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P. G. Romeo
Mikhail V. Volkov
A. R. Rajan *Editors*

Semigroups, Categories, and Partial Algebras

ICSAA 2019, Kochi, India, December 9–12

Springer

Springer Proceedings in Mathematics & Statistics

Volume 345

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Editors

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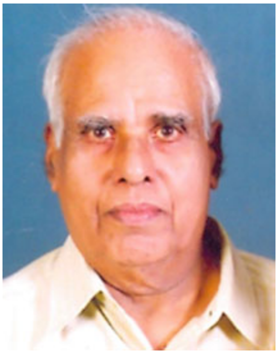
ISSN 2194-1009 ISSN 2194-1017 (electronic)
Springer Proceedings in Mathematics & Statistics
ISBN 978-981-33-4841-7 ISBN 978-981-33-4842-4 (eBook)
https://doi.org/10.1007/978-981-33-4842-4

Mathematics Subject Classification: 60-xx, 06-xx, 18-xx, 13-xx, 22-xx

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore



Komanamana Sankaranarayanan Subramonian Nambooripad (1935–2020) was the founder of semigroup theory research in the state of Kerala, India. With his brilliant findings, he opened a new direction in the structure theory of semigroups and was very visible in the international semigroup community. He was an academician of high repute serving in the University of Kerala from 1979 to 1995. He was an extremely active player in the mathematics scene of Kerala and was inspirational in the organization of the International Conference on Semigroups and Applications (ICSAA-2019). This volume is dedicated to his memory.

Preface

The International Conference on Semigroups and Applications (ICSAA-2019) held at the Department of Mathematics, Cochin University of Science and Technology (CUSAT), Kochi, Kerala, India on December 09–12, 2019 was focused on the recent advances in semigroup theory and its applications. The scientific program of ICSAA-2019 emphasized the recent activities related to the structure theory of semigroup and its applications, semigroups of operators, partial algebras, and category theory.

Leading researchers from 15 different countries working in these areas participated in ICSAA-2019. A total of more than 100 delegates including Ph.D. students from these countries actively participated in discussions and deliberations. The following plenary lectures were given:

- *Scrambling index and synchronized automata*,
  Alexander E. Guterman, Moscow State University, Russia;
- *Inverse semigroups and Leavitt path algebras*,
  John C. Meakin, University of Nebraska-Lincoln, USA;
- *Word problem for K-terms over some pseudovarieties of semigroups with commuting idempotents*,
  José Carlos Costa, University of Minho, Portugal;
- *One-relator groups, monoids and inverse semigroups*,
  Robert D. Gray, University of East Anglia, UK;
- *Intermediate quotients of the Booleanization of an inverse semigroup*,
  Ganna Kudryavtseva, University of Ljubljana, Slovenia;
- *Simplicity of inverse semigroup algebras*,
  Nora Szakacs, University of York, UK;
- *The star-free closure*,
  Marc Zeitoun, University of Bordeaux, France;
- *On the multiplicity-free plethysms* $p_2[s_\lambda]$,
  Luisa Carini, University of Messina, Italy;
- *An introduction to radicals in module theory*,
  Nguyen Van Sanh, Hue University, Vietnam;
- *Random walks on modules of finite commutative rings*,
  Arvind Ayyer, Indian Institute of Science Bangalore, India;

- *Stone pseudovarieties*,
  Jorge Almeida, University of Porto, Portugal;
- *Normal distributions of finite Markov chains*,
  Anne Schilling, University of California Davis, USA;
- *On Morita theory on semirings*,
  Sujit Kumar Sardar, Jadavpur University, India;
- *Max-plus automata and the tropical semiring*,
  Laure Daviaud, City, University of London, UK;
- *Commutators for semigroups*,
  Peter Mayr, University of Colorado Boulder, USA.

The daily program consisted of lectures, paper presentations, and discussions held in an open and encouraging atmosphere. In addition to the above speakers, there were invited talks and paper presentations. We are grateful to all participants for their valuable contributions and for making the ICSAA-2019 a successful event. Moreover, we would like to thank the National Board for Higher Mathematics, DAE, New Delhi, SERB-DST, New Delhi, Mathematics and Statistical Sciences Trust, Thiruvananthapuram, Kerala for providing us with financial support. We are also thankful to the Cochin University of Science and Technology, India, for additional support and practical assistance related to the preparation and organizing of the conference.

The present volume, the proceedings of ICSAA-2019, is an outcome of the conference. The chapters of this volume include papers presented at ICSAA-2019 and papers contributed by the speakers of ICSAA-2019 and their collaborators. We wholeheartedly acknowledge the support received from Profs. Mark V. Lawson, José Carlos Costa, Maria B. Szendrei, and Mahim Ranjan Adhikari for refereeing some of the chapters. We also thank Sneha K. K., Alanka Thomas, and Riya Jose (Ph.D. students of Prof. P. G. Romeo) for their assistance and support in preparing these proceedings. This volume is dedicated to the memory of Prof. K. S. S. Nambooripad who passed away in January 2020 within days after ICSAA-2019. He was a source of strength for the organizers of this conference though due to ill health he could not attend the conference.

