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A Functional Analytic Approach

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To our families

Preface

This book is the first introductory and self-contained presentation of a method for the analysis of singularly perturbed boundary value problems that we have called the Functional Analytic Approach. The method was proposed some 20 years ago by the second author and, since then, it has developed in many directions and been implemented in linear and nonlinear problems, including problems that arise from Continuum Mechanics and Material Sciences.

Before the publication of this book, the presentation of the theoretical aspect and of the applications of the Functional Analytic Approach was disseminated in a number of different journal articles. This book is the first comprehensive introduction to the topic, and it covers both the theoretical material that stands at the basis of the Functional Analytic Approach and its applications to a series of problems that ranges from simple illustrative examples to more involved research results.

The Functional Analytic Approach makes constant use of Potential Theory, of the integral representation method for the solutions of boundary value problems, of the Theory of Analytic Functions both in finite and infinite dimension, and of Nonlinear Functional Analysis. All theoretical results used in the Functional Analytic Approach are collected in the first seven chapters of the book and in an appendix at the end of the book. Despite the fact that these initial chapters and the appendix are mostly self-contained, we did not include proofs that can be found in widely available textbooks and for which we cannot claim any paternity. Some classical results have, however, been revisited in a way that fits our purposes, and the corresponding proofs have been presented in full detail. We begin describing the Functional Analytic Approach in Chapter 8, where we show its application to a Dirichlet problem for the Laplace equation in a domain with a hole that shrinks to a point. In the chapters that follow Chapter 8, we study different kinds of boundary conditions, including nonlinear boundary conditions, and different geometric settings, including unbounded domains with periodic sets of holes or inclusions.

To keep the length of the book within reasonable limits, we excluded some of the directions along which the Functional Analytic Approach has developed. In particular, we do not treat differential equations other than the Laplace and Poisson equations and we stick to domains with shrinking holes. For the analysis involving different equations and, in particular, elliptic systems, and for the study of different domain perturbations, we refer the reader to the works of the authors and their collaborators in the bibliography.

Throughout the book we have tried to keep the presentation at a level which we think adequate for the graduate student. As for prerequisites, we expect the reader to be familiar with basic concepts of Lebesgue integration for functions of several variables and of L^p spaces, with some elementary Functional Analysis, and with Differential Calculus in normed spaces. We have tried to keep the use of Distribution Theory to a minimum. In addition, the book provides references to all accessory results.

The book is designed to serve various purposes.

• The large introductory part can be used as a reference textbook for graduate courses on classical Potential Theory and its applications to boundary value problems for the Laplace and Poisson equations. Students and instructors can find, here, material on Lipschitz and Schauder functions and sets (Chapter 2), Green formulas and layer potentials (Chapter 4), Potential Theory for the Laplace equation in multi-connected domains (Chapter 6), and volume potentials and their applications to the Poisson equation (Chapter 7). The periodic counterpart of these results of Potential Theory is presented later in the book, in Chapter 12.

• The initial chapters also contain results that are rarely presented in the literature and may also, therefore, attract the interest of more expert readers. For example, in Section 4.3 we present a theorem on the Hölder continuity of singular integrals of convolution type. This result was originally proven by Carlo Miranda [212], but the corresponding paper was published in Italian and, to the best of our knowledge, has never appeared in English and in the “modern” form included in the book.

• In Chapter 8 we start introducing the Functional Analytic Approach. A reader looking for a quick introduction to the method can find simple illustrative examples specifically designed for this purpose in Section 8.3, where we study the Dirichlet problem in dimension three or bigger; Section 9.2, where we consider a mixed problem for the Laplace equation; Section 9.3, about a mixed problem for the Poisson equation; Section 10.3, on a problem with two “moderately close” holes; and Section 11.2, for the analysis of a nonlinear problem.

• More expert readers will find a comprehensive presentation of the Functional Analytic Approach, which allows a comparison between our approach and the more classical expansion methods of Asymptotic Analysis and offers insights on the specific features of the approach and its applications to linear boundary value problems (Chapters 8, 9, and 10), nonlinear boundary value problems (Chapter 11), and problems in unbounded periodic domains (Chapter 13).

• Finally, applications of the method to problems that arise in the study of composite or porous materials, and more specifically in the study of the average properties of such materials, can be found in Section 13.5, on the longitudinal flow along a periodic array of thin cylinders, and in Sections 13.6 and 13.7, on the effective conductivity of a composite with a thermal resistance at the materials interface.

We would like to thank the students and colleagues that helped us in writing this book.

In particular, we acknowledge the help received from Prof. Alberto Cialdea, Prof. David Natroshvili, Prof. Sergei V. Rogosin, and Prof. Promarz M. Tamrazov on technical issues related to the use of integral equations and Potential Theory for the solution of boundary value problems. We are indebted to Prof. Joan Verdera for several recent references on issues related to the theory of singular integrals in Hölder spaces. We also mention the help received from Prof. Mark L. Agranovsky on technical issues related to the theory of Analytic Functions, and the help of Prof. Gérard Bourdaud, Prof. Victor I. Burenkov, and Prof. Winfried Sickel on issues related to Function Space Theory.

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Chapter 1
Introduction

This book is devoted to the analysis of the basic boundary value problems for the Laplace equation in singularly perturbed domains. More precisely, we will focus on domains with small holes and develop an alternative approach to those of Asymptotic Analysis.

While one could develop most of the book's material for Lipschitz domains and a large class of elliptic operators, we have decided to write a primer of sorts, one easily accessible to beginners. Therefore, we have tried to explain our point of view under the most elementary conditions possible: we treat domains of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$ (the classical Lyapunov sets) and the model of elliptic operators, i.e., the Laplace operator.

In a typical problem found in this book, we have a bounded open subset Ω^o of class $C^{1,\alpha}$ of $\mathbb{R}^n$ and we assume that Ω^o contains the origin. Then we consider a certain boundary value problem in Ω^o, which we assume has a unique solution u^o. Here the superscript 'o' stands for 'outer.' (Fittingly, the superscript 'i' will stand for 'inner'.)

