Muriel Seltman
Robert Goulding

# Thomas Harriot's Artis Analyticae Praxis

## An English Translation with Commentary

Springer

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# Preface

The *Artis analyticae praxis* was published in 1631 in Latin. Until relatively recently, an English edition would not have been considered necessary since most of the people who might have been expected to be interested in the text would have been able to read it in Latin. That is no longer the case and it is fitting that an English edition should be published.

It is also the case that a considerable proportion of the readership of the present volume will not be professional mathematicians, so we have tried to produce a translation that makes the mathematical content accessible to the modern reader. This is not because the algebra is intrinsically difficult but because it is not the kind of mathematics which is a part of today's secondary school curriculum.

A further problem lies in the fact that the book in the form in which it was published in 1631 may very well not conform to Harriot's intentions for the publication of his mathematical manuscript papers. This is why the book is accompanied by a Commentary which attempts to compare it with the appropriate passages in the surviving manuscript papers.

The present work is a translation of the original text and not intended as a facsimile. The original has well over 300 errors (there may very well be more) and we have listed these at the end. In the interests of mathematical accessibility we have tried to produce a mathematically "clean" copy.

Again, for ease of reading (and printing) we have altered Harriot's sign for equality in his manuscripts to the modern version and omitted the two vertical lines between the parallels. Where it has been necessary to use the inequality signs seen in Harriot's manuscripts, we have used the modern version for the same reasons. Similarly, we have not included the ubiquitous dots appearing in the original work, which were common at the time and which separated the numerical from the literal part of an algebraic term, thus 2.x for the modern 2x.

The translation was the responsibility of Robert Goulding and Muriel Seltman was responsible for the Commentary. Both the translation and the Commentary were originally based on an M.Sc. Dissertation presented at University College, London by Muriel Seltman, but as work proceeded these influences have disappeared without trace and the present book is totally new.

We would like to acknowledge the assistance of the British Library, Lambeth Palace Library (London), and Liverpool University Library. Our grateful thanks are due to Dr. J. V. Field who read the entire commentary and made valuable suggestions. The *Praxis* relied less on formal proof than on the immediate evidence of the equations arranged on the page. We have taken pains to preserve the visual impact of the *Praxis* — and this would not have been possible without William Adams' expertise in LaTeX. In particular, he typeset the most challenging part of the *Praxis*, the Numerical Exegesis.

At an early stage of the project, Mordechai Feingold offered invaluable advice and support, and encouraged us to submit our manuscript to Springer for publication in the series in which it now appears. This turned out to be an excellent fit for our book, and we are also grateful to the series editor Jed Buchwald and to Mark Spencer, our editor at Springer.

Above all, we would like to express our thanks to the British Society for the History of Mathematics and the Harriot Seminar. Each provided a generous grant which not only helped financially but was also valuable as a moral support for our work.

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# Introduction

Revolution in mathematics means the birth of the new but not the demise of the old, only its obsolescence. And in mathematics, as in any other field, the particular aspects undergoing such change must be specified: for example, symbolism, methodology, type of problem, method of proof, axiomatic structure, level of abstraction, or, perhaps, methods of computation.

For reason of brevity, I assume the possibility of a model for the history of mathematics in western Europe, the defining characteristic of which is progressive abstraction. There may be others. I would argue that in the sixteenth and early seventeenth centuries, algebra underwent changes that involved genuine novelty, a revolution one might say, rendering previous assumptions, symbolisms, methodology, goals, and so on, or any combination of these, obsolescent but not invalidating them. And Thomas Harriot (c. 1560–1621) undeniably, played a considerable role in this transformative process.

In the algebraic work of Thomas Harriot, it was above all his notation that was revolutionary. His algebra was the first to be totally expressed in a purely symbolic notation (traditionally, using letters and operational signs), and this was the case in both his manuscripts and in the work published under his name as *Artis analyticae praxis* (1631, London). There appears in his work for the first time ever, the possibility of algebraic logic embodied in the very notation itself, which renders such logic manifest. In Harriot's algebra, we can check that the symbolic manipulation obeys the rules for manipulating algebraic quantities set out at the beginning of the *Praxis* (pp. 11). The rigour for so long associated only with Euclidian geometry now has a new field of operation—algebra.

Yet, Harriot is known in general histories of mathematics principally for certain technical innovations in algebra—for the invention of the inequality signs, for equating the terms of a polynomial equation to zero, and for generating such equations from the product of binomial factors, thereby displaying their structure. It is only since the late 1960s and 1970s with the work of R. C. H. Tanner, Jon Pepper, D. T. Whiteside, and others, that his work has received serious and scholarly attention.

Undoubtedly, his work in algebra was overshadowed in its own time by that of Descartes (1596–1650), whose *La Géométrie*, published only six years after Harriot's posthumous work, would go beyond that of Harriot in achievement and potentiality for future development, but nothing can diminish the credit due to Harriot for his own achievement. It is the contribution that his book made to the ongoing revolution in mathematics of the late Renaissance that justifies the publication, for the first time, of an English translation of the *Artis analyticae praxis*, making it more accessible to modern readers. Such a translation is, in our view, long overdue.

Thomas Harriot was born into a world in which traditional ideas were under intense challenge. Dee's pupil, Thomas Digges (1546–1595), was the first Englishman to publicize Copernicanism and did this in the vernacular. His father, Leonard Digges (c. 1520–1559), advocated teaching mathematics to artisans. There was in fact considerable rapport between the leading scholarly mathematicians and mathematical practitioners in the England of that time. The fact that Digges is published in English suggests a relatively high level of literacy.

The economic context for this collaboration was the rapid emergence of English mercantile capitalism (or, perhaps, imperialism). Dee was technical advisor to the Muscovy Company as Harriot was later to be a member of the Virginia Company. Harriot was friendly with Dee and Hakluyt, corresponded with Kepler (1571–1630) on optics and the telescope, and even made telescopes for sale during the final twelve years of his life. We cannot do better than quote D. T. Whiteside's summary of his accomplishments.

"Harriot in fact possessed a depth and variety of technical expertise which gives him good title to have been England's—Britain's—greatest mathematical scientist before Newton. In mathematics itself he was the master equally of the classical synthetic methods of the Greek geometers Euclid, Apollonius, Archimedes and Pappus, and of the recent algebraic analysis of Cardano, Bombelli, Steven and Viète. In optics he departed from Alhazen, Witelo and Della Porta to make first discovery of the sine-law of refraction at an interface, deriving an exact, quantitative theory of the rainbow, and also came to found his physical explanation of such phenomena upon a sophisticated atomic substratum. In mechanics he went some way to developing a viable notation of rectilinear impact, and adapted the measure of uniform deceleration elaborated by such medieval 'calculators' as Heytesbury and Alvarus Thomas correctly to deduce that the ballistic path of a projectile travelling under gravity and a unidirectional resistance effectively proportional to speed is a titled parabola—this years before Galileo had begun to examine the simple dynamics of unresisted free fall. In astronomy he was as accurate, resourceful and assiduous an observer through his telescopic 'trunks'—even anticipating Galileo in pointing them to the Moon—as he was knowledgeable in conventional Copernican theory and wise to the nuances of Kepler's more radical hypotheses of celestial motion in focal elliptical orbits. He further applied his technical expertise to improving the theory and practice of maritime navigation; determined the specific gravities and optical dispersions of a wide variety of liquids and some solids; and otherwise busied himself with such more conventional occupations of the Renaissance *savant* as making alchemical experiment and creating an improved system of 'secret' writing". [*Hist. Sci.*, xiii (1975), (61–70)]

## Life and Reputation

Thomas Harriot is thought to have been born about 1560, probably near Oxford. He certainly graduated from St Mary's Hall, Oxford, part of Oriel College, around 1580, after which he entered the service of Walter Ralegh, who remained his patron until the 1590s.

