Lecture Notes in Mechanical Engineering

S. R. Mishra
T. N. Dhamala
O. D. Makinde *Editors*

Recent Trends in Applied Mathematics

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ISSN 2195-4356 ISSN 2195-4364 (electronic)
Lecture Notes in Mechanical Engineering
ISBN 978-981-15-9816-6 ISBN 978-981-15-9817-3 (eBook)
https://doi.org/10.1007/978-981-15-9817-3

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

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A Multi-item Deteriorating Inventory Model Under Stock Level-Dependent, Time-Varying, and Price-Sensitive Demand

Abhijit Barman and P. K. De

Abstract This paper advocates a multi-item deteriorating inventory model where shortages are not allowed. Here, we have proposed a single-stage EOQ model for deteriorating items where the demand function is depending on nonlinear selling price, nonlinear time, and inventory stock. The model is developed under a known initial inventory. The main objective of this model is to determine the selling price and time length until the inventory reaches zero for each item. To demonstrate our model, one numerical example has been given which is followed by a sensitivity analysis of the major parameters involved in this model.

Keywords Multi-item inventory · Deteriorating items · Selling price · Order quantity · Hessian matrix

1 Introduction

In real-life situations, it is observed that demand for an inventory model changes for the number of items increases in the stocks. That is why companies or any firm owners deal with the multi-item inventory system. The present paper presents a multi-item inventory system over a single period with a finite time horizon. The product deteriorates with the passes of time under the different deteriorating rates. Most of the items that undergo decay over time are medicine, blood banks, volatile liquids, vegetables, etc. Demand for the items is deterministic which depends on inventory label, selling price, and time-varying. The main goal of this model is to determine the unit selling price of a product and the length of the period up to zero inventory that maximizes the overall profit of a retailer or any inventory warehouse.

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S. R. Mishra et al. (eds.), *Recent Trends in Applied Mathematics*, Lecture Notes in Mechanical Engineering, https://doi.org/10.1007/978-981-15-9817-3_1

Even though sufficient literature is available in the area of deteriorating items, but still very less literature is available on multi-item inventory system with deterioration.

The first effort to illustrate the optimum order policies for deteriorating items was established by Chare and Schratures [1]. They introduced an EOQ model for an exponentially decaying inventory system. Later, Covert and Philip [2] extended this model by incorporating variable deterioration rate with two parameters Weibull distribution. Bhattacharya [3] proposed a new method for deteriorating items with linear stock-dependent demand rate in a two items inventory system. Dye et al. [4] discussed pricing and ordering policy for deteriorating items with shortages where the deterioration and demand rate are continuous as well as a differentiable function of time and price, respectively. Pal et al. [5] established a multi-item EOQ model with nonlinear price-dependent and price break-sensitive demand. In the case of non-instantaneous deteriorating items, a joint pricing and inventory model has been established by Maihami and Kamalabadi [6]. Linear price-sensitive and nonlinear time-dependent demand functions have been considered to develop this model with partially backlogging. Sarkar et al. [7] established an inventory model for deteriorating items considering time-sensitive demand with a finite production rate. The selling price and component cost are considered at a continuous rate of time. Yang [8] studied an EOQ model where the holding cost is stock-dependent and the demand rate is also stock-dependent with relaxed terminal environments under shortages. The prime goal of this model is profit maximization by determining optimum order quantity and level of ending inventory. Janssen et al. [9] reviewed 393 articles that are published from January 2012 to December 2015 and categorized the articles based on the different demand characteristics and the deterioration of the items. Feng et al. [10] used the demand as a multivariate function of stock, price, and freshness in an EOQ model. Chen et al. [11] discovered an inventory model for time elapse deteriorating items with a short lifecycle. This model is designed for the stock label, time-varying, and price-sensitive deterministic demand in a finite horizon multi-period setting.

This paper address an EOQ model for n numbers of different items in a finite time horizon. For each item, an initial inventory stock depending on store capacity has been taken separately. The deterministic demand function is taken in a pattern of the nonlinear selling price, exponential time-varying, and linear stock-dependent. Shortages of products are not allowed in this multi-item inventory system. Thus, this paper determines the optimum selling price, time length for which the inventory reaches zero for each item and the overall profit.

The rest of the paper is organized as follows. In Sect. 2, we describe the notations and assumptions used throughout the model. We inaugurate the mathematical model with necessary and sufficient conditions in Sect. 3. In Sect. 4, a numerical example has been provided to illustrate the solution procedure. In Sect. 5, a sensitivity analysis of the optimum solutions concerning different parameters has also been provided. Finally, the summarized findings and some future research suggestions are discussed in Sect. 6.