Next we make in Ω^o a hole of size $\epsilon > 0$ around the origin. For example, we can remove from Ω^o a ball centered at the origin and with radius ϵ. We denote by $\Omega(\epsilon)$ the perforated domain that we obtain and we consider in $\Omega(\epsilon)$ a boundary value problem which is, in a sense, of the same nature of the problem on Ω^o . For example, we can have a Dirichlet problem for the same equation both in $\Omega(\epsilon)$ and in Ω^o. We assume that the problem in $\Omega(\epsilon)$ also has a unique solution u_ϵ.

At this point, we can say that the problem in Ω^o is 'unperturbed', and that the problem in $\Omega(\epsilon)$ has been perturbed by the presence of the hole.

It is then natural to ask how the solution u_ϵ, or its energy integral, or other functionals associated with u_ϵ behave when ϵ approaches 0. The difficulty is that the limiting set $\Omega^o \setminus \{0\}$ has a nature which is different from that of $\Omega(\epsilon)$ and, in particular, $\Omega^o \setminus \{0\}$ is no longer a set of class $C^{1,\alpha}$.

As can be seen in a large body of mathematical literature, problems of this type are not only interesting in themselves, but also have importance in Continuum Mechanics and in applied sciences and engineering. For example, they find several applications in shape and topological optimization problems and in the inverse prob-

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lems related to nondestructive testing. A description of such applications can be found in the monographs of Novotny and Sokołowski [240] and of Ammari and Kang [8], which contain many further references on the topic.

In an effort to approximate u_ϵ or related functionals, a commonly accepted way to tackle the problem is to figure out an asymptotic expansion of u_ϵ for $\epsilon > 0$ small. So for example, we fix a point p of the unperturbed set, with that point being away from the hole, and we write an equality of the type

$$u_\epsilon(p) = a(\gamma_1(\epsilon), \ldots, \gamma_s(\epsilon)) + R(\epsilon) \qquad \text{as } \epsilon \text{ tends to } 0^+ , \tag{1.1}$$

where s is a natural number, $a(\cdot, \ldots, \cdot)$ is an 'approximating' function of s real variables defined in a neighborhood of the origin in $\mathbb{R}^s$, $\gamma_1, \ldots, \gamma_s$ are s explicitly known functions which tend to zero as ϵ tends to 0^+, such as

$$\epsilon\,, \epsilon \log \epsilon\,, \log^{-1} \epsilon\,, \text{ and so on,}$$

and $R(\epsilon)$ is a reminder term. Here p is kept fixed and thus we did not include it as a variable in the right-hand side of (1.1).

Now the task is to show that the expansion we have written is actually of some use. For example, to show that (1.1) gives an approximation of the unknown $u_\epsilon(p)$ by means of $a(\gamma_1(\epsilon), \ldots, \gamma_s(\epsilon))$, we have to prove that the reminder term $R(\epsilon)$ tends to zero more rapidly than certain functions of $\gamma_1(\epsilon)$,..., $\gamma_s(\epsilon)$ as ϵ tends to zero. The faster $R(\epsilon)$ tends to zero, the better is our approximation.

As is well known, guessing the explicit form of the functions γ_1,..., γ_s and of the function a is a part of the problem that may be utterly nontrivial, especially if the problem we are looking at is nonlinear. In some cases, one can expect that a is a polynomial and that (1.1) takes the form

$$u_\epsilon(p) = \sum_{|\beta| \le r} a_\beta \gamma_1(\epsilon)^{\beta_1} \ldots \gamma_s(\epsilon)^{\beta_s} + R_r(\epsilon) \qquad \text{as } \epsilon \text{ tends to } 0^+. \tag{1.2}$$

If that is the case, then the coefficients a_β can usually be computed with means at our disposal, after which we wish to show that the reminder $R_r(\epsilon)$ converges to zero faster than order r with respect to $\gamma_1, \ldots, \gamma_s$.

There has been an enormous effort in the mathematical literature to accomplish the plan described above, not only for problems with a shrinking hole, but also for singular perturbations of Ω^o generated by the formation of peaks, wedges, slits, thin bridges, and so on.

The most common approach adopted in the literature is that of the expansion methods of the so-called Asymptotic Analysis. The large number of authors contributing to the development of such methods prevents us from attempting any complete body of references. Nevertheless, we want to mention the early results in the monographs of Cherepanov [48] and [49] on the formation of cracks and in the books of Nayfeh [234], Van Dyke [270], and Cole [57], which present an extensive review of the expansion methods known at the time. For the rigorous description of the method of matching outer and inner asymptotic expansions we refer to the book of Il'in [127], and for the Compound Expansion Method (also known as Multi-Scale Expansion Method) we mention the two volumes of Mazya et al. [203, 204] where, among other results, the authors introduce a systematic approach for analyzing general Douglis and Nirenberg elliptic boundary value problems in domains with perforations and corners.

Different domain perturbations can be found in Kozlov et al. [146], where the authors consider boundary value problems in domains depending on a small parameter ϵ in such a way that the limit regions as ϵ tends to zero consist of subsets of different space dimensions. For the analysis of problems in domains with two holes that collide into one another while shrinking in size we mention the works of Bonnaillie-Noël et al. [32] and Bonnaillie-Noël et al. [31] and for the analysis of problems in a domain containing 'clouds' of small holes we refer to the book of Maz'ya et al. [200], where the authors exploit Meso-Scale Asymptotic Approximations to compute the asymptotic expansion of Green kernels. We also mention Dauge et al. [85] where the authors consider self-similar perturbations of a corner domain and make a comparison between the Multi-Scale and the Matched Asymptotic Expansion Methods.

Further references on expansion methods and their applications can be found in the books and papers mentioned above. Expansion methods are not, however, the only means available for dealing with domain perturbations. For example, in the two dimensional case one can resort to complex analytic techniques, as in the works of Mityushev et al. [221] and [224] and Rogosin and Vaitekhovich [251].