Harriot had been sent by Ralegh on Sir Richard Grenville's expedition to Virginia (the territory in question is now North Carolina) in 1585 as surveyor, and subsequently published *A briefe and true report of the new found land of Virginia*. This was to be his only publication during his lifetime.

In the early 1590s, Henry Percy, the ninth Earl of Northumberland, became Harriot's patron. This association lasted his lifetime, outliving Percy's imprisonment following the Gunpowder Plot (1605–1621). This membership of the Earl's household brought Harriot into contact with Walter Warner (c. 1557–1643), Robert Hues (1553–1632), and Nathaniel Torporley (1564–1632).

Nathaniel Torporley, a cleric and very reputable mathematician of his time, had considerable admiration for Harriot and was to play a part in the publication of the *Praxis* on Harriot's death. Hues published *a Treatise on Globes* in 1594, was associated with Gresham College, and was appointed by Harriot in his Will to oversee the pricing of his books when they should come to be sold.

The details of the dispersal and disposal of Harriot's papers after his death have been studied and treated in detail in the secondary literature.[1]

Harriot's Will gave Torporley the task of editing his mathematical writings asking him to "peruse and order and to separate the chief of them from my waste papers, to the end that after he doth understand them he may make use in penning such doctrine that belongs unto them for public uses. . . . " If Torporley did not understand the notation, he was to confer with Warner or Hues. Failing this, Protheroe and Aylesbury should be asked to help (Tanner, *History of Science, 6*, 1967, p. 5). Finally, after this, the papers should be put into Percy's library and the key to the trunk holding them should be held by Henry Percy, the ninth Earl of Northumberland, Harriot's patron at the time.

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[1] Rosalind C. H. Tanner, "Thomas Harriot as Mathematician: a Legacy of Hearsay". *Physis* IX, 1967, pp. 235–256, 257–292.
Rosalind C. H. Tanner, "The Study of Thomas Harriot's Manuscripts", I. Harriot's Will, *History of Science, 6*, 1967, pp. 1–16.
Jon V. Pepper, "The Study of Thomas Harriot's Manuscripts", II. Harriot's unpublished papers, *History of Science, 6*, 1967, pp. 17–40.
R. Cecilia H. Tanner, "Nathaniel Torporley and the Harriot Manuscripts", *Annals of Science*, 25, 1969, pp. 339–349.
Jacqueline A. Stedall, "Rob'd of Glories: the Posthumous Misfortunes of Thomas Harriot and his Algebra", *Arch. Hist. Exact. Sci. 54* (2000), pp. 455–497.
Jacqueline A. Stedall, *The Greate Inventions of Algebra: Thomas Harriot's Treatise on Equations*, Oxford University Press, 2003.

Torporley's copies of some of Harriot's mathematical papers, the *Congestor analiticus* and a compilation called the *Summary* (see J. Stedall, 2003, pp. 19–20, 24–26), are in Lambeth Palace Library, London, having been transferred in 1996 from the library of Sion College where Torporley spent his final years. *The Congestor Analyticus* is Torporley's incomplete attempt at a treatise and the *Summary* of what Torporley thought an edition of Harriot's algebra should contain. After the publication of the *Praxis*, Torporley wrote a criticism of the work, entitled: *Corrector analyticus artis posthumae Thomae Harrioti*. Harriot's own papers, mathematical and others, are in various libraries, but those that are relevant to the contents of the *Praxis* are in the British Library under the catalog numbers Add. MSS 6782-9, almost all in 6782-4. The pages that deal with algebra constitute only a fraction of this material, and the pages that are certainly relatable to the *Praxis* (concerned with the theory of equations and the solution of polynomial equations with numerical coefficients) are interspersed with pages on other mathematical and scientific topics and some sheets that are undoubtedly "waste."

Torporley planned, but never completed, the task assigned to him by Harriot. Within a few years of Harriot's death, the work appears to have been shared between Torporley and Warner. Whatever notes Warner used to put together the *Praxis*, whether he worked from the manuscript papers or a later draft by Harriot, now lost, he certainly interfered with what Harriot had intended and rearranged, altered, reduced, and augmented, as we know from Torporley's copy of Harriot's notes. In a recent study, Jacqueline Stedall has discussed in considerable depth the relationship of the text of the *Praxis* to Harriot's surviving manuscripts in the light of Torporley's *Congestor* and *Summary*. Although the present authors do not fully concur with Stedall's arguments, it is clear that her book presents a "treatise"as close as is possible to what Harriot might have published had he lived to do so (J. Stedall, *The Greate Invention of Algebra*, OUP, 2003).

In the end, it was Warner who edited Harriot's papers and redrafted and rearranged them for the *Praxis*. Whether he did this from the papers now in the British Library, from the Torporley papers or from a further re-working by Harriot, or from other copies, we cannot tell. The history of the papers in the few centuries after Harriot's death has been speculatively traced by several writers, notably R. C. H. Tanner [see *Annals of Science*, 25, 1969, pp. 339–49].

The influence of Harriot's mathematical writings on later English mathematicians such as John Pell (1611–1685), Charles Cavendish (1592–1664), and (indirectly upon) John Wallis (1616–1703) was considerable. Certainly, he had the highest reputation among his contemporaries. Warner's commendation of Harriot took the form of saying: "Harriot... ought truly to be considered to have completely perfected numerical Exegesis, an art which is instrumental in all Mathematical arts and on that account the most useful" (*Praxis*, Preface, p. 4).

In what sense was Harriot's art new? Principally for the much more convenient and practical character of the numerical exegesis previously presented by Viète. The specific means that Harriot used were "a literal notation: that is, the letters of the alphabet, either by themselves or in any combination, according to the needs of the calculation of the reasoning" (*Praxis*, Preface, p. 4). This (says Warner) is to be compared with the work of Viète, who had proposed a "logistic which was to be exercised through (verbally) interpreted signs; although this perhaps was useful for understanding the new discipline, it was subsequently found to be inconvenient for normal practical use." Hence, the first object of Warner's praise was Harriot's symbolism. Processes hitherto "somewhat irksome and rather ungainly" had "been bought to the utmost simplicity and lucidity. (For a description of Viète's notation, see below, p. xx).