Kochi, India — P. G. Romeo
Yekaterinburg, Russia — Mikhail V. Volkov
Thiruvananthapuram, India — A. R. Rajan

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Ordering Orders and Quotient Rings

Alexander Guterman, László Márki, and Pavel Shteyner

Dedicated to the memory of K.S.S. Nambooripad, in highest esteem

Abstract In the present paper, we introduce a general notion of quotient ring which is based on inverses along an element. We show that, on the one hand, this notion encompasses quotient rings constructed using various generalized inverses. On the other hand, such quotient rings can be viewed as Fountain–Gould quotient rings with respect to appropriate subsets. We also investigate the connection between partial order relations on a ring and on its ring of quotients.

Keywords Order relations · Quotient rings

Mathematics Subject Classification (2020) 06A06 · 06F25 · 16H20

L. Márki's research was partially supported by the Hungarian National Research, Development and Innovation Office, NKFIH, grant no. 119934. P. Shteyner is grateful to the BASIS Foundation grant 19-8-2-35-1.

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P. G. Romeo et al. (eds.), *Semigroups, Categories, and Partial Algebras*,
Springer Proceedings in Mathematics & Statistics 345,
https://doi.org/10.1007/978-981-33-4842-4_1

1 Introduction

Let R be an associative, not necessarily commutative ring. The classical notion of a ring of left quotients Q of its subring R is well known. To be a quotient ring, it is necessary for the ring Q to have an identity element. Then all its elements can be written as "left fractions" $a^{-1}b$, where $a, b \in R$ and every element of R which is not a zero divisor in R should be invertible in Q.

Starting from similar investigations in semigroups, Fountain and Gould introduced in [10] a new generalization of classical quotient rings based on the notion of group inverse. These new quotient rings have been described for some special classes of rings in subsequent research. In particular, such quotient rings need not have an identity.

The procedure of assigning inverses to certain elements is called localization. It can be carried out, more generally, by considering other generalized inverses, for example, Moore–Penrose, Drazin, and others. In particular, for rings with involution, rings of quotients with respect to Moore–Penrose inverses were studied in [4, 21]. An important feature of the notion, introduced in [4], is that it is equipped with an additional parameter, namely, one can specify the elements that are required to have inverses. This leads to certain interesting and useful properties of quotient rings and their orders.

Recently, the general concept of an inverse along an element which covers and generalizes the notion of outer generalized inverse was introduced and developed in [15], see also [9]. This notion generalizes all classical outer inverses and unifies many classical notions connected to generalized inverses. In particular, partial order relations on semigroups such as Nambooripad order, sharp order, star order, and others, can be defined in terms of outer inverses, see [11, 12].

In the present paper, we introduce a general notion of quotient rings which is based on inverses along an element. We show that, on the one hand, this notion encompasses quotient rings constructed using various generalized inverses. On the other hand, these quotient rings can be viewed as Fountain–Gould quotient rings with respect to appropriate subsets (in the sense of [4]).

Our paper is organized as follows. Sections 2, 3, and 4 contain general information on Green's relations, generalized inverses and inverses along an element, and partial orders on semigroups, correspondingly—much of this is a recapitulation of known results. Section 5 deals with quotient rings. In Sect. 6, we investigate the connection between partial order relations on a ring and on its ring of quotients.

2 Green's Relations

Let S be a semigroup. As usual, S^1 denotes the monoid generated by S, and $E(S)$ denotes the set of idempotents of S. Firstly, we recall some results on Green's relations that we need in the sequel. For more information see, for instance, [14] or [6].

Definition 2.1 For elements a and b of S, relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ are defined by

1. $a\mathcal{L}b$ if and only if $S^1a = S^1b$.
2. $a\mathcal{R}b$ if and only if $aS^1 = bS^1$.
3. $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$.

That is, a and b are $\mathcal{L}$*-related* ($\mathcal{R}$*-related*) if they generate the same left (right) principal ideal, and $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. As is well known, one can rewrite the $\mathcal{L}$- and the $\mathcal{R}$-relation over a monoid S^1 in terms of equations by substituting Item 1. and Item 2., respectively.

Lemma 2.2 *1. $a\mathcal{L}b$ if and only if there exist $m, n \in S^1$, such that $ma = b$ and $a = nb$.*
2. $a\mathcal{R}b$ if and only if there exist $m, n \in S^1$, such that $am = b$ and $a = bn$.

