \frac{w^m \mathrm{dA}_\lambda(w)}{(1-z\overline{w})^{2+\lambda}(1-w)^{2+\lambda+m-\alpha}},\end{aligned}$$

for $z \in \mathbb{D}$. Now for $\lambda > -1$ the formula (26) follows by comparing the above formulas. The constructions in both sides of the formula (26), as functions of λ are holomorphic in $\mathbb{C}$ except for the poles $\lambda = -2, -3, \dots$. Therefore, by arguments of analytic continuation, the formula (26) remains valid for all $\lambda \in \mathbb{R}$ except for $\lambda = -2, -3, \dots$. Finally, the formula (27) follows by (22). □

6 Some Mapping Results in $L_\lambda^p(\mathbb{D})$

6.1 Young Type Theorem

Theorem 6.3 *Let* $f \in \mathcal{A}_\lambda^p(\mathbb{D})$, $g \in L_\lambda^q(\mathbb{D})$, $\lambda > -1$. *Let* $1 \leqslant p, q, r \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} \geqslant 1$, *then*

$$\|\mathbb{K}_g f\|_{r,\lambda} \leqslant C_\lambda \|g\|_{q,\lambda} \|f\|_{p,\lambda}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1, \tag{28}$$

where the constant does not depend on f *and* g.

Proof The case $r = p = \infty$ is obvious with $C_\lambda = 1$:

$$\|\mathbb{K}_g f\|_\infty \leqslant \|g\|_{1,\lambda}\|f\|_\infty. \tag{29}$$

Let now $r = p = 1$. By Fubini theorem we have

$$\begin{aligned}
\|\mathbb{K}_g f\|_{1,\lambda} &\leqslant \int_{\mathbb{D}} |g(w)|\mathrm{dA}_\lambda(w) \int_{\mathbb{D}} |f(z\overline{w})|\mathrm{dA}_\lambda(z)\\
&= (\lambda+1)\int_{\mathbb{D}} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|}|f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&= (\lambda+1)\int_{|w|<\frac{1}{2}} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|}|f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&+(\lambda+1)\int_{\frac{1}{2}<|w|<1} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|}|f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z).
\end{aligned}$$

Due to known estimate $|f(z)| \leqslant \frac{\|f\|_{p,\lambda}}{(1-|z|^2)^{\frac{2+\lambda}{p}}}$, $z \in \mathbb{D}$ we have $|f(z)| \leqslant \left(\frac{4}{3}\right)^{2+\lambda}\|f\|_{1,\lambda}$ for $|z| < \frac{1}{2}$, hence

$$\begin{aligned}
&(\lambda+1)\int_{|w|<\frac{1}{2}} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|}|f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&\leqslant \left(\frac{4}{3}\right)^{2+\lambda}\|f\|_{1,\lambda}\int_{|w|<\frac{1}{2}}|g(w)|\mathrm{dA}_\lambda(w) \leqslant \left(\frac{4}{3}\right)^{2+\lambda}\|f\|_{1,\lambda}\|g\|_{1,\lambda}.
\end{aligned}$$

For the second term similar estimate is also trivial:

$$\begin{aligned}
&(\lambda+1)\int_{\frac{1}{2}<|w|<1} \frac{|g(w)|}{|w|^2}\mathrm{dA}_\lambda(w)\int_{|z|<|w|}|f(z)|\left(1-\left|\frac{z}{w}\right|^2\right)^\lambda \mathrm{dA}(z)\\
&\leqslant 4(\lambda+1)\int_{\frac{1}{2}<|w|<1}|g(w)|\mathrm{dA}_\lambda(w)\int_{|z|<|w|}|f(z)|\left(1-|z|^2\right)^\lambda \mathrm{dA}(z)\\
&\leqslant 4\|f\|_{1,\lambda}\|g\|_{1,\lambda}.
\end{aligned}$$

Hence, we obtain

$$\|\mathbb{K}_g f\|_{1,\lambda} \leqslant \left(4+\left(\frac{4}{3}\right)^{2+\lambda}\right)\|g\|_{1,\lambda}\|f\|_{1,\lambda}. \tag{30}$$

In view of (29) and (30), applying Riesz-Thorin-Stein-Weiss interpolation theorem [23] we obtain

$$\|\mathbb{K}_g f\|_{p,\lambda} \leqslant C_\lambda \|g\|_{1,\lambda} \|f\|_{p,\lambda}, \quad 1 \leqslant p \leqslant \infty. \tag{31}$$

Here C_λ does not depend on f and g.

By Hölder inequality

$$\|\mathbb{K}_g f\|_\infty \leqslant \|g\|_{p',\lambda} \|f\|_{p,\lambda}, \quad 1 \leqslant p \leqslant \infty. \tag{32}$$

Combining (31) with (32) and again interpolating between 1 and p' we finally obtain (28). □

As an anonymous reviewer kindly noticed, in fact Theorem 6.3 states that the bilinear operator $B(f; g) = f \times g$ is bounded (with respect to the norm $\| \cdot \|_{r,\lambda}$ on its rage) on the corresponding product of spaces.

6.2 Sobolev Type Theorem

Besides the operator $\mathbb{I}^\alpha_\lambda$, consider also

$$\mathbb{I}^{\alpha,+}_\lambda f(z) = \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{2+\lambda-\alpha}} \mathrm{dA}_\lambda(w), \quad \alpha > 0, \quad \lambda > -1, \quad z \in \mathbb{D},$$

and the following two more general operators as well

$$T^\alpha_{a,b} f(z) = (1 - |z|^2)^a \int_{\mathbb{D}} \frac{f(w)}{(1 - z\overline{w})^{2-\alpha+a+b}} \mathrm{dA}_b(w),$$
$$S^\alpha_{a,b} f(z) = (1 - |z|^2)^a \int_{\mathbb{D}} \frac{f(w)}{|1 - z\overline{w}|^{2-\alpha+a+b}} \mathrm{dA}_b(w).$$

The following Sobolev type theorem is valid.

Theorem 6.4 *Let* $0 < \alpha < 2$, $1 < p < \frac{2}{\alpha}$, $a + b \geqslant 0$, *and* $b > -\frac{1}{p}(1 - \frac{\alpha}{2})$. *Then*

$$\|T^\alpha_{a,b} f\|_{q,bq} \leqslant C \|f\|_{p,bp}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2},$$

$$\|S^\alpha_{a,b} f\|_{q,bq} \leqslant C \|f\|_{p,bp}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{2}.$$