The Functional Analytic Approach of this book can be considered as a complement to expansion methods, though we take, in a sense, a different standpoint. We no longer look for asymptotic expansions such as (1.1) but, in the example above, we look for representation formulas of the form

$$u_\epsilon(p) = \mathcal{A}(\gamma_1(\epsilon), \ldots, \gamma_s(\epsilon)) \qquad \text{for } \epsilon > 0 \text{ small enough}, \tag{1.3}$$

where $\mathcal{A}(\cdot, \ldots, \cdot)$ is a real analytic function of s real variables defined in a neighborhood of the origin in $\mathbb{R}^s$ and $\gamma_1, \ldots, \gamma_s$ are explicitly known functions which tend to zero as ϵ tends to 0^+ (just as in (1.1)). In particular, formula (1.3) implies that

$$u_\epsilon(p) = \sum_{\beta \in \mathbb{N}^s} \frac{D^\beta \mathcal{A}(0, \ldots, 0)}{\beta!} \gamma_1(\epsilon)^{\beta_1} \ldots \gamma_s(\epsilon)^{\beta_s}$$

for $\epsilon > 0$ sufficiently small, and that accordingly $u_\epsilon(p)$ has an asymptotic expansion in terms of the powers of $\gamma_1(\epsilon), \ldots, \gamma_s(\epsilon)$. Namely, for each natural r, we have

$$u_\epsilon(p) = \sum_{|\beta| \le r} c_\beta \gamma_1(\epsilon)^{\beta_1} \ldots \gamma_s(\epsilon)^{\beta_s} + R_r(\epsilon) \tag{1.4}$$

for $\epsilon > 0$ sufficiently small, where $R_r(\epsilon)$ converges to zero faster than order r with respect to $\gamma_1, \ldots, \gamma_s$, and where the coefficients c_β are delivered by the equality $c_\beta = D^\beta \mathcal{A}(0, \ldots, 0)/\beta!$.

We emphasize that the main point of this book is demonstrating the existence of the function $\mathcal{A}$ for a suitable choice of $\gamma_1, \ldots, \gamma_s$. The matter of computing $\mathcal{A}$ or the coefficients c_β is secondary from our point of view, although we regard such a problem as being of great importance in various applications, and we do also consider it in some cases.

We also note that a formula such as (1.3) conveys the message that if we look for an asymptotic expansion of the form (1.2) and if we wish to make the approximation actually converge to $u_\epsilon(p)$ when r tends to $+\infty$ and ϵ is fixed, then an effective choice of the functions $\gamma_1, \ldots, \gamma_s$ is that given by formula (1.3).

Especially in the case of nonlinear problems, such information may be of interest for people who look for asymptotic expansions by means of Asymptotic Analysis techniques.

1.1 An Example

To better illustrate our point of view, we now introduce a concrete example where our 'unperturbed' problem is a Dirichlet boundary value problem on Ω^o with a boundary datum $g^o \in C^{1,\alpha}(\partial\Omega^o)$ (cf. [163], [166]). Namely,

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^o, \\ u = g^o \quad \text{on } \partial\Omega^o \,. \end{cases} \tag{1.5}$$

As is well known, problem (1.5) has one and only one classical solution $u^o \in C^{1,\alpha}(\overline{\Omega^o})$.

In addition to the assumptions introduced above, we assume that the domain Ω^o has a connected exterior. To put it simply, Ω^o has no holes. Then we perturb problem (1.5) by introducing a hole in Ω^o. To do so, we fix another domain Ω^i which satisfies the same assumptions as Ω^o. The set Ω^i will serve as a reference domain for the hole, the letter 'i' standing for 'inner.' Indeed, for $\epsilon_0 > 0$ small enough we have

$$\epsilon\overline{\Omega^i} \subseteq \Omega^o \qquad \forall \epsilon \in [-\epsilon_0, \epsilon_0]$$

(we recall that Ω^i and Ω^o contain the origin). Accordingly, for $\epsilon \in [-\epsilon_0, \epsilon_0]$ we can define the perforated domain

$$\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i} \,.$$

We observe that the boundary $\partial\Omega(\epsilon)$ consists of two connected components,

$$\epsilon\partial\Omega^i \quad \text{and} \quad \partial\Omega^o \,.$$

Also, if ϵ shrinks to 0, then $\Omega(\epsilon)$ degenerates to the punctured domain

$$\Omega(0) = \Omega^o \setminus \{0\}.$$

Our next step is to introduce a Dirichlet problem in $\Omega(\epsilon)$. To assign the Dirichlet datum on the component $\epsilon\partial\Omega^i$ of the boundary, we fix a function $g^i \in C^{1,\alpha}(\partial\Omega^i)$ and we rescale it. Then, for each $\epsilon \in]0, \epsilon_0[$ the 'perturbed' problem is given by

$$\begin{cases} \Delta u = 0 & \text{in } \Omega(\epsilon)\,, \\ u(x) = g^i(x/\epsilon) & \forall x \in \ \epsilon\partial\Omega^i\,, \\ u = g^o & \text{on } \partial\Omega^o\,. \end{cases} \tag{1.6}$$

Like problem (1.5), problem (1.6) also has a unique classical solution in $C^{1,\alpha}(\overline{\Omega(\epsilon)})$. We denote it by u_ϵ and we wish to study its dependence upon ϵ.

To do so, we fix a point p in $\Omega^o \setminus \{0\}$. Then we observe that, possibly shrinking ϵ_0, we can assume that

$$p \in \Omega(\epsilon) \qquad \forall \epsilon \in]0, \epsilon_0[\,.$$

Accordingly, $u_\epsilon(p)$ is defined for all $\epsilon \in]0, \epsilon_0[$ and we can ask the question:

What can we say about $u_\epsilon(p)$ as $\epsilon > 0$ approaches 0?