Warner next gives the reader what he considers to be Harriot's two chief discoveries (which Warner sees as relying on what he calls "the dexterity of this *Arithmetic*"): first, the generation from binomial roots of Canonical Equations so that, when they are "applied to" common equations, the roots of the latter are revealed. ("...a most ingenious discovery," writes Warner); second the derivation of Canonical Polynomials.[2]

The "uniform and continuous" application of certain rules or Canons enables the mathematician to work with ease and certainty. The latter, thinks Warner, is Harriot's most important discovery and a mere glance at corresponding cases in the *Numerical Exegesis* (below) and Harriot's manuscripts, on the one hand, and Viète's *De numerorum potestatum resolutione* (1600), on the other hand, makes this quite clear to us. Hence, Harriot was a master mathematician and Analyst in the opinion of his editor, Walter Warner. And Warner was not alone in his high

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[2] A Canonical Equation was one expressed in a standard form produced by the multiplication of binomial factors. What Harriot did may be summarised (for a quadratic equation) in modern notation, as follows:

If $a$ is to be a root of the equation, $x = a$,
then $x - a = 0$
If $b$ is a root, $x = b$
then $x - b = 0$
It follows that $(x - a)(x - b) = 0$
i.e., $x^2 - (a + b)x + ab = 0$
and $x^2 - (a + b)x = -ab$

The expression in the final line is the Canonical Equation and it follows that any equation in such a form has roots $a, b$.

A Canonical Polynomial may be explained (again, in modern notation) as follows: Problem 2 of the Numerical Exegesis solves an equation of the form:

$$a^2 + da = f^2$$

in the particular case:

$$a^2 + 432a = 13584208$$

Put $a = b + c$
Then $(b + c)^2 + d(b + c) = f^2$
i.e. $b^2 + 2bc + c^2 + db + dc = f^2$
i.e. $db + b^2 + dc + 2bc + c^2 = f^2$

The Canonical Polynomials are $db + b^2$ and $dc + 2bc + c^2$, which are used in tabular fashion to perform the computation. (See n.2 on Numerical Exegesis).

opinion of Harriot. William Lower, a friend of Harriot's is well-known for having urged Harriot to publish (W. Lower, from J. W. Shirley, *Thomas Harriot*, Oxford, 1983, p. 400).

Despite the waning in Harriot's influence in the eighteenth and nineteenth centuries, he was always remembered. In 1777, Lagrange (1736–1813) credited Harriot with having been the first to show the conditions for a cubic equation lacking the second term to have real roots (Tanner, *Physis*, 1967, p. 283). In 1883, a letter from the algebraist J. J. Sylvester (1814–1897) to Arthur Cayley (1821–1895) referred to Harriot: "It was gratifying, however, to see the handwriting of the man who first introduced the Algebraic Zero into Analysis, the father of current Algebra" (John Fauvel, *Harrioteer*, September 1996). A full appreciation of his work is not possible without considering its historical context, which, in turn, leads naturally into a discussion of the *Praxis* itself.

## Historical Background

The second half of the sixteenth and the first half of the seventeenth centuries saw the burgeoning of new methodologies, notations, objects, and relationships in algebra, accompanied by a movement toward generality. Over and above this came a small move toward extending and generalizing the concept of number with Cardano's acknowledgment of negative roots in *Ars Magna* (1545). The "irreducible" case was noted, that is, the case of the cubic equation with three real roots, two of which are reached via conjugate complex numbers. But, although Cardano solved the cubic by a "recipe" resembling those used by the Babylonians (with only an implicit formula), his "demonstration" (a form of proof) of the solution was a geometric one. In Cardano there is no overt demonstration of the algebraic logic leading to the solution. Algebraic logic is not the same as (Aristotelian) formal logic, but it does include it. Manipulation of the real numbers is governed by distributive, commutative, and associative laws, which are axiomatic, but the inferential links are those of formal logic.

Viète is the first algebraist to use a literal sign for a general number, and algebra becomes the "ars analytice," which has a geometrical aspect due to its being related to magnitude.

In the course of the Renaissance, a number of different notations had emerged, to a large extent embodying different concepts, but four principal strands may be distinguished. All of them were in use in the first half of the seventeenth century.

"Cossic" numbers were used by Luca Pacioli (1445–1514) in the *Summa de arithmetica, geometrica, proportioni et proportionalita* of 1494 and were used by Michael Stifel (c. 1487–1567) in the *Arithmetica Integra* of 1544. The latter notation uses single signs (taken from Old German script) for the unknown and each of its powers, e.g., 𝔵 corresponds to $x$, ʒ corresponds to $x^2$, ɔ corresponds to $x^3$, and ʒʒ to $x^4$. This notation pays its respects to the three-dimensional constraint inherited from the Greeks. There is no common reducibility in the Cossic sign-system, just as there was none in Diophantus (i.e., $x^2$, $x^3$ have a common "base" absent in the Cossic system). Moreover, there is no exponent acting as an operator (as in $x^2$).

A second strand, going back to Nicolas Chuquet (c. 1500) and Rafael Bombelli (c. 1526–1573), is embodied principally in Simon Stevin (1548–1620) and Albert Girard (1590–1663). Hindu–Arabic numerals play the decisive role in this tradition, acting as coefficients and as indicators of the powers of numbers, if not exponents in the modern manner. Stevin, in the *Arithmetic*, uses numerals inside circles: ①, ②, ③ represent (our) $x$, $x^2$, and $x^3$, respectively.

It would be a mistake to see this notation as "lacking the unknown," at least subjectively for those using it. Certainly, this notation has a strong bias toward an operational exponent and a uniform quantitative base rooted in unity conceived as a number.

The third strand, found as early as Michael Stifel (1486–1567) and culminating first in Harriot and then in Descartes' *La Géométrie* (1637), ultimately carried the day, becoming standard traditional algebraic notation. In its earliest form (1, 1A, 1AA, 1AAA...) (M. Stifel, *Die Coss Christoffs Rudolfs*, V, 1553) no exponent is overtly used, but from the start it represents a significant departure from the Cossic and other geometrically expressed systems. This form of notation is used by Harriot, although Descartes in his correspondence used a wide variety of notations as the fancy took him or, perhaps, according to the person to whom he was writing.

The fourth strand is represented by François Viète (1540–1603) whose notation in all works, apart from the *De numerosa potestatum resolutione* (1600), in which numerical equations are solved, appears thus:

> Sit data B differentia duorum laterum, & datum quoque D adgregatum eorumdum. Oportet inuenire latera.
>
> Latus minus esto A, maius igitur erit A + B. Adgregatum ideo laterum A bis + B. At idem datum est D. Quare A bis + B aequatur D. et per antithesim, A bis aequabitur D − B, & omnibus subduplatis, A aequabitur D semissi, minùs B semisse.
>
> Vel, latus maius esto E. Minus igitur erit E − B. Adgregatum ideo laterum. E bis, minùs B. At idem datum est D. Quare E bis minus B aequabitur D. & per antithesim, E bis aequabitur D + B, & omnibus subduplatis E aequabitur D semissi, plùs B semisse.