These are equivalence relations on S, and we denote the $\mathcal{L}$-class ($\mathcal{R}$-class, $\mathcal{H}$-class) of an element $a \in S$ by $\mathcal{L}_a(\mathcal{R}_a, \mathcal{H}_a)$. The $\mathcal{L}$ ($\mathcal{R}$) relation is right (left) compatible, that is, for any $c \in S^1$, $a\mathcal{L}b$ implies $ac\mathcal{L}bc$ ($a\mathcal{R}b$ implies $ca\mathcal{R}cb$).

In parallel with these equivalence relations, we have the preorder relations:

1. $a \leq_{\mathcal{L}} b$ if and only if $S^1a \subseteq S^1b$;
2. $a \leq_{\mathcal{R}} b$ if and only if $aS^1 \subseteq bS^1$;
3. $a \leq_{\mathcal{H}} b$ if and only if $a \leq_{\mathcal{L}} b$ and $a \leq_{\mathcal{R}} b$.

3 Generalized Inverses

We start by recalling several basic notions.

Definition 3.1 Let $a \in S$.

1. We say that a is *(von Neumann) regular* if $a \in aSa$.
2. A particular solution to $axa = a$ is called an *inner inverse* of a and is denoted by a^-.
3. A solution of the equation $xax = x$ is called an *outer inverse* of a and is denoted by $a^=$.
4. An inner inverse of a that is also an outer inverse is called a *reflexive inverse* and is denoted by a^+.

The set of all inner (resp. outer, resp. reflexive) inverses of a is denoted by $a\{1\}$ (resp. $a\{2\}$, resp. $a\{1, 2\}$).

Definition 3.2 A semigroup S is *regular* if all its elements are regular.

The definitions of group, Moore–Penrose and Drazin inverses are standard and can be found in the literature (see, for example, [5, 14]). We provide them here for completeness.

Definition 3.3 Let $a \in S$.

1. The element a is *group invertible* if there is $a^{\#} \in a\{1, 2\}$ that commutes with a.
2. The element a has a *Drazin inverse* a^D if a positive power a^n of a is group invertible and $a^D = (a^{n+1})^{\#} a^n$.
3. If $*$ is an involution in S, then a is *Moore–Penrose invertible* if there is $a^{\dagger} \in a\{1, 2\}$ such that $aa^{\dagger}$ and $a^{\dagger}a$ are symmetric with respect to $*$.

Each of these inverses is unique if it exists.

3.1 Inverses Along an Element

In this section, we recall the definition of inverse along an element, which was introduced in [15], and several useful properties of this inverse.

Lemma 3.4 ([15, Lemma 3]) *Let $a, b, d \in S$. Then the following are equivalent.*

1. *$b \leq_{\mathcal{H}} d$, and $d = dab = bad$,*
2. *$b = bab$ and $b\mathcal{H}d$.*

Definition 3.5 We say that b is *an inverse of a along d* (denoted as $b = a^{-d}$) if b satisfies the equivalent conditions of Lemma 3.4.

Theorem 3.6 ([15, Theorem 6]) *Let $a, d \in S$. If a^{-d} exists, then it is unique.*

Theorem 3.7 ([15, Theorem 7]) *Let $a, d \in S$. Then the following are equivalent:*

1. *a^{-d} exists.*
2. *$ad\mathcal{L}d$ and $\mathcal{H}_{ad}$ is a group.*
3. *$da\mathcal{R}d$ and $\mathcal{H}_{da}$ is a group.*
   In this case, $(ad)^{\#} \in S$ exists and $a^{-d} = d(ad)^{\#} = (da)^{\#}d$.

Denote by $C(X) = \{y \in S \mid xy = yx \text{ for all } x \in X\}$ the centralizer of $X \subseteq S$.

Theorem 3.8 ([11, Lemma 3.31]) *Let $a, d \in S$. If a^{-d} exists, then $a^{-d} \in C(C(\{a, d\}))$.*

Theorem 3.9 ([15, Theorem 11]) *Let $a \in S$. Then*

1. *$a^{\#} = a^{-a}$,*
2. *$a^D = a^{-a^m}$ for some integer m,*
3. *in case S is a $*$-semigroup, $a^{\dagger} = a^{-a^*}$.*

3.2 Properties of the Group Inverse

The following statements provide some commutativity relations for group invertible elements of a semigroup and belong to folklore. We include them here with proofs for completeness.

Lemma 3.10 *Let S be an arbitrary semigroup, $u, a \in S$, and u be group invertible. Then $ua = au$ if and only if $au^{\#} = u^{\#}a$.*

Proof Let $au = ua$. By Theorem 3.9 $u^{\#} = u^{-u}$. Then, by Theorem 3.8, $u^{\#} \in C(C(\{u\})$. Since $au = ua$ we have $a \in C(\{u\})$. It follows that $u^{\#} \in C(\{a\})$, that is, $u^{\#}a = au^{\#}$.