The answer given in the book accords with our introductory explanation and turns out to be dependent on the dimension n. In Chapter 8, we will prove that for $n \geq 3$ the map which takes ϵ to $u_\epsilon(p)$ has a real analytic continuation in a neighborhood of zero. Namely, there exist $\epsilon_p > 0$ and a real analytic function $\mathcal{U}_p$ from $]-\epsilon_p, \epsilon_p[$ to $\mathbb{R}$ such that

$$u_\epsilon(p) = \mathcal{U}_p(\epsilon) \qquad \forall \epsilon \in]0, \epsilon_p[. \tag{1.7}$$

Then, with reference to formula (1.3), we have the case where $s = 1$ and $\gamma_1(\epsilon) = \epsilon$. If instead the dimension n is two, then we will show that there are three real analytic functions $\mathcal{U}_p$, $\mathcal{W}_p$, and $\mathcal{R}$ from $]-\epsilon_p, \epsilon_p[$ to $\mathbb{R}$ such that

$$u_\epsilon(p) = \mathcal{U}_p(\epsilon) + \frac{\mathcal{W}_p(\epsilon)}{\mathcal{R}(\epsilon) + \frac{\log\epsilon}{2\pi}} \qquad \forall \epsilon \in]0, \epsilon_p[\,. \tag{1.8}$$

The presence of the logarithmic term implies that the right-hand side cannot be continued analytically in a neighborhood of zero, unless we consider some special cases where $\mathcal{W}_p(\epsilon) = 0$. However, if we divide both the numerator and the denominator of the quotient by $\log\epsilon$, we obtain

$$u_\epsilon(p) = \mathcal{U}_p(\epsilon) + \frac{\mathcal{W}_p(\epsilon)\frac{1}{\log\epsilon}}{\mathcal{R}(\epsilon)\frac{1}{\log\epsilon} + \frac{1}{2\pi}} \qquad \forall \epsilon \in]0, \epsilon_p[\,.$$

Then we observe that for $t_p > 0$ small enough we have

$$|\mathcal{R}(\epsilon)t| < \frac{1}{2\pi} \qquad \forall (\epsilon, t) \in]-\epsilon_p, \epsilon_p[\times]-t_p, t_p[$$

and accordingly

$$\mathcal{R}(\epsilon)t + 1/(2\pi) \neq 0 \qquad \forall(\epsilon, t) \in]-\epsilon_p, \epsilon_p[\times]-t_p, t_p[\,.$$

It follows that the function $\mathcal{A}$ from $]-\epsilon_p, \epsilon_p[\times]-t_p, t_p[$ to $\mathbb{R}$ which takes a pair (ϵ, t) to

$$\mathcal{A}(\epsilon, t) \equiv \mathcal{U}_p(\epsilon) + \frac{\mathcal{W}_p(\epsilon)t}{\mathcal{R}(\epsilon)t + \frac{1}{2\pi}}$$

is real analytic. Also, possibly shrinking ϵ_p, we can assume that

$$\frac{1}{\log \epsilon} \in]-t_p, t_p[\qquad \forall \epsilon \in]0, \epsilon_p[\,.$$

Then we conclude that

$$u_\epsilon(p) = \mathcal{A}\left(\epsilon, \frac{1}{\log \epsilon}\right) \qquad \forall \epsilon \in]0, \epsilon_p[\,.$$

With reference to formula (1.3), we now have the case where $s = 2$, $\gamma_1(\epsilon) = \epsilon$, and $\gamma_2(\epsilon) = 1/\log \epsilon$. As a consequence, the function $u_\epsilon(p)$ can be written as an analytic function of ϵ and $1/\log \epsilon$. So, the message conveyed to us here is that, if we want to write an asymptotic expansion associated with a convergent series, we should try using an expansion of powers of ϵ and $1/\log \epsilon$.

In Chapter 8, we explain how to compute the Taylor coefficients of the analytic functions which appear in the above formulas for $u_\epsilon(p)$ by following the ideas of the paper [82] with Rogosin. In particular, we shall see that rather than using the expansion in ϵ and $1/\log \epsilon$, it is convenient to consider a rearrangement of it. Indeed, it turns out that an expansion in powers of ϵ and

$$\frac{1}{r_0 + \frac{\log \epsilon}{2\pi}}$$

is more efficient. Here r_0 is a real constant that depends only on Ω^o and Ω^i. Then the results that we obtain can be compared with those of Il'in [127] and Maz'ya et al. [203, 204].

1.2 A Selection of Problems

We now indicate some of the problems that have been investigated by means of the Functional Analytic Approach.

1.2.1 Perturbation Problems for the Riemann Map

The first problem to which the Functional Analytic Approach was applied concerns the Riemann map in planar perforated domains. Indeed, when $n = 2$, the set Ω^o and the inner hole $\epsilon\Omega^i$ that we introduced above are Jordan domains. Then the Riemann Mapping Theorem implies that there exist a unique $r_\epsilon \in]0, 1[$ and a unique (suitably normalized) holomorphic diffeomorphism from an annulus with inner radius equal to r_ϵ and outer radius equal to 1 to $\Omega(\epsilon)$ (the so-called Riemann map). By the Functional Analytic Approach one can analyze the behavior of both r_ϵ and of the Riemann map as ϵ tends to zero, even in the case where the Jordan curves which bind the domains Ω^o and Ω^i are considered as variables and we have a problem of nonlocal nature. In particular, one can prove that r_ϵ admits a real analytic continuation for small and negative values of ϵ (see [160], [161]).

1.2.2 Linear Elliptic Boundary Value Problems

The analysis of the Dirichlet problem for the Poisson equation, which develops what we have illustrated above for the Laplace operator, has been presented in [162] and [167]. In these papers we have also analyzed the dependence of the solution upon suitable diffeomorphisms that parametrize the boundaries of Ω^o and Ω^i. Then in [74] we have investigated the meaning of equality (1.7) when ϵ is negative. Indeed, one can observe that the solution u_ϵ is defined only for positive values of ϵ, whereas the map $\mathcal{U}_p$ is defined in an open neighborhood of $\epsilon = 0$. Thus, it makes sense to ask what happens for $\epsilon < 0$. In [79] we have considered an analogous question for the case of dimension two and we have investigated the logarithmic behavior that appears in (1.8). The Functional Analytic Approach has also been applied to linear elliptic systems of differential equations. In particular, it has been used to deal with problems for the Lamé equations of linearized elasticity [66], [67] and for the Stokes system, which describes the steady-state flow of an incompressible viscous fluid [62].