This is in the 1600 edition of the Viète work, which was followed in 1646 by its appearance as part of *Opera Mathematica*, edited by F. Schooten (1615–1660 or 61). The sign − is used in 1600 as well as the word minus for the subtractive operation, and whenever the word is used thus, it is changed by Schooten to − in 1646. *De Numerosa Potestatum Resolutione* (1600) uses − throughout, although the sign is usually slightly broken due to the exigencies of printing. The word "bis," used in 1600 is changed to 2 by Schooten, i.e., A bis becomes A2. "D semissi plus B semissi" in the above extract becomes $D\frac{1}{2} + B\frac{1}{2}$ in 1646 and "D semissi minus B semissi" becomes $D\frac{1}{2} - B\frac{1}{2}$ in 1646, as a result of Schooten's editorship.

In Viète's logistica speciosa, "A quadratum" corresponds to, but is not to be identified with, A-squared ($A^2$). "A quadratum" denotes "a square," the "side" of which is A, following the Greek tradition. A, B, C, etc. were known as "species".

"A quadratum" is a development upon Diophantus' square number denoted by the first two letters of δυναμισ (dunamis). Harriot would write simply *aa*.

François Viète (1540–1603) saw himself as "restoring" the mathematics of the classical world and saw his *Introduction to the art of analysis* (*In artem analyticen isagoge*), as part of the *restoration of mathematical analysis* (*opus restitutae mathematicae analyseos*). However, despite his subjective view of his own project, what emerges objectively in his work is a decisive transformation in which Analysis (and Synthesis) will, for the first time, be identified with algebra as well as geometry. The title of Viète's book varies in different editions. Those of 1591, 1624, and 1631 have *In artem analyticum isagoge*, the 1635 edition has *In artem analyticam isagoge* and that of 1646 has *In artem analyticum isagoge* (Witmer, 1983, p. 11).

Perhaps Viète's most outstanding contribution to algebra lies in the new level of generality engendered by his notation in which, for example, A represents a general positive number. On the other hand, the link with a geometrical base is never broken and even *On the Numerical Solution of Equations* is followed by *Canonical Rescension of Geometrical Constructions* (Tours, 1593), which (as Mahoney wrote) "showed how symbolic quantities may be interpreted as line segments and symbolic operations as geometric construction procedures" (Mahoney, 1994, p. 38).

It is, however, of more importance that it is made clear in the *Isagoge*, especially in Chapters III and IV, that the entities being referred to are magnitudes, and this is so in all of Viète's work except for the (numerical) solution of equations with numerical coefficients. Magnitudes are, in fact, geometrical quantities, lengths, areas, or volumes. Such a magnitudinal base accounts for the retention by Viète of Greek geometrical constraints. Chapter III opens: "The prime and perpetual law of equations or proportions which, since it deals with their homogeneity, is called 'the law of homogeneous terms' is this: 'Homogeneous terms must be compared with homogeneous terms' " (Witmer, 1983, p. 15). Viète's algebra is thus both geometrically and numerically based and he builds on the work of Pappus and Diophantus.

Diophantus had flourished c.250 AD and Pappus a half a century later. The first provided Viète with a basis in (arithmetical) algebra and the analytic solution of algebraic problems and Pappus provided geometrical analysis. Viète used both these classical authors and subsumed them into his new analytic art. He did not, however, see himself as an innovator but as a renewer of the classical tradition.

## Viète and Harriot

Unlike Viète, Harriot did not base his notation upon magnitude even though he applied algebra to geometrical problems. (See, for example, British Library Add. MS 6784, ff. 19–28.) For both authors, however, the distinction between "Numerical" and "Specious" Logistic (Arithmetic) is given as that between representation by numerical signs or by letters of the alphabet. The first mention of *logistice* *speciosa* by Viète is in Chapter I of *In artem analyticen isagoge*: “It no longer limits its reasoning to numbers, a shortcoming of the old analysts, but works with a newly discovered symbolic logistic ... ” (Witmer, 1983, p. 13). The full definition by Viète comes at the beginning of Chapter 4. Numerical logistic is [a logistic] that employs numbers, symbolic logistic is one that employs symbols or signs for the things as, say, the letters of the alphabet (Witmer, 1983, p. 17). Such wording accords very well with the practice of Harriot and accords with Definition 1 of the *Praxis*. It would also accord with the later views of Wallis, as expressed in his *Algebra* (1685, p. 12). (See Witmer, 1983, p. 13, fn. 8 for full discussion.)

Harriot studied Viète closely and was affected by him both instrumentally and conceptually. In the manuscripts, Harriot’s solutions of polynomial equations with numerical coefficients are all accompanied by a reference to the corresponding solution by Viète and, moreover, all the pages are headed *De numerosa potestatum resolutione* (*On the Numerical Resolution of Powers*) the title of Viète’s own published work (1600). Harriot’s debt to Viète is clear in the algorithmic outline of the solution. The Definitions demonstrate a conceptual debt to Viète but, importantly, the manner in which his notation transcends that of Viète.

Harriot’s algebra was expressed in a completely symbolic notation unlike that of Viète, which had included linguistic elements. In a sense, Harriot’s algebra by-passed the species of Viète, which had their roots in Diophantus and were bound to Greek ideas about the necessity for homogeneity. Homogeneity was retained throughout the Harriot manuscripts, as it was in the *Praxis* except for the cases in which a collection of terms was equated to zero.

It was Descartes (who said that he had not read Viète before writing *La Géométrie*) who dealt with the issue of homogeneity by the use of “dummy” units. As early as the *Regulae ad directionem ingenii, Rules for Right direction of the mind* (posth. 1684), Descartes pointed out in Rule XV the need for an arbitrary unit and followed this up in Rule XVIII in connection with a problem involving continued proportion. The Rule continues with an “algebra of magnitudes” in which multiplication (product) is defined by taking two magnitudes $a$ and $b$ and asking the reader to imagine them as the sides of a rectangle. Then, “if we wish to multiply $ab$ by $c$,... we conceive $ab$ as a line, viz. $ab$.”

Descartes had already distinguished conceiving from imagining in Rule XIV. Imagination involves a mental image but conception is a faculty of understanding that requires us only “to attend to what is signified by the name”. Thus, he has given meaning to the product $ab$ as a line by postulating it as a theoretical construct but having a basis in what can be visualized. Hence, even in the *Regulae* Descartes had confronted this issue.

Things are dealt with differently in the *Gèomètrie*, where unity is used directly to dispose of the problem of homogeneity. First, he uses proportions in similar triangles, having taken one side as unity, and simply shows that another side is equal to the product of the remaining sides. A little further on, the issue is tackled algebraically when he considers $a^2\ b^2 - b$ and we must consider the first term divided by unity once and the second quantity multiplied twice by unity. The important thing is that Rule XV of the *Regulae* had said of unity that it is “an object extended in every direction and admitting of countless dimensions". So, in effect, each "dimension of unity" can deal with a different magnitude of dimension.