Now let $u^{\#}a = au^{\#}$. Therefore, by the above, $ua = (u^{\#})^{\#}a = a(u^{\#})^{\#} = au$. □

Corollary 3.11 *Let S be an arbitrary semigroup and $u, a \in S$ be group invertible. Then the following statements are equivalent:*

1. $ua = au$.
2. $au^{\#} = u^{\#}a$.
3. $a^{\#}u = ua^{\#}$.
4. $a^{\#}u^{\#} = u^{\#}a^{\#}$.

Proof Items 1., 2. and 3. are equivalent by Lemma 3.10. If we apply this lemma to $a^{\#}$ and u, we obtain that Items 3 and 4 are also equivalent. □

Lemma 3.12 *Let S be an arbitrary semigroup. Suppose that $a, b \in S$ are group invertible and $ab = ba$. Then $(ab)^{\#} = a^{\#}b^{\#}$.*

Proof By Corollary 3.11, a, b, $a^{\#}$ and $b^{\#}$ are mutually commutative. Then $ab(a^{\#}b^{\#}) = (a^{\#}b^{\#})ab$. Also $ab(a^{\#}b^{\#})ab = aa^{\#}abb^{\#}b = ab$. Finally,

$$(a^{\#}b^{\#})ab(a^{\#}b^{\#}) = a^{\#}aa^{\#}b^{\#}bb^{\#} = a^{\#}b^{\#}.$$

Thus $(ab)^{\#} = a^{\#}b^{\#}$. □

4 Order Relations on General Semigroups and Their Properties

Let S be an arbitrary semigroup, $a, b \in S$. Following Drazin [8], Nambooripad [18], and Petrich [20] we introduce several useful partial orders on S.

Definition 4.1
- $a <^{-} b$ if and only if $a^{-}a = a^{-}b$ and $aa^{-} = ba^{-}$ for some $a^{-} \in a\{1\}$ (*minus order*);
- $a\mathcal{N}b$ if and only if $a = axa = axb = bxa$ for some $x \in S$ (*Nambooripad order*);
- $a\mathcal{M}b$ if and only if $a = xb = by$ and $xa = a$ for some $x, y \in S^1$ (*Mitsch order*);

Due to the following result, if the semigroup S is regular then the above partial orders coincide.

Lemma 4.2 ([17, Lemma 1]) *For a regular semigroup S, the following conditions are equivalent:*

1. $a = eb = bf$ *for some* $e, f \in E(S)$*;*
2. $a = aa'b = ba''a$ *for some* $a', a'' \in a\{1, 2\}$*;*
3. $a = aa'b = ba'a$ *for some* $a' \in a\{1, 2\}$*;*
4. $a'a = a'b$ *and* $aa' = ba'$ *for some* $a' \in a\{1, 2\}$*, see also [13];*
5. $a = ab'b = bb'a$*,* $a = ab'a$ *for some* $b' \in b\{1, 2\}$*;*
6. $a = axb = bxa$*,* $a = axa$*,* $b = bxb$ *for some* $x \in S$*;*
7. $a = eb$ *and* $aS \subseteq bS$ *for some idempotent* e *such that* $aS = eS$*, see also [18];*
8. $a = xb = by$*,* $xa = a$ *for some* $x, y \in S$*.*

The minus order and the Nambooripad order coincide on any semigroup.

Lemma 4.3 ([12, Lemma 3]) *The minus order $<^-$ and the Nambooripad order $\mathcal{N}$ are equivalent on any semigroup S.*

The *sharp partial order* on a semigroup is defined in [16] in the following way.

Definition 4.4 Let S be a semigroup, $a, b \in S$, and a be group invertible. We put $a <^\# b$ if and only if $aa^\# = ba^\# = a^\# b$.

The following lemma provides some useful properties of the sharp order. We present a proof of these properties for completeness.

Lemma 4.5 *Let $a, b \in S$, where S is an arbitrary semigroup.*

1. *Let $a <^\# b$. Then $ab = ba$.*
2. *Suppose that b is group invertible. Then $a <^\# b$ if and only if $a^\# <^\# b^\#$.*

Proof 1. By the definition of the sharp order we have $aa^\# = ba^\# = a^\# b$. Then $aa(aa^\#) = aa(a^\# b)$, that is, $aa = ab$. Also $(aa^\#)aa = (ba^\#)aa$, that is, $aa = ba$.

2. By the definition of the sharp order we have $aa^\# = ba^\# = a^\# b$. By Item 1 we have $ab = ba$.

Let us prove that $a^\# a = b^\# a$. Indeed, $b^\# a = b^\#(aa^\#)a = b^\# ba^\# a = b^\# b(aa^\#) = b^\# b(ba^\#) = (b^\# bb)a^\# = ba^\# = aa^\#$, therefore, $b^\# a = a^\# a$. Dually, $ab^\# = aa^\#$.

Note that $(x^\#)^\# = x$ for any group invertible $x \in S$. Then $b^\# a = ab^\# = a^\# a$ implies that $a^\# <^\# b^\#$.