2 Notations and Assumptions

The following notations and assumption are adopted to develop the model:

Notations

M_i	Manufacturing cost per item for *i*th product
O_i	Ordering cost for *i*th product
h_i	Holding cost per unit time for *i*th product
Q_i	Initial order quantity for *i*th product
p_i	Unit selling price for *i*th item
R_i	Demand rate for each product
θ_i	Deterioration rate for *i*th product
$a_i, b_i, c_i, \mu_i, \alpha_i, \lambda_i$	Demand and stock elasticity parameters
$I(p_i, t)$	Inventory level for *i*th product at time t
T_i	Time length up to zero inventory
TP	Total profit

Assumptions

- The model is considered for n number of different types of products for deteriorating items in a single stage.
- Shortages are not considered in this inventory model i.e. $I_i(p_i, t) \geq 0$ for $i = 1, 2, 3 \ldots n$.
- The replenishment rate is infinite and lead time is negligible.
- Deterioration rate θ_i is constant for *i*th product.
- Demand rate R_i is deterministic in nature and a function of inventory level $I_i(p_i, t)$ with nonlinear selling price $(a_i - b_i p_i - c_i p_i^2)$ and exponentially time varying. For $i = 1, 2, 3 \ldots n$ with considering $a_i \gg b_i \gg c_i$, R_i is represented by

$$R_i = (a_i - b_i p_i - c_i p_i^2)\alpha_i e^{\lambda_i t} + \mu_i I(p_i, t).$$

3 Mathematical Model Formulation and Solution Methodology

At the beginning of the cycle, the system starts with inventory Q_i for the ith product. Over the course of the period, the inventory level down due to both demand and deterioration until it reaches zero at time T_i. During the time interval $[0, T_i]$, the following differential equation represents the inventory status for the ith product

$$\frac{dI_i(p_i,t)}{dt} + \theta_i I(p_i,t) = -R_i \tag{1}$$

with two boundary conditions, $I_i(p_i, 0) = Q_i$ and $I_i(p_i, T_i) = 0$ for $i = 1, 2, \ldots n$. Solving the inventory system and using the boundary conditions, we get the level of inventory of ith item at time t is

$$I_i(p_i,t) = Q_i e^{-(\theta_i+\mu_i)t} + \frac{(a_i - b_i p_i - c_i p_i^2)\alpha_i}{(\theta_i + \mu_i + \lambda_i)}\left[e^{-(\theta_i+\mu_i)t} - e^{\lambda_i t}\right] \tag{2}$$

From the second boundary condition, we have

$$T_i = \frac{1}{(\theta_i + \mu_i + \lambda_i)} Log\left[\frac{Q_i(\theta_i + \mu_i + \lambda_i)}{(a_i - b_i p_i - c_i p_i^2)\alpha_i} + 1\right] \tag{3}$$

Next, for $i = 1, 2, 3 \ldots n$, the total profit in the whole cycle consists of the following five elements:

- Total ordering cost for the ith product is given by

$$OC_i = O_i \tag{4}$$

- Inventory holding cost for the ith product is given by

$$HC_i = h_i \int_0^{T_i} I(p_i,t)dt = h_i K_i \tag{5}$$

- Total manufacturing cost for the ith product is

$$MC_i = M_i Q_i \tag{6}$$

- Deteriorating cost for the ith product is given by

$$DC_i = M_i \int_0^{T_i} \theta_i I(p_i, t)dt = M_i \theta_i K_i \tag{7}$$

- Sales revenue for the *i*th product is written by:

$$\begin{aligned} SR_i &= p_i \int_0^{T_i} R_i(p_i, t) \\ &= \frac{1}{\lambda_i} p_i \alpha_i (a_i - b_i p_i - c_i p_i^2)(e^{\lambda_i T_i} - 1) + p_i \mu_i K_i \end{aligned} \tag{8}$$

where

$$\begin{aligned} K_i = \int_0^{T_I} I_i(p_i, t)dt = \frac{Q_i}{(\theta_i + \mu_i)}[1 - e^{-(\theta_i + \mu_i)T_i}] \\ + \left\{ \frac{(a_i - b_i p_i - c_i p_i^2)\alpha_i}{(\theta_i + \mu_i + \lambda_i)} \left[\frac{1}{\lambda_i} + \frac{1}{(\theta_i + \mu_i)} - \frac{e^{\lambda_i T_i}}{\lambda_i} - \frac{e^{-(\theta_i + \mu_i)}}{(\theta_i + \mu_i)} \right] \right\} \end{aligned} \tag{9}$$

Therefore, the total profit of the retailer for all the items in the whole cycle is

$$\begin{aligned} TP(p_1, p_2, \ldots p_n) &= \sum_{i=1}^{n} [SR_i - (OC_i + HC_i + MC_i + DC_i)] \\ &= \sum_{1}^{n} \left[\frac{1}{\lambda_i} p_i \alpha_i (a_i - b_i p_i - c_i p_i^2)(e^{\lambda_i T_i} - 1) + (p_i \mu_i - h_i - M_i \theta_i) K_i - O_i - M_i Q \right]_i \end{aligned} \tag{10}$$

TP $(p_1, p_2, \ldots p_n)$ is function of $p_1, p_2, \ldots p_n$. So, for some T_i (*from Eq.* 3), the necessary conditions for the overall profit function come from $\frac{\partial TP(p_1, p_2, \ldots p_n)}{\partial p_i} = 0$ for $i = 1, 2, 3 \ldots n$.