1.2.3 Eigenvalues Problems

By the Functional Analytic Approach one can also study the asymptotic behavior of eigenvalues and eigenfunctions in domains with a small hole. For the analysis of a Neumann eigenvalue problem for the Laplace operator, we refer to [169], and for the analysis of the Steklov eigenvalue problem, we refer to the papers [113] and [114] with Gryshchuk and to paper [171].

1.2.4 Nonlinear Boundary Value Problems

Further applications of the Functional Analytic Approach concern nonlinear boundary value problems in domains with small holes and inclusions. Typically, in the applications of the Functional Analytic Approach to such problems, one first shows the existence of an ϵ dependent family of solutions u_ϵ of the perturbed problem in $\Omega(\epsilon)$ (which may not be the only one). Then one looks for representation formulas such as (1.3), again for u_ϵ or related functionals. This being accomplished, one can investigate (local) uniqueness properties of the family of solutions. This approach has been used to study nonlinear boundary value problems for linear differential operators with nonlinear boundary conditions, and certain nonlinear boundary value problems for quasi-linear equations that can be reduced to linear differential equations with nonlinear boundary conditions. For example, a problem for the Laplace equation with a nonlinear Robin boundary condition has been analyzed in [164] and nonlinear transmission conditions are studied in [168], in the paper [72] with Molinarolo, and in the paper [225] of Molinarolo. Nonlinear traction problems for the equations of the linearized elastostatics have been considered in [66] and nonlinear traction problems in which the boundary data are allowed to depend singularly on the parameter ϵ have been studied in [63], [64], and [67].

1.2.5 Problems in Periodic Domains

Along with problems in bounded perforated domains, we have considered problems in periodic domains with an infinite number of holes or inclusions. As a first step, we have considered problems where the periodicity cell is fixed and each of the holes is shrinking to a point. We mention, for example, [228], which concerns a Dirichlet problem for the Laplace equation, and [179], where we consider a Neumann problem for the Poisson equation. Nonlinear Robin boundary conditions have been studied in [176] and [68] and quasilinear heat transmission problems have been investigated in [177] and [70]. Then we have turned to problems in which the periodicity cell is also shrinking: for the analysis of a linear problem, we refer to [178] and [181], and for a nonlinear problem to [182]. Then we have turned to problems in which the periodicity cell is also shrinking: for the analysis of a linear problem, we refer to [178] and [181], and for a nonlinear problem to [182, 183].

1.2.6 Different Boundary Perturbations

Some papers concern applications of the Functional Analytic Approach to boundary perturbations other than holes or inclusions that shrink to interior points. For example, the case of two shrinking holes colliding into one another has been considered in [78], [79], and [80]. The case of a small hole that approaches the outer boundary of a domain has been studied in the paper [29] with Bonnaillie-Noël and Dambrine, and self-similar perturbations of a plane sector domain have been analyzed in [60] with Costabel and Dauge.

1.2.7 Perturbation Results for Integral Operators

To carry out the analysis of singularly perturbed elliptic boundary value problems using the approach introduced in this book, we need to understand the dependence of certain integral operators upon perturbations both of the density and of the support. In this sense, we mention the paper [184] with Preciso on the Cauchy integral; the papers [185] and [186] with Rossi on the layer potentials associated with the fundamental solution of the Laplace and of the Helmholtz operator; the papers [62], [65], and the paper [89] with Dondi on layer potentials associated with a fundamental solution of general elliptic operators with constant coefficients; the paper [174] on periodic layer potentials associated with a fundamental solution of general elliptic operators with constant coefficients; the paper [69] on volume potentials corresponding to parametric families of fundamental solutions; and the papers [172], [173] with Luzzini on the layer potentials associated with the fundamental solution of the heat equation. In addition, special families of parameter-dependent fundamental solutions are introduced and studied in [61] and in paper [73] with Morais, which deal with the case of general elliptic operators with constant real and quaternion coefficients, respectively, and in [180], where we consider the periodic case and we investigate the dependence upon the periodic structure. We also mention the work on nonlinear integral operators with analytic kernels in [175].

1.3 Structure of the Book

In Chapter 2 we present some preliminary material, mainly on spaces of Hölder continuous functions and Schauder spaces, as well as some basics about regular subsets of $\mathbb{R}^n$. Although this part is mostly self-contained, we sometimes refer to other textbooks for basic results of Calculus and of Real and Functional Analysis.

In Chapter 3 we summarize some basic properties of harmonic functions and for most of the proofs we refer to major monographs, in particular to Evans [95], Folland [102], and Gilbarg and Trudinger [107].

In Chapter 4 we present the Green Identities both in a bounded domain and in the corresponding unbounded exterior domain. Then we introduce the notion of single and double layer potentials and show certain mapping properties of these potentials. Although these properties can be found in the classical monographs of Günter [115] and of Kupradze et al. [151], here we prove the corresponding statements with optimal Hölder exponents. To do so, we exploit a result of Miranda [212] on the Hölder continuity of singular integrals of convolution type. We also provide a complete proof of Miranda's result (that was originally published in Italian).

In Chapter 5 we introduce some basic properties of Fredholm operators and the Fredholm Alternative Theorem both in its classical setting and in the version of the duality pairs developed by Wendland [273], [274].

In Chapter 6, we obtain existence and uniqueness results for the classical boundary value problems for the Laplace equation in Schauder spaces using a Potential Theoretic Method and the Fredholm Alternative Theorem. For recent contributions on integral equation methods for the solution of boundary value problems we refer to the monographs of Constanda [58], Constanda et al. [59], Gewinner and Stephan [116], Hsiao and Wendland [126], Maz'ya and Soloviev [205], McLean [206], Medková [208], Mitrea and Mitrea [215], Kohr and Pop [143], Sauter and Schwab [254].