By the above, $a^\# <^\# b^\#$ implies $(a^\#)^\# <^\# (b^\#)^\#$. That is, $a <^\# b$. □

The following example shows that $a <^\# b$ does not imply $b^\# <^\# a^\#$.

Example 4.6 Let $S = T_4$ be the semigroup which consists of all maps from the set $\{1, 2, 3, 4\}$ to itself with the composition operation. As usual, we represent such a map by two rows, where the first row contains the elements and the second row contains their images.

Let $a = \begin{pmatrix} 1&2&3&4 \\ 1&1&1&1 \end{pmatrix}$, $b = \begin{pmatrix} 1&2&3&4 \\ 1&3&4&2 \end{pmatrix}$. Then $a^\# = a$ and $b^\# = b^{-1} = \begin{pmatrix} 1&2&3&4 \\ 1&4&2&3 \end{pmatrix}$. Also $a = a^\# a = a^\# b = ba^\#$. Then $a <^\# b$. But $b^\#(b^\#)^\# = \begin{pmatrix} 1&2&3&4 \\ 1&2&3&4 \end{pmatrix} \neq a^\#(b^\#)^\# = a$, that is, $b^\# \not<^\# a^\#$.

If in addition we assume that our semigroup is commutative, then the generalized inverses and the orders under consideration have several additional properties.

Let us prove them here for completeness.

Lemma 4.7 *Let S be a commutative semigroup.*

1. *Let $a \in S$ be regular. Then a is group invertible and $a^\# = a^- a a^-$ for any $a^- \in a\{1\}$.*
2. *For $a, b \in S$ it holds that $a <^- b$ if and only if $a <^\# b$.*
3. *For $a, b \in S$ it holds that $a\mathcal{M}b$ if and only if $a = xa = xb$ for some $x \in S$.*

Proof 1. Let $a^- \in a\{1\}$. We prove that $a^- a a^- \in a\{1, 2\}$. Indeed, we have $(a^- a a^-)a(a^- a a^-) = a^-(aa^- a)a^- aa^- = a^-(aa^- a)a^- = a^- aa^-$.
Also $(aa^- a)a^- a = aa^- a = a$. Since S is commutative, we obtain, by the definition of the group inverse, that a is group invertible and $a^\# = a^- aa^-$.

2. Since $a^\#$ is an inner inverse of a, we obtain that $a <^\# b$ implies $a <^- b$ in general. Suppose that $a <^- b$, that is, $aa^- = ba^-$ for some $a^- \in \{1\}$. Then by multiplying by aa^- on the right we obtain $(aa^-)(aa^-) = a(a^- aa^-) = b(a^- aa^-)$. By Item 1 a is group invertible and $a^\# = a^- aa^-$. Finally, $aa^\# = ba^\# = a^\# b$.
3. By the definition of the Mitsch order $a\mathcal{M}b$ means that $a = xa = xb = by$ for some $x, y \in S^1$. In particular, $a = xa = xb$.

Now suppose that $a = xa = xb$, therefore $bx = xb$ since S is commutative. Then $a\mathcal{M}b$ by the definition. $\square$

The following example shows that there are different Mitsch-compatible elements even in commutative semigroups.

Example 4.8 Let $S = \mathbb{Z}_+$ with multiplication. Then $0 = 0n = n0 = 00$ for any $n \in S$. That is, $0\mathcal{M}n$.

5 Quotient Rings Along Functions

Let R be an associative ring and $f : R \to R$ be a given function.

Definition 5.1 $a \in R$ is *left f-cancellable* if $\forall x, y \in R \cup \{1\}$, $af(a)x = af(a)y$ implies $f(a)x = f(a)y$.

Right f-cancellable elements are defined dually. *f-cancellable* means both left and right f-cancellable.

The set of all f-cancellable elements of R is denoted by $\mathcal{C}_f(R)$.

For special types of the function f and special classes of rings, f-cancellable elements and corresponding quotient rings (see Definition 5.4) are widely investigated, see, for instance, [1–4, 10, 21]. In the following example, we collect some of the most useful such functions.

Example 5.2 • If $f(a) = a^*$, then f-cancellable means $*$-*cancellable*.
- If $f(a) = a$, then f-cancellable means *square-cancellable* and we use the notation $\mathfrak{C}(R)$ for $\mathcal{C}_f(R)$ in this case.
- If $f(a) = d$ for all $a \in R$ and some $d \in R$, then we call f-cancellable elements d-*cancellable*, and use the notation $\mathcal{C}_d(R)$ in this case.