$$
\begin{aligned}
\frac{\partial TP(p_1, p_2, \ldots p_n)}{\partial p_i} &= -\frac{p_i Q_i(-b_i - 2c_i p_i)\left[1 + \frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i - b_i p_i - c_i p_i^2)\alpha_i}\right]^{-1+\frac{\lambda_i}{\theta_i+\mu_i+\lambda_i}}}{(a_i - b_i p_i - c_i p_i^2)} + \frac{\mu_i Q_i\left[-1 + \left(1 + \frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i - b_i p_i - c_i p_i^2)\alpha_i}\right)^{\frac{-(\theta_i+\mu_i)}{(\theta_i+\mu_i+\lambda_i)}}\right]}{(\theta_i + \mu_i)} \\
&+ \frac{\alpha_i[\mu_i(a_i - b_i p_i - c_i p_i^2) + (p_i\mu_i - h_i - M_i\theta_i)(-b_i - 2c_i p_i)]\left[\frac{1}{\lambda_i} + \frac{1}{\theta_i+\mu_i} - \frac{\left(1+\frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i-b_i p_i - c_i p_i^2)\alpha_i}\right)^{\frac{\lambda_i}{\theta_i+\mu_i+\lambda_i}}}{\lambda_i} - \frac{\left(1+\frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i-b_i p_i - c_i p_i^2)}\right)^{\frac{-(\theta_i+\mu_i)}{\theta_i+\mu_i+\lambda_i}}}{\theta_i+\mu_i}\right]}{\theta_i + \mu_i + \lambda_i} \\
&+ \frac{(p_i\mu_i - h_i - M_i\theta_i)\alpha_i(a_i - b_i p_i - c p_i^2)\left[\frac{Q_i(-b_i - 2c_i p_i)\left(1+\frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i-b_i p_i-c_i p_i^2)\alpha_i}\right)^{-1+\frac{\lambda_i}{\theta_i+\mu_i+\lambda_i}}}{(a_i - b_i p_i - c_i p_i^2)^2\alpha_i} + \frac{Q_i(-b_i-2c_i p_i)(-\theta_i-\mu_i)\left(1+\frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i-b_i p_i-c_i p_i^2)\alpha_i}\right)^{-1+\frac{-(\theta_i+\mu_i)}{(\theta_i+\mu_i+\lambda_i)}}}{(a_i - b_i p_i - c_i p_i^2)\alpha_i(\theta_i+\mu_i)}\right]}{(\theta_i + \mu_i + \lambda_i)} \\
&+ \frac{(p_i\mu_i - h_i - M_i\theta_i)Q_i^2(-b_i - 2c_i p_i)(-\theta_i - \mu_i)\left(1 + \frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i - b_i p_i - c_i p_i^2)\alpha_i}\right)^{-1+\frac{-\theta_i-\mu_i}{(\theta_i+\mu_i+\lambda_i)}}}{(a_i - b_i p_i - c_i p_i^2)^2\alpha_i(\theta_i + \mu_i)} \\
&+ \frac{\alpha_i[p_i(-b_i - 2c_i p_i) + (a_i - b_i p_i - c_i p_i^2)]\left[-1 + \left(1 + \frac{Q_i(\theta_i+\mu_i+\lambda_i)}{(a_i - b_i p_i - c_i p_i^2)\alpha_i}\right)^{\frac{\lambda_i}{(\theta_i+\mu_i+\lambda_i)}}\right]}{\lambda_i}
\end{aligned}
\tag{11}
$$

This gives a system of nonlinear equations with n number of unknowns p_i for $i = 1,2\ldots n$.

Concavity test by the Hessian matrix

To check whether the profit function (10) is concave, we have to determine the hessian matrix

$$H = \begin{bmatrix} \frac{\partial^2 TP}{\partial p_1^2} & \frac{\partial^2 TP}{\partial^2 p_1 \partial p_2} & \cdots & \frac{\partial^2 TP}{\partial p_1 \partial p_n} \\ \frac{\partial TP}{\partial p_2 \partial p_1} & \frac{\partial^2 TP}{\partial p_2^2} & \cdots & \frac{\partial^2 TP}{\partial p_2 \partial p_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial^2 TP}{\partial p_n \partial p_1} & \frac{\partial^2 TP}{\partial p_n \partial p_2} & \cdots & \frac{\partial^2 TP}{\partial p_n^2} \end{bmatrix}$$

We have to show all the principal minors of the hessian matrix will alternate their sign starting with a negative sign. Since the expression of the second-order derivatives is highly nonlinear, we will check the result numerically with some graphical representation. We have shown numerically as well as graphically the concavity of profit function $TP(p_1, p_2, \ldots p_n)$ in the given numerical example.

4 Numerical Investigation

Let us consider a storehouse problem with two different items with different demand rates and deterioration rates are