In Chapter 7, we introduce some basic properties of the volume potentials in Schauder spaces and prove existence results for the classical boundary value problems of the Poisson equation. Then we also consider the case of Roumieu spaces. Although the properties of volume potentials in Schauder spaces can be found in the classical monographs of Günter [115] and of Kupradze et al. [151], here we prove the corresponding statements with optimal Hölder exponents by following a proof of Miranda [212] and by exloiting a known lemma (cf. Majda and Bertozzi [196, Prop. 8.12, pp. 348–350]) for which we provide a proof of Mateu et al. [198].

In Chapter 8, we begin to explain the Functional Analytic method of the present book and we do so in detail for the Dirichlet problem for the Laplace operator in a domain with a single hole which shrinks to a point, as we have outlined above.

Then in Chapter 9 we turn to a mixed boundary value problem for the Laplace and Poisson equations and the Steklov eigenvalue problem.

In Chapter 10 we illustrate in detail the case of two holes which shrink to a common point. We focus on the Dirichlet problem for the Laplace equation.

In Chapter 11 we consider the case of nonlinear boundary conditions in domains with a small hole. We do so for nonlinear Robin and nonlinear transmission conditions for the Laplace equation.

In Chapter 12 we consider boundary value problems for the Laplace equation in periodic domains obtained by removing from $\mathbb{R}^n$ a periodic set of holes. To do so, we develop a periodic version of Potential Theory based on layer potentials which are defined by replacing the fundamental solution of the Laplace equation with a periodic analogue.

In Chapter 13, we consider both linear and nonlinear boundary value problems in the entire space with a periodic set of perforations which shrink to points.

In the appendix, we have included a number of known results that are necessary for our purposes throughout the book.

Chapter 2
Preliminaries

Abstract In this chapter we review some preliminary material from Calculus and Functional Analysis and also recall the basic properties of real analytic functions. We introduce the notion of Roumieu classes, then summarize the classical properties of Hölder continuous functions. Next, we introduce the notion of coordinate cylinders and of sets that are hypographs of Cartesian functions locally around the boundary points, including the case of Lipschitz sets. Finally, we turn to Schauder spaces and the corresponding embedding theorems and to the subsets of $\mathbb{R}^n$ of class $C^{m,\alpha}$. Although this part of the book is mostly self-contained, we sometimes refer to other textbooks for basic results of Calculus and of Real and Functional Analysis.

2.1 Basic Notation

In this section we introduce some symbols that we use all throughout the book. We begin with the symbol $=$ that we use for equalities and equations and the symbol $\equiv$ that we use for definitions (instead of $:=$ that other authors use).

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, the set of natural numbers including zero, the set of integer numbers, the set of real numbers, and the set of complex numbers, respectively. We denote by $\mathbb{K}$ either the field $\mathbb{R}$ or $\mathbb{C}$.

Let X be a set. Then we denote by D_X the diagonal of $X \times X$, i.e., we set

$$D_X \equiv \{(x_1, x_2) \in X \times X : x_1 = x_2\} .$$

If Y is also a set then we denote by Y^X the set of maps from X to Y.

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces on $\mathbb{K}$. Let U and V be subsets of X and Y, respectively. Then $\overline{U}$ denotes the closure of U, ∂U denotes the boundary of U, and $\operatorname{diam}(U)$ denotes the diameter of U, i.e.,

$$\operatorname{diam}(U) \equiv \sup \{\|x - y\|_X : x, y \in U\} .$$

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We denote by either U^+ or $\mathring{U}$ the interior of U and by U^- the exterior of U, i.e., we set

$$U^+ \equiv \mathring{U}\,, \qquad U^- \equiv X \setminus \overline{U}\,. \tag{2.1}$$

In particular, $U^+ = U$ if U is open.

We say that a function f from U to Y is locally constant provided that for each $u_0 \in U$ there exists a neighborhood W_0 of u_0 in U such that f is constant in W_0.

We denote by $C^0(U,V)$ the set of continuous functions from U to V. We denote by $B(U,V)$ the set of bounded functions from U to V. As it is well known, if Y is complete then the space $B(U,Y)$ with the sup-norm

$$\|f\|_{B(U,Y)} \equiv \sup_{x\in U} \|f(x)\|_Y$$

is complete and the subspace $C^0_b(U,Y) \equiv C^0(U,Y) \cap B(U,Y)$ of $B(U,Y)$ is also complete. When no ambiguity can arise, we simply write $B(U)$ instead of $B(U,\mathbb{K})$ and similarly for other function spaces defined on U.

If U is a subset of $\mathbb{R}$ and if f is a function from U to $\mathbb{R}$, we say that f is increasing provided that $f(\rho_1) \leq f(\rho_2)$ whenever ρ_1, $\rho_2 \in U$ and $\rho_1 < \rho_2$. Then we say that f is strictly increasing provided that $f(\rho_1) < f(\rho_2)$ whenever ρ_1, $\rho_2 \in U$ and $\rho_1 < \rho_2$. Similarly, we define the decreasing functions and the strictly decreasing functions.

The inverse function of an invertible function f is denoted $f^{(-1)}$, as opposed to the reciprocal of a real-valued function g, or the inverse of a matrix A, which are denoted g^{-1} and A^{-1}, respectively. The preimage of a set E under a function f is denoted by $f^{\leftarrow}(E)$.