Harriot did none of this (as far as we know) and homogeneity is maintained in the manuscripts even so far as showing the product of eight zeros (Add. MS 6783, f. 187). For Harriot, however, letters stand for particular unknown numbers (and occasionally lengths) and the need for a unity of Descartes' type just does not arise.

Descartes bases his algebraic geometry on a combination of magnitude and number. Magnitudes are numerically defined via the letters of their names that also stand for numbers, for example, the magnitude *a* is a length designated by "*a*," which is a general number and used algebraically. It is the combination of this with their use in a locus problem implicitly involving motion that will enable curves to be expressed by equations and facilitate the emergence of the real variable. Harriot's notation satisfies the necessary conditions for an algebraic symbolism but is insufficient, because of the superfluity of unknowns arising from the homogeneity requirement, unlike the algebra that follows from the usage of Descartes.

The Preface and Definitions of the *Praxis* are unique in having no corresponding pages in the Harriot manuscripts. We do not know, therefore, whether the Definitions are the unaided work of Warner or whether he was using a draft by Harriot, which has since been lost. The major interest in the Definitions lies in the first twelve, which refer to such terms as Logistice Speciosa, Analysis, Synthesis, Zetetic, Poristic, and Exegetic, all of which had been used by Viète. An assessment may be made of the conceptual connections between Harriot and Viète by a study of these Definitions, but this is so only if Warner's version can in some way be identified with Harriot's. No trace of the Definitions has come to light in any of Warner's papers, nor in Torporley's copy at Lambeth Palace, and as we have already remarked, nor is it to be found in the Harriot manuscripts themselves. However, Warner knew Viète's *Isagoge* and Add. MS 4394, f. 108 has a table, in the left-hand column of which is a list of mathematical works and in each corresponding place in the right-hand column is a name. Against Viète's *Isagoge* is the name Warner and, although there is no hint of the nature of the connection (ownership?), it is clear that some form of association is attested to by this.

Pages in Harriot's manuscripts, particularly Add. MS 6784 ff. 19–28 (not used in the *Praxis*) show all the above terms (albeit with reference to work not directly relevant to the *Praxis* itself). There is, therefore, sufficient evidence to permit us to connect Harriot and Warner conceptually with Viète. A detailed comparison of the Definitions with appropriate passages appears in the textual comments on the Definitions (see below). It is clear that, in the reference to Logistice Speciosa in the *Praxis* in Definition One, what is meant by Speciosa (specious) is the use of letters rather than numbers and Harriot favors a notation in which a letter of the alphabet represents a number seen as composed of numerical units. It will be seen below that the word was used simply to designate literal notation by Warner, Harriot's editor. Furthermore, it will be seen that analysis and synthesis (analysis in the three forms mentioned by Viète, Zetetic, Exegetic, and Poristic) all become exclusively algebraic in Harriot's hands in the manuscripts and in the *Praxis*. It also becomes clear that different sections of the *Praxis* correspond to each of the processes. What, then, are the contents and achievements of the *Praxis*?

## Contents and Achievements of the *Artis Analyticae Praxis*

As the Contents page indicates, the book is in two parts, of which the first, up to the end of Section 6, is on theory of equations and the second, the Numerical Exegesis, is given over to the solution of polynomial equations with numerical coefficients. The *Preface* provides a broad outline history of Analysis. In fact, the historical background is used by Warner to provide a backdrop for a eulogy to Harriot, seen as "a new Viète," who is himself credited with the invention of numerical Exegesis and "specious" (i.e., in letters) Arithmetic. However, Viète's logistic is found by the author to be "inconvenient for normal practical use" and it is Harriot's (purely) literal notation which remedies this and brings Analysis to "the utmost simplicity and lucidity."

Moreover, it is not surprising that the polynomials are especially welcomed, providing as they do a general, algorithmic, purely calculative and non-verbal method of solution for equations with numerical coefficients by successive approximation. This method may now be outdated, but such algorithmic rules constitute a watershed between the situation when verbal directions were written in prose to describe a procedure and the framing of such directions in a purely quantitative form like a calculation in arithmetic.

Such polynomials were, according to Warner, Harriot's second major discovery. The first had been to generate Canonical equations from "binomial rootes" (that is, factors, see Section Two, pp. 12–28, *Praxis*) so that the roots of common equations may be revealed by comparison. Such determination of roots would be done in Section 5 of the *Praxis* for which the manuscript evidence, apart from the inequalities, is extremely sketchy. Yet, Warner states this as Harriot's achievement with confidence and some evidence appears in the Torporley papers (now in Lambeth Palace Library but previously in Syon House, Arc. L.40.2 L.40), which were taken from the Harriot manuscripts and are really abbreviated notes.

On 44v and 45r there is some algebra, which is difficult to decipher, accompanied by an inequality replicating an inequality in Section Five of the *Praxis*:

$$\left.\begin{array}{c}\frac{qqr+qrr}{2}\\[4pt] \frac{qqr+qrr}{2}\\ \hline\end{array}\right| < \left.\begin{array}{c}\frac{qq+qr+rr}{3}\\[4pt] \frac{qq+qr+rr}{3}\\[4pt] \frac{qq+qr+rr}{3}\\ \hline\end{array}\right|$$

There seems to be some comparison of equations but no determination of roots can be seen. It follows that either certain manuscript papers have been lost or that Torporley did have access to a draft or set of notes which went further than Harriot's work in the manuscripts.

We may also make another deduction. It is in the *Preface* that one might have expected a reference to the omission of non-positive roots in the work. But the author writes, "any uncertainty of the roots ... is revealed and dispelled". Yet hundreds of cases exist in the manuscript papers in which negative (and occasionally also imaginary) roots are given. It must be noted that in almost all manuscript pages, negative roots are recognized. There are about 400 equations, quadratic, cubic, and biquadratic, all with numerical coefficients, displayed systematically in lists and accompanied by a simple statement of their roots, but without algebraic derivation. In only five cases are negative roots not given and these are imaginary.

Cases exist, however, of positive and negative imaginaries AD MS 6783, f. 49, f. 156, and f. 301 that display negative imaginary and complex roots and see p. 221 for a summary list of different roots, dealing with all equations shown in the manuscripts apart from those pages which are obvious waste.