Lemma 5.3 *Let R be an arbitrary ring and let $a, d \in R$. If a^{-d} exists, then $a \in \mathcal{C}_d(R)$.*

Proof Let us consider $x, y \in R \cup \{1\}$ satisfying $adx = ady$. Then multiplying by a^{-d} on the left we have $a^{-d}adx = a^{-d}ady$, which implies $dx = dy$ since $a^{-d}ad = d$ by Item 1 of Lemma 3.4. Dually, $xda = yda$ implies $xd = dy$, and then $a \in \mathcal{C}_d(R)$. □

Definition 5.4 Let Q be a ring, R be a subring of Q. Let $f : R \to R$ be a function. We say that R is a left *order along* f in Q and Q is a left *quotient ring of R along* f with respect to a subset $\mathcal{C} \subseteq \mathcal{C}_f(R)$ if:

1. every $u \in \mathcal{C}$ is invertible along $f(u)$ in Q,
2. every $q \in Q$ can be written as $q = u^{-f(u)}b$ for some $u \in \mathcal{C}$ and $b \in R$.

Right orders can be defined dually. In the subsequent discussion we shall omit the word "left".

Some well-known quotient rings are examples of the introduced general construction, see [2, 4, 10].

Example 5.5 1. Let $f(a) = a$. Then a *quotient ring along* f is a *Fountain–Gould left quotient ring*.
2. Let $f(a) = a^*$. Then a *quotient ring along* f is a *Moore–Penrose quotient ring*.

Remark 5.6 Note that in [4, 21], a slightly different definition of the Moore–Penrose inverse was used. Namely, there b is called a Moore–Penrose inverse of a if $ab^*a = a$, $ba^*b = b$, $a^*b = b^*a$ and $ba^* = ab^*$. Then $a^\dagger = b^*$, where $(a^\dagger)$ is a Moore–Penrose inverse of a in the sense of Definition 3.1 (see [5, 19]). This is more convenient for the considerations below, since $a^\dagger$ is an outer inverse, and as a consequence $a^\dagger$ is a particular case of an inverse of a along an element, and b is not. Let us also mention that both definitions are common and clearly connected. Also in [4], it is assumed that $\mathcal{C} = \mathcal{C}^*$ in the definition of the Moore–Penrose quotient ring with respect to $\mathcal{C}$. In this case, $\{x^\dagger \,|x \in C\} = \{(x^\dagger)^* \,|x \in C\}$.

We provide here the definition of a Fountain–Gould quotient ring to be investigated below.

Definition 5.7 Let R be a subring of a ring Q. We say that R is a *Fountain–Gould left order in Q* and *Q is a Fountain–Gould left quotient ring of R with respect to a subset $\mathcal{C}$ of* $\mathfrak{C}(R)$ if

1. every $u \in \mathcal{C}$ is group invertible in Q,
2. every $q \in Q$ can be written as $q = u^{\#}b$ for some $u \in \mathcal{C}$ and $b \in R$.

The following three statements allow us to construct the same quotient ring using different functions.

Lemma 5.8 *Let $d_1, d_2 \in R$ be such that $d_1 \mathcal{H} d_2$. Then $\mathcal{C}_{d_1}(R) = \mathcal{C}_{d_2}(R)$.*

Proof Since $d_1 \mathcal{H} d_2$, by Lemma 2.2 we have $d_2 = d_1 u$ for some $u \in R \cup \{1\}$. Consider $a \in \mathcal{C}_{d_1}(R)$ and suppose that, for some $x, y \in S^1$, $ad_2x = ad_2y$. Then $ad_1(ux) = ad_1(uy)$ and as a consequence $d_1ux = d_1uy$, that is, $d_2x = d_2y$.

By dual arguments $xd_2a = yd_2a$ implies $xd_2 = yd_2$. Thus $a \in \mathcal{C}_{d_2}(R)$ and $\mathcal{C}_{d_1}(R) \subseteq \mathcal{C}_{d_2}(R)$. Dually, $\mathcal{C}_{d_2}(R) \subseteq \mathcal{C}_{d_1}(R)$. □

Corollary 5.9 *Let $f_1, f_2 : R \to R$ be such that $f_1(x) \mathcal{H} f_2(x)$ in R for every $x \in R$. Let $\mathcal{C} \subseteq \mathcal{C}_{f_1}(R) \cup \mathcal{C}_{f_2}(R)$. Then R is an order along f_1 in Q with respect to $\mathcal{C}$ if and only if R is an order along f_2 in Q with respect to $\mathcal{C}$.*

Proof By the same arguments as in Lemma 5.8, we have $\mathcal{C}_{f_1}(R) = \mathcal{C}_{f_2}(R)$. Then $f_1(u) \mathcal{H} f_2(u)$ implies $u^{-f_1(u)} = u^{-f_2(u)}$ for every $u \in \mathcal{C}$ by Item 2 of Lemma 3.4 and the uniqueness of the inverse along an element (Theorem 3.6). □

Corollary 5.10 *Let R be a subring of a ring Q, $f_1, f_2 : R \to R$ be constant functions: $f_1(x) = d_1$, $f_2(x) = d_2$ for every $x \in R$ such that $d_1 \mathcal{H} d_2$ in R. Let $\mathcal{C}$ be a subset of $\mathcal{C}_{d_1}(R) \cup \mathcal{C}_{d_2}(R)$. Then R is an order along f_1 in Q with respect to $\mathcal{C}$ if and only if R is an order along f_2 in Q with respect to $\mathcal{C}$.*

If f is a constant function, then the quotient ring along f has the following additional properties:

Lemma 5.11 *Let f be a constant function: $f(x) = d$ for every $x \in R$ for some $d \in R$ and let Q be a quotient ring along f of R with respect to a subset $\mathcal{C}$ of $\mathcal{C}_d(R)$. Then*

1. *$\{\mathcal{C}\}^{-d} \subseteq \mathcal{H}_d \subseteq Q$.*
2. *If $q \in Q$ and $q = a^{-d}x$ for some $a \in \mathcal{C}$ and $x \in R$, then $daq = dx$.*
3. *$Q = \mathcal{H}_d R$.*

Proof 1. Follows from Lemma 3.4.
2. It holds that $d = daa^{-d} = a^{-d}ad$. Thus, if $q = a^{-d}x$, then $daq = daa^{-d}x = dx$.
3. Follows from Definition 5.4 and Item 1. □

A quotient ring along a function is a generalization of the notion of a classical quotient ring.

Definition 5.12 Recall that a ring Q is a *(classical) ring of left quotients of its subring* R, or that R is a *(classical) left order in* Q, if the following three conditions are satisfied:

1. Q has an identity element.
2. Every element of R which is not a zero divisor is invertible in Q.
3. Every $q \in Q$ can be written as $q = a^{-1}b$ where $a, b \in R$ and a^{-1} is the inverse of a in Q.

Lemma 5.13 *Let S be an arbitrary semigroup with identity 1, and suppose that $a \in S$ is invertible along* 1. *Then the inverse of a along* 1 *is the classical inverse of a.*

Proof By Lemma 3.4 (Item 1) a is indeed invertible, since $d = 1$ implies $1 = a^{-1}a1 = 1aa^{-1}$. □

Lemma 5.14 *Let $f(x) = 1$ for all $x \in R$ and let Q be a quotient ring along* 1 *of R with respect to $\mathcal{C} = \mathcal{C}_1(R)$. Then Q is the classical ring of left quotients of R.*

Proof Since $f : R \to R$, we automatically assume that $1 \in R$. By Lemma 5.13, Definition 5.4 (Item 2) implies Definition 5.12 (Item 3).

By Definition 5.4 (Item 1) every 1-cancellable element is invertible in Q. Thus, we only need to show that every element of R which is not a zero divisor is 1-cancellable.

Indeed, let $a \in R$, where a is not a zero divisor. Suppose that $a1x = a1y$ for some $x, y \in R$. Then $ax - ay = a(x - y) = 0$, that is $x = y$. In other words, a is left 1-cancellable. By dual arguments, a is also right 1-cancellable. □

Corollary 5.15 *Let $f(x)\mathcal{H}1$ in R for all $x \in R$ and let Q be a quotient ring along f of R with respect to $\mathcal{C} = \mathcal{C}_f(R)$. Then Q is the classical ring of left quotients of R.*

Proof This is a direct corollary of Lemma 5.14 and Corollary 5.9. □

The next lemma deals with pairs of elements of a quotient ring along a constant function.

Lemma 5.16 *Let $d \in R$ be fixed and $f(x) = d$ for all $x \in R$. Let Q be a quotient ring of R along f with respect to $\mathcal{C} = \mathcal{C}_f(R)$.*

Then for any $p, q \in Q$ there exist $u, w \in \mathcal{C}$, $a, b, c \in R$ and $x, y \in Q$ such that $p = u^{-d}a$ and $q = u^{-d}xb = yu^{-d}b = u^{-d}w^{-d}c$.

Proof By Definition 5.4, we have $p = u^{-d}a$ and $q = v^{-d}b$ for some $u, v \in \mathcal{C}, a, b \in R$. By Lemma 3.4 (Item 2) $v^{-d}\mathcal{H}d$ and $d\mathcal{H}u^{-d}$ in Q. Then by Lemma 2.2 there exist $x, y \in Q$ such that $v^{-d} = u^{-d}x = yu^{-d}$. Then $q = u^{-d}xb = yu^{-d}b$. Since $x \in Q$, there exist $w \in \mathcal{C}$ and $c' \in R$ such that $x = w^{-d}c'$. If $c = c'b$, then $q = u^{-d}w^{-d}c$. □

The next theorem shows that every quotient ring in the sense of Definition 5.4 can be viewed as a Fountain–Gould left quotient ring with respect to some set.

For $f : R \to R$ and $\mathcal{C} \subseteq \mathcal{C}_f(R)$, we introduce the following notation:

- $\mathcal{D}^l_{\mathcal{C}} = \{f(u)u \mid u \in \mathcal{C}\}$

- $\mathcal{D}_{\mathcal{C}}^{l\infty} = \{f(u)u, (f(u)u)^2, \dots \mid u \in \mathcal{C}\}$
- $\mathcal{D}_{\mathcal{C}}^{r} = \{uf(u) \mid u \in \mathcal{C}\}$
- $\mathcal{D}_{\mathcal{C}}^{r\infty} = \{uf(u), (uf(u))^2, \dots \mid u \in \mathcal{C}\}$
- $\mathcal{D}_{\mathcal{C}} = \mathcal{D}_{\mathcal{C}}^{l} \cup \mathcal{D}_{\mathcal{C}}^{r}$
- $\mathcal{D}_{\mathcal{C}}^{\infty} = \mathcal{D}_{\mathcal{C}}^{l\infty} \cup \mathcal{D}_{\mathcal{C}}^{r\infty}$

Theorem 5.17 *Let $f : R \to R$ and let Q be a quotient ring along f of R with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. Then Q is a Fountain–Gould left quotient ring of R with respect to any set $\mathcal{D}$ satisfying $\mathcal{D}_{\mathcal{C}}^{l} \subseteq \mathcal{D} \subseteq \mathcal{D}_{\mathcal{C}}^{\infty}$. In this case, $\mathcal{D} \subseteq \mathfrak{C}(R)$.*

Proof 1. By Definition 5.4, every $u \in \mathcal{C}$ is invertible along $f(u)$ in Q. Then, by Theorem 3.7, $f(u)u$ and $uf(u)$ are group invertible in Q. As a consequence, $(f(u)u)^n$ and $(uf(u))^n$ are also group invertible for any $n \geq 1$. Finally, every $v \in \mathcal{D} \subseteq \mathcal{D}_{\mathcal{C}}^{\infty}$ is group invertible in Q. In addition, $\mathcal{D} \in \mathfrak{C}(R)$ by Lemma 5.3.

2. By Definition 5.4, every $q \in Q$ can be written as $q = u^{-f(u)}b$ for some $u \in \mathcal{C}$ and $b \in R$. Again, by Theorem 3.7, $f(u)u$ is group invertible and $u^{-f(u)} = (f(u)u)^{\#}f(u)$. It follows that $q = u^{-f(u)}b = (f(u)u)^{\#}f(u)b$. Note that $u, f(u), b \in R$ and $f(u)u \in \mathcal{D}_{\mathcal{C}}^{l} \subseteq \mathcal{D}$.

Finally, every $q \in Q$ can be written as $v^{\#}c$ for some $v \in \mathcal{D}$ and $c \in R$.
Thus Q is a Fountain–Gould quotient ring of R with respect to $\mathcal{D}$. □

For right quotient rings along f the following result can be proved dually.

Lemma 5.18 *Let $f : R \to R$ and let Q be a right quotient ring of R along f with respect to a subset $\mathcal{C}$ of $\mathcal{C}_f(R)$. Then Q is a Fountain–Gould right quotient ring of R with respect to any set $\mathcal{D}$ satisfying $\mathcal{D}_{\mathcal{C}}^{r} \subseteq \mathcal{D} \subseteq \mathcal{D}_{\mathcal{C}}^{\infty}$. In this case, $\mathcal{D} \subseteq \mathfrak{C}(R)$.*

5.1 Properties of Fountain–Gould Quotient Rings

Lemma 5.19 *Let Q be a Fountain–Gould left quotient ring of R with respect to $\mathcal{C} \subseteq \mathfrak{C}(R)$.*

Then Q is a Fountain–Gould left quotient ring of R with respect to some $\mathcal{C}' \subseteq \mathfrak{C}(R)$, $\mathcal{C} \subseteq \mathcal{C}'$, where $\mathcal{C}'$ is such that $u^2 \in \mathcal{C}'$ for any $u \in \mathcal{C}'$.

Proof Let $\mathcal{C}' = \bigcup_{n=1}^{\infty} \{u^n | u \in \mathcal{C}\}$. It is easy to verify that $u^2 \in \mathcal{C}'$ for any $u \in \mathcal{C}'$. Also $\mathcal{C} \subseteq \mathcal{C}'$ and as a consequence every $q \in Q$ can be written as $q = u^{\#}a$ for some $a \in R$ and $u \in \mathcal{C}' \supseteq \mathcal{C}$.

Finally, every element of $\mathcal{C}'$ is group invertible since group invertibility of $u \in \mathcal{C}$ implies group invertibility of u^n for any $n \geq 1$. Thus Q is a Fountain–Gould left quotient ring of R with respect to $\mathcal{C}'$. □

Theorem 5.17 allows us to use some known properties of Fountain–Gould quotient rings (with respect to a subset). In this section, we provide several such properties. They were originally proved in [2] for the variant of the Fountain–Gould quotient