Let A be a matrix. Then A^t denotes the transpose matrix of A and A_{ij} denotes the (i,j)-entry of A. If A is invertible, we set

$$A^{-t} \equiv (A^{-1})^t\,.$$

Let $n \in \mathbb{N} \setminus \{0\}$. We denote by $\mathbb{M}_n(\mathbb{K})$ the set of $n \times n$ matrices with entries in the field $\mathbb{K}$. We denote by $\delta_{l,j}$ the Kronecker symbol. Namely,

$$\delta_{l,j} \equiv \begin{cases} 1 & \text{if } l = j\,, \\ 0 & \text{if } l \neq j\,. \end{cases}$$

We denote by I_n or simply by I the identity matrix $I \equiv (\delta_{lj})_{l,j=1,\dots,n}$. We denote by

$$\mathbb{O}_n(\mathbb{K}) \equiv \left\{ A \in \mathbb{M}_n(\mathbb{K}) : \ A^t A = I = A A^t \right\}$$

the set of the orthogonal matrices of $\mathbb{M}_n(\mathbb{K})$.

If S is a set, I_S denotes the identity map from S to itself. If S is clear from the context, we simply write I instead of I_S. We denote by $\mathrm{supp} f$ the support of a complex valued function f. If $x \in \mathbb{R}$, we set

$$\mathrm{sgn}(x) \equiv \begin{cases} 1 & \text{if } x > 0\,, \\ 0 & \text{if } x = 0\,, \\ -1 & \text{if } x < 0\,. \end{cases}$$

For all $\rho > 0$, $x \in \mathbb{R}^n$, x_j denotes the j-th coordinate of x, $|x|$ denotes the Euclidean modulus of x in $\mathbb{R}^n$, i.e.,

$$|x| \equiv \sqrt{\sum_{j=1}^{n} x_j^2} \qquad \forall x \equiv (x_1, \ldots, x_n) \in \mathbb{R}^n\,.$$

Unless otherwise specified, we always equip $\mathbb{R}^n$ with the Euclidean norm $|\cdot|$. Then we set

$$\mathbb{B}_n(x, \rho) \equiv \{y \in \mathbb{R}^n : |x - y| < \rho\}\,.$$

We often abbreviate the unit ball $\mathbb{B}_n(0, 1)$ as $\mathbb{B}_n$. We denote by $\{e_1, \ldots, e_n\}$ the canonical basis of $\mathbb{R}^n$. We denote by a dot the scalar product of vectors.

If $(Z, \mathcal{M}, \mu)$ is a measured space, we retain standard abbreviations as

$$\mu - \text{a.a.} = \mu - \text{almost all}\,, \qquad \mu - \text{a.e.} = \mu - \text{almost everywhere}\,.$$

We retain the standard notation for the Lebesgue space $L^p_\mu(Z)$ of p-summable (equivalence classes of) measurable functions for $p \in [1, +\infty]$. In particular, we set

$$\|f\|_{L^p_\mu(Z)} \equiv \left(\int_Z |f|^p \, d\mu\right)^{1/p} \qquad \forall f \in L^p_\mu(Z)$$

if $p \in [1, +\infty[$ and

$$\|f\|_{L^p_\mu(Z)} \equiv \mathrm{ess\ sup}_Z |f| \qquad \forall f \in L^\infty_\mu(Z)$$

if $p = +\infty$ and we abbreviate $\|\cdot\|_{L^p_\mu(Z)}$ as $\|\cdot\|_p$ whenever there is no ambiguity on the space $L^p_\mu(Z)$. For basic inequalities such as the Hölder inequality for Lebesgue spaces, we refer to textbooks such as Folland [103, Chap. 6].

Also, if X is a vector subspace of $L^1_\mu(Z)$, we find convenient to set

$$X_0 \equiv \left\{f \in X : \int_Z f \, d\mu = 0\right\}\,. \tag{2.2}$$

If $A \in \mathcal{M}$ and $\mu(A) \in]0, +\infty[$, we set

$$\fint_A \equiv \frac{1}{\mu(A)} \int_A\,.$$

By the known inclusions of the $l^p(\mathbb{N}) \equiv L^p_\mu(\mathbb{N})$ spaces with μ equal to the counting measure, we have the known inequality

$$\left(\sum_{j=1}^{\infty}|a_j|^q\right)^{1/q} \leq \left(\sum_{j=1}^{\infty}|a_j|^p\right)^{1/p} \tag{2.3}$$

for all $p, q \in]0, +\infty[$ such that $p \leq q$ and for all sequences $\{a_j\}_{j\in\mathbb{N}}$ of real numbers such that the right-hand side is finite (cf., e.g., Folland [103, Prop. 6.11]). Instead the convexity of the p-power function for $p \in [1, +\infty[$ implies that

$$\left(\sum_{j=1}^{m}|a_j|\right) \leq m^{1-(1/p)} \left(\sum_{j=1}^{m}|a_j|^p\right)^{1/p} \tag{2.4}$$

for all $m \in \mathbb{N} \setminus \{0\}$ and $(a_1, \ldots, a_m) \in \mathbb{R}^m$, and the concavity of the p power function for $p \in]0, 1]$ implies that

$$\left(\sum_{j=1}^{m}|a_j|\right) \geq m^{1-(1/p)} \left(\sum_{j=1}^{m}|a_j|^p\right)^{1/p} \tag{2.5}$$

for all $m \in \mathbb{N} \setminus \{0\}$ and $(a_1, \ldots, a_m) \in \mathbb{R}^m$.

We denote by $\mathcal{L}_n$ the σ-algebra of Lebesgue measurable sets of $\mathbb{R}^n$ and by m_n the Lebesgue measure in $\mathbb{R}^n$. If μ is the Lebesgue measure in $\mathbb{R}^n$, we normally omit the letter μ in the symbol L^p_μ for the Lebesgue spaces and in abbreviations such as $\mu -$ a.a. and $\mu -$ a.e..

A subset M of $\mathbb{R}^n$ is a differential manifold of dimension $s \in \{1, \ldots, n\}$ and of class C^1 embedded in $\mathbb{R}^n$ if, for every $p \in M$, there exists an open neighborhood W of p in $\mathbb{R}^n$ and a map $\psi \in C^1\left(\overline{\mathbb{B}_s(0,1)}, \mathbb{R}^n\right)$ such that ψ is a homeomorphism of $\mathbb{B}_s$ onto $W \cap M$, $\psi(0) = p$, and the Jacobian matrix $D\psi$ has rank s at all points of $\overline{\mathbb{B}_s(0,1)}$, i.e., a ψ is parametrization for M around p. By homeomorphism we understand a continuous bijection which has a continuous inverse. Instead, for the definition of a C^1 function on the closed set $\overline{\mathbb{B}_s(0,1)}$, we refer to Section 2.3.