Warner takes no account of this. Perhaps he had never seen the pages with negative (and complex) roots? Perhaps Warner did not recognize them? One wonders whether such issues may not have been discussed between Warner and Harriot. And what were Harriot's own views? The disparity between manuscript pages in which non-positive roots were given and those in which they were ruled out suggests the possibility that Harriot's views changed in the course of his life. It is unlikely that different papers were intended for different audiences since he goes to the trouble to "prove" by substitution in his papers on the Generation of Canonical Equations that the positive roots are the only possible ones. Moreover, factors corresponding to negative roots are meticulously avoided. Especially noteworthy is the page showing a biquadratic equation solved completely, in which all four roots are given (two real and two complex) in P. Rigaud, *Miscellaneous Works and Correspondence of the Rev. James Bradley, D.D., F.R.S., 2 vols. (Oxford, 1832–3). Supplement—with an account of Harriot's astronomical papers, Plate V.*

Warner must surely be given the benefit of the doubt as regards his motives and must be assumed as having acted, however mistakenly (and perhaps, incompetently), in good faith. The only other alternative is to suppose him to have been guilty of perverse falsification. And we must bear in mind that the Torporley copy itself followed the manuscript pages in precluding negative roots. Hence, it may very well have been Harriot's own intention to ignore negative roots, such intention springing from his belief at the time or for pedagogical reasons. (Or, perhaps, he just did not want to stir up controversy?)

The Definitions have been touched upon above (see p. 9, above) and are dealt with in detail below. They are given, as the author indicates, to make clear the precise meanings of terms in common use. We might speculate, however, that a further reason for their inclusion is to provide Euclidean "credentials" for the work as a whole, a practice which was not at all uncommon then and for a considerable time to come.

Section 1 gives lists of examples of the four rules and some rules for the manipulation of equations. In Section 2, binomial factors are multiplied to produce (with a little manipulation) Primary Canonical Equations, and this leads straight into Section 3 in which one or two terms are removed from such equations in order to reduce them to what are called Secondary Canonical Equations, which are in a form more suitable for solving. The removal of one term is achieved simply by equating the appropriate coefficient to zero. However, the removal of two terms creates difficulties for Warner (see below, pp. 46, *Praxis*). The results obtained in Sections 2 and 3 are then used in Section 4 for the designation of (positive) roots as a preparation for Section 5. In this section, what are called common equations, are compared with the Canonicals and, on the basis of the results of Section 4, conclusions are drawn on the numbers of their roots.

The final Section of the theoretical part of the book removes the next-to-highest term from polynomial equations (cubic and biquadratic) following the method initiated by Cardano and followed by Viète and, in the cases of Problems 12 and 13, actually solves the equations. However, Section 6 is exceedingly repetitious, using the same method time after time and we can only suppose that the intention was to provide practice for the reader or to present all possible cases, however repetitious.

Up to Section 5 there seems to be a rationale for the ordering of the sections. It is only with Section 6 that this seems not to be the case. Why should such a section be inserted after Section 5 and before the Numerical Exegesis when it has no immediate connection with either? It can be shown that Warner rearranged the material he was working on in a way which contradicted Harriot's intentions (see below Appendix, p. 277) resulting in a structure which was not wholly coherent. Warner's rearrangement explains the apparent inappropriateness of the structure of the *Praxis*.

The "practical" part comes next, the numerical solution of equations by the method of successive approximation. The method is distinguished from all previous presentations by the use of polynomials, written out in lists in a beautifully clear symbolism designed for computation by the reader, which are called by Warner, Rules for Guidance. As the *Preface* and the final words of Section 6 had explained, this section is the principal object of the whole work.

The particular achievements ("discoveries" as the *Preface* calls them) all refer to and are dependent on the new notation. Multiplication is denoted by $\underline{\phantom{xx}}|$ and when the binomial factors are multiplied to generate equations, all is very clear on the basis of the notation:

$$(a - b)\,(a - c) = a^2 - (b + c)\,a + bc \text{ is written } \left.\underline{\begin{array}{c} a - b \\ a - c \end{array}}\right| \,{=}{=}{=}\, \begin{array}{r} aa - ba \\ - ca + bc \end{array}$$

Here, the work is set out as in simple arithmetic. Christopher Clavius (1537–1612) had done something similar, but the Cossic notation in which he expressed everything did not lend itself to being directly related to simple arithmetic. Harriot has been credited with being the first to equate all the terms of an equation to zero. He may have been able to do this because there is less likelihood of seeing the side "opposite" to the zero as "something" equated to "nothing" when the entire equation is seen abstractly and symbolically. However, Harriot never leaves the equation in this form and a final line is always added equating all to the "homogene" even if this is negative. (Viète called the "constant" the "homogenea comparationis". See *Isagoge*, Chapter V, *On the Rules of Zetetics*.)

Section 5 calls for special mention for it is in Section 5 that a general method is postulated as providing conditions for determining the roots of literal cubic (and one biquadratic) equation(s) based upon the results of Section 4. In several cases, a common equation is compared with a Canonical equation on the basis of their having comparable inequality relations and the number of roots of the former are then determined in terms of those in the Canonical obtained in Section 4.

The idea is a good one but the reasoning is flawed. First, the inequality condition for the identity of the roots is simply asserted without justification. Second, the conditions given are necessary but not sufficient.

Cecily Tanner ("*Thomas Harriot as mathematician: a legacy of hearsay. Part 2*," *Physis* 9, 1967a, p. 283) quotes Lagrange as having written in 1777, "Harriot seems to me to have truly been the first to show directly and analytically that third degree equations without a second term cannot have [all] their roots real unless the cube of one third of the coefficient of the third term, taken with opposite sign, is greater than the square of half of the last term". [Harriot me parait être proprement le premier qui ait démontre d'une manière directe et analytique que les equations du troisième degré sans second term ne sauraient avoir leurs raciness réelles, à moins que le cube du tiers du coefficient du troisième terme, pris avec une signe contraire, ne soit plus grande que le carré de la moitié du dernier terme.] With these words, Lagrange claims just enough for the *Praxis* and implies by his use of the word "unless" that only the necessary condition is given.

Finally, Edmund Halley, who investigated the matter geometrically ("*On the Numbers and Limits of the Roots of Cubic and Quadratic Equations*" (Phil. Trans. Roy. Soc. (translated), 1686–7, 16, 395–407, esp. 398) concluded from a consideration of the equation

$$z^3 - bz^2 + pz - q = 0,$$

that "Also Prop. 5, Sect 5 of our countryman Harriot's Art Analytica and Prob. 18 of Viète's Numer. Potest. Resol. is hardly founded."

## Assessment of the Significance of the Praxis

How must the *Praxis* be judged overall in respect of its content and structure? First, its style is fundamentally Euclidean insofar as it begins with Definitions as if setting out an axiomatic system and this, together with the verbosity so unlike Harriot's own work, alters the very shape from a terse algebraic presentation to a sequence of Propositions presented as if they were theorems. Perhaps Warner did this in order to render the work more acceptable and perhaps the extra words (especially in the Numerical Exegesis) might be thought to make the work more accessible. Also, most mathematical works to hand (particularly Viète) were wordy and in Latin prose but, in the light of the vernacular textbooks of Recorde and in the context of the practical proclivities of Harriot himself, it is possible to speculate that Warner might well have aimed at an aristocratic rather than a plebeian readership (or, perhaps, an international one).

Putting such speculations aside, however, the particular achievements of the *Praxis* were many and clear; the exposure of the structure of polynomial equations through their generation by multiplication of binomial factors; equation of all terms to zero; the display of the coefficients in terms of roots for all to see; the treatment of inequality as operational and the establishment in the Numerical Exegesis of the algorithmic method by means of a transparent notation. For the first time, the focus is on the study of the (quantitative) relational structure of equations.

Harriot did not state what has come to be known as Descartes' Rule of Signs, that is, an equation can have as many true [positive] roots as it contains changes of sign from + to – or from – to +; and as many false [negative] roots as the number of times two + or two – signs are found in succession, nor did he state the general relationship between roots and coefficients and did not assert the equality between the degree of an equation and the number of roots. Although many examples in the manuscripts are solved for the correct number of roots, there are a very large number in which this is not the case, particularly biquadratics with two complex (conjugate) roots. (See MS Add 6783, f. 49.) At various times these omissions have been treated as shortcomings but such value-judgments are inappropriate historically. In the history of mathematics we risk absurdities by not judging work by the standards of its own time.

What is indisputable is that Harriot was first in the field to use a truly symbolic notation, and the *Praxis* is the first published algebraic work to be purely symbolic, even though his retention of homogeneity rendered it unsuitable for modern needs. Symbolism of any kind arises from the need to embody in a visualizable form that which is essentially unvisualizable. We recognize today that a mathematical sign-system should embody only the quantitative aspect of what is being represented symbolically and do this totally. The system should be fully cipherized, embracing unknowns, knowns, and operational signs. Such a system carries with it the quantitative logic of the argument or computation involved.

The lack of exponential notation in Harriot's algebra undoubtedly renders his notation incomplete. Nevertheless, the resulting inconvenience is only minor and the need to count the number of letters, though tedious, is itself a quantitative procedure and may have been a godsend in Harriot's day as it did away with the possibility of ambiguity in Cossic notation when multiplying powers.

In Harriot's work we can see that a new zone of mathematical existence has come into being, self-dependent and detached from the substantive base to which it is, nevertheless, connected. Perhaps, the most powerful characteristic of this new zone is its potential for unlimited representation, which can be known only with hindsight. Harriot's notation is unique, unambiguous, number-based and non-linguistic. Harriot has been rightly deemed to be a Renaissance figure with his multiplicity of interests but he fulfills himself primarily by taking algebra out of the Renaissance and into the modern world. And this is not only in manuscript but in print in the *Praxis* in 1631, six years before Descartes' *Geometry*.

It is necessary to distinguish between any mathematical development in its own right, on the one hand, and its role in the historical process on the other. Any assessment of Harriot must credit him unqualifiedly with his achievements independently of comparison with the towering figure of Descartes.

Whatever Descartes acquired from Harriot, no-one could accuse him of plagiarism, however, if only because his "geometrism" contrasts with the numerically based algebra of Harriot. If he absorbed some of Harriot's ideas and transformed them "in his own image," who would blame him? None of us lives in an intellectual vacuum and if Harriot's thought contributed to that of Descartes, it greatly increases Harriot's stature and the measure of his contribution to the process of historical development of mathematics.

Whatever his faults in carrying out the task he was left with, we cannot but feel gratitude to Warner for actually *doing* it. Without the *Praxis*, would Harriot's reputation have continued? Would anyone have bothered to retrieve his papers? We can never be sure. For this alone, we owe a debt to Warner. Harriot's work was indeed well worth preserving and its being passed on to later mathematicians did make a difference, if only to encourage and stimulate mathematical thinking, especially through familiarity with his (relatively) advanced notation.

# The Practice of the Analytic Art

For solving Algebraic equations by a new, convenient and general method:

A Treatise

Transcribed with the utmost accuracy and care from the last papers of Thomas Harriot, the celebrated Philosopher and Mathematician:

and Dedicated to the Most Illustrious Lord
Lord Henry Percy
Earl of Northumberland

Who ordered this work to be newly revised, transcribed, and published for the general use of Mathematicians—a work which was first composed for his own use, under the auspices of his Generosity and Patronage, and therefore a dedication which is most richly deserved.

London

by Robert Barker, Royal Printer:
And Successor to John Bill, in the Year 1631.

# Preface to Analysts

It was the Frenchman François Viète—an eminent man and, due to his outstanding skill in the mathematical sciences, an ornament to his nation—who first, in a remarkable undertaking and by an unprecedented effort, set about the restitution of the analytic art (the subject of this book) which had long lain ignored and neglected since the learned age of Greece. He has left to posterity indisputable evidence of his noble mental endeavour in the various treatises which he wrote, both elegantly and acutely, to advance this subject. But although he laboured hard to restore the analysis of the ancients—this indeed was the task he set himself—it seems that he passed on to us not so much an analysis restored, as one augmented and embellished by his own discoveries, an analysis new and (one might almost say) entirely his own. This is to put it in general terms, and it must be explained in more detail: I shall show what it was that Viète first did in furthering his goal, so that then it will be possible to gain a better idea of what our most learned author Thomas Harriot (who was the next contender in the analytical arena) afterwards achieved.

And so, to begin at the very beginning: all the ancient practitioners, in seeking the solutions of problems in which the reasoning did not exceed the limits of the quadratic order, generally employed Analysis. This is obvious in practice in their various works, and they themselves also clearly state this to be the case. It is quite certain, then, that the mathematical sciences which we have received from them were enriched with very many additions by the aid of this investigative art. For, by an Analytical process, they first brought the Problem to the stage of solution, so that it was tractable and simple; then, retracing their steps in the Analysis, they constructed a proof synthetically; and finally, they dispensed with the Analysis, and attached this constructed proof to the Problem. But they were only able to do this within the limits of the common Elements,[1] or (as they themselves put it) while they were dealing with a plane locus.[2] But when they had attempted Analysis, and happened to meet with formulas of higher orders (especially cubics), a solution followed less successfully than they had hoped; and so that they would not seem deprived of all resource in their art to furnish a solution in some Geometric form, they would take refuge either in solid loci (by which name is to be understood Conic sections) or what they called linear loci, (for example Spirals, Conchoids, Quadratrices and similar forms of this kind)—necessary supplements, as it were, of a defective art. Those supplements, however, are complicated lines, mechanically described by compound motions; it is quite impossible to make any calculation or reasoning beyond immediate consequences from the assumptions in their original construction.[3] The result of this was that, by calling in the assistance of these aids, the desired solution of the problem had to be devised in a merely technical manner, with the help of the hand and the eye. Thus the analytical capacity of the ancient Greeks in the solution of problems languished during the whole period in which the study and profession of the mathematical arts flourished among them.

But when Greece was at last conquered by barbarian arms and reduced to slavery, the whole learning of Greece passed over into the Arab schools. There, throughout the succeeding period, it was cultivated and developed to a remarkable degree by the studies of an ingenious people. Now, although in the other branches of philosophy many useful discoveries (and some rather more obscure ones too) were made by their skillful investigations and have come down to us; and although the Arabic name of algebra itself, their own coinage, is strong evidence that the study and practice of this art flourished among them (as are the very few writings of theirs which exist in that field); nevertheless, it is DIOPHANTUS the Greek analyst, the last-born of the ancient line of Mathematicians, who alone forbids us from regarding ourselves as indebted to the Arabs either for the actual invention of algebra, or for anything subsequently added to the discoveries of the Greeks which serves to perfect or extend analysis.

And so the analysis of the Greeks has remained in its original form—in the same state of imperfection, that is, in which they themselves had left it—right up to our own times, passing unmodified through the hands of the Arabs. Until, that is, the Italians CARDANO and TARTAGLIA, celebrated mathematicians of a previous age and devoted students of algebra, basing their work on a foundation of geometry (and strongly disputing between themselves for the honour of the discovery), tried to advance the art to the demonstrative solution of equations of the cubic degree by solving some special cases—accurately enough, but in a form rather complicated by binomial roots; and only special cases, because their basis of solution is not general and absolute. After them, others returned their invention to the anvil. Of these, STEVIN the Belgian in his general Arithmetic handled this subject best and most carefully of all. First, he proposed a mode of solution of those equations of the cubic type which, by their nature and primary form, are soluble (inasmuch as their solution can be constructed immediately from a basic substitution). Secondly, he reduced and solved those forms of cubic equations which can, by their very condition, be reduced to primary forms. Thirdly, he also reduced and solved in the same way those biquadratics which could be reduced to primary cubics. But by his tacit exclusion of the remainder—equations of both the Cubic and the Biquadratic type to which these conditions do not apply—he condemned them (the great majority of the total number) as insoluble, much to the detriment of the art. And this was the extent of the progress of this Italian invention, limited not so much by human ignorance as by the nature of the matter itself.

At last, however, Viète appeared, that great architect in analytic studies. But although he employed the aid of Supplements, Recognitions and Angular Sections,[4] and tried everything to overcome, with the engines of his intellect as it were, this stubborn irregularity in the art of analysis, he does not appear to have advanced the subject much beyond the limit reached by his predecessors. Until, that is, after trying geometry in vain, he applied himself to the arithmetical approach and arrived at the happy invention of his numerical exegesis. After this discovery, he was able to state with confidence that haughty and universal problem of his: to solve every problem. For this is the art that nature itself has ordained for the solution of all equations of every degree and form by a general, uniform and infallible method. And since the solutions of problems are ultimately achieved by the solution of equations, when Viète perceived the tremendous power of this exegesis in the solution of equations, he deemed that a universal solution of problems was possible by its means, and so chose to emphasize that problem by stating it in such a magnificent form. The invention of this exegesis is foremost in dignity among all the elements of Viète's projected work of restitution; so, too, let it take first place in the order of this account.

There remains the second of his own inventions, which he introduced into Mathematics under the title of Specious (symbolic) Arithmetic. This is less essentially pertinent to the restitution of Analysis than is numerical Exegesis; yet, because it far outstrips the other in natural priority, and certainly in the generality of its application, it ought not to be less valued. The ancients were at a great disadvantage in not having this specious Arithmetic—as anyone will recognise who has experienced its incredible aptness for handling Mathematical matter succinctly and lucidly, in comparison with the wordy tedium of the ancient style. Since it is agreed that Viète has enriched the art by these two developments—specious Arithmetic and numerical Exegesis (and no trace whatever exists of these in the writings of antiquity)—he deserves, as has been said, to be commended for creating a new art, at least for the most part, rather than for restoring an ancient one.

It is that numerical Exegesis which we offer here, taken from the papers of our Thomas Harriot; not as it was shaped by Viète's first deliberations, but as it was subsequently reshaped by those of Harriot—and reshaped to such an extent, that if Viète seems to have created in some sense a new Analysis by his invention of the Exegesis, then Harriot, by his revision of Exegesis, has produced a new Viète, certainly one which ought to be admired for its new, much more convenient and practical character, as anyone will judge who compares the nature of each when it is put into practice.

But in order to complete this revision of Exegesis, he first had to alter the form of Viète's Logistic as well. For Viète proposed, both by precept and example, a logistic which was to be exercised through (verbally) interpreted signs; although this perhaps was useful for understanding the new discipline, it was subsequently found to be inconvenient for normal practical use.

Consequently Harriot used only a literal notation: that is, the letters of the alphabet, either by themselves or in any combination, according to the needs of the calculation or the reasoning. The numerous examples in the present treatise clearly demonstrate that, by this appropriate change, the practical use of specious Arithmetic, which was formerly somewhat irksome and rather ungainly, has been brought to the utmost simplicity and lucidity. Relying on the dexterity of this Arithmetic, Harriot set about reshaping the Exegesis chiefly through two discoveries of his. First, he set down certain equations generated from Binomial roots, which he calls Canonical. When these are applied to common equations, any uncertainty of the roots which remains in these common equations is revealed and dispelled, through the equivalence of these canonical equations—a most ingenious discovery. Secondly, when he came to the actual application of the numerical Exegesis, he derived certain polynomials from the expressions themselves of the equations which were to be solved; these he also calls Canonical. For, in fact, certain Canons or rules exist within the resolution itself, and by their uniform and continuous application, the process of the work of Analysis is conducted from start to finish with such ease and certainty that Harriot—by the discovery of this device alone, more than all his other discoveries in this field—ought truly to be considered to have completely perfected numerical Exegesis, an art which is instrumental in all Mathematical arts and on that account the most useful. These are almost all of our Author's achievements in his labour to reshape Exegesis; here given only in summary form, but they are explained thoroughly and in detail in the following treatise, to the great benefit of Analysts.

# Definitions

## Certain definitions, which (in lieu of an introduction) may assist in understanding the terms in common use in the art, as well as those peculiar to the present treatise 1

### Definition 1 [1]

Specious Logistic: This type of Arithmetic is frequently used and is absolutely necessary in these writings on Analysis; it is a sibling of Arithmetic, through participation in the same genus. For Arithmetic is numerical Logistic. The distinction between them goes no further than that signified by their names: in Arithmetic, the quantities of measurable things are expressed and reckoned by characters or figures peculiar to the art, by numerals, as in measurement generally; in the former, however, the quantities themselves are indicated and in every way handled through written signs—the letters of the alphabet, that is—‘speciously’, as it were (borrowing the term ‘specious’ from commercial usage). Hence it has received the name Specious.

### Definition 2

*Equation* is used in its common sense for any sort of equality of two or more quantities; but as a special term of this art, it is the clearly determined equality of the sought quantity with some given quantity, when a comparison has been made of one with the other. The part which is sought is a simple or affected (conditioned) power but the part given is commonly called the given homogeneous term of the comparison or equation.

### Definition 3 [2]

In propositions of any sort and in drawing up theorems or problems from them scientifically, the best method of proof and an entirely natural way, is that by

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Bold numbers in the margin refer to the pages of the 1631 printed edition of the *Praxis*. The preface was unpaginated.