We denote by $\mathcal{L}_M$ and m_M (or simply m_s) the σ-algebra of Lebesgue surface measurable sets and the Lebesgue surface measure on a manifold M of dimension s embedded in $\mathbb{R}^n$, respectively. We denote by $d\sigma$ the area element of M (cf., e.g., Naumann and Simader [232]). If μ is the Lebesgue surface measure on M, we normally omit the subscript μ in the notation for the Lebesgue spaces and in abbreviations such as a.a. and a.e..

As is well known, if $\alpha \in]0, +\infty[$ then the function $e^{-t}t^{\alpha-1}$ is integrable in $]0, +\infty[$. Then the Euler Gamma function is defined by

$$\Gamma(\alpha) \equiv \int_0^{+\infty} e^{-t}t^{\alpha-1}\,dt \qquad \forall \alpha \in]0, +\infty[\,.$$

By integrating by parts, we have

$$\Gamma(\alpha+1) = \alpha\Gamma(\alpha) \qquad \forall \alpha > 0\,.$$

By a simple inductive argument, we then have

$$\Gamma(n+1) = n! \qquad \forall n \in \mathbb{N}\,.$$

By elementary Calculus, it is known that

$$\Gamma(1/2) = \sqrt{\pi}\,.$$

We denote by s_n the surface measure of $\partial\mathbb{B}_n(0,1)$ and by ω_n the n-dimensional measure of the ball $\mathbb{B}_n(0,1)$. As is well known,

$$\omega_n = \frac{\pi^{n/2}}{\Gamma((n/2)+1)}\,, \qquad s_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}\,, \qquad s_n = n\omega_n\,,$$

for each $n \in \mathbb{N} \setminus \{0\}$ and

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad \forall x,y \in]0,+\infty[\tag{2.6}$$

(cf., e.g., Lebedev [187, §1.5]). We also mention that if $x \in \mathbb{R}^n$ and $\rho \in]0,+\infty[$, then

$$\int_{\partial\mathbb{B}_n(x,\rho)} f(y)\,d\sigma_y = \int_{\partial\mathbb{B}_n(0,1)} f(x+\rho\xi)\rho^{n-1}\,d\sigma_\xi$$

for each complex-valued integrable function f on $\partial\mathbb{B}_n(x,\rho)$.

2.2 Preliminaries of Linear Functional Analysis

If X and Y a vector spaces on the field $\mathbb{K}$, $L(X,Y)$ denotes the set of linear functions from X to Y.

If X is a vector space on the field $\mathbb{K}$, then a linear topology on X is a topology $\mathcal{T}$ on X such that both the sum from $X \times X$ to X and the product from $\mathbb{K} \times X$ to X are continuous.

We will mainly consider the case in which X is endowed by a norm $\|\cdot\|_X$. As it is well known, a norm on a space X generates a linear topology on X.

A series $\sum_{j=0}^{\infty} x_j$ with terms in a normed space X is said to be normally convergent provided that

$$\sum_{j=0}^{\infty} \|x_j\|_X < +\infty\,.$$

If X is complete, then it is known that a normally convergent series is also convergent in X.

Theorem 2.1. *Let $(X, \|\cdot\|_X)$ be a normed space. Let V be a closed subspace of X. Then the norm on the quotient X/V defined by*

$$\| [x] \|_{X/V} \equiv \inf_{v \in V} \|x+v\|_X \qquad \forall [x] \in X/V$$

generates the quotient topology on X/V, i.e. the strongest topology on X/V such that the canonical projection π of X onto X/V is continuous.

If X is complete, then $(X/V, \|\cdot\|_{X/V})$ is complete. If Y is a normed space and if T is a linear map from X/V to Y, then T is continuous if and only if $T \circ \pi$ is continuous.

For a proof, we refer for example to Schaefer [255, p. 42].

Let X, Y be normed spaces. Then we endow the product space $X \times Y$ with the norm defined by

$$\|(x,y)\|_{X\times Y} \equiv \|x\|_X + \|y\|_Y \qquad \forall (x,y) \in X \times Y\,,$$

while we use the Euclidean norm for $\mathbb{R}^n$. If T is a linear operator from X to Y, then we set

$$\operatorname{Ker} T \equiv \{x \in X : \, T[x] = 0\}$$

for the kernel (or null space) of T and

$$\operatorname{Im} T \equiv \{T[x] : \, x \in X\}$$

for the image (or range) of T. A linear operator T from X to Y is well known to be continuous if and only if it maps bounded subsets of X to bounded subsets of Y. It is customary to say that a linear operator T from X to Y is bounded provided that T maps bounded subsets of X to bounded subsets of Y. With such a terminology, a linear operator T from X to Y is continuous if and only if it is bounded. Then

$$\mathcal{L}(X,Y)$$

denotes the space of linear and continuous operators from X to Y, and we set

$$\|T\|_{\mathcal{L}(X,Y)} \equiv \sup_{x \in B_X(0,1)} \|T[x]\|_Y \qquad \forall T \in \mathcal{L}(X,Y)\,,$$

where

$$B_X(p,r) \equiv \{x \in X : \, \|x-p\|_X < r\}$$

for all $p \in X$ and $r \in]0,+\infty[$. If Y is complete, then $\big(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}(X,Y)}\big)$ is well known to be a Banach space. We often refer to the following classical result. For a proof, we refer for example to Brezis [35, Theorem 2.6 and Corollary 2.7 p. 35].

Theorem 2.2 (of the Open Mapping). *Let X, Y be Banach spaces. If $T \in \mathcal{L}(X,Y)$ is surjective, then T is open, i.e., T maps open subsets of X to open subsets of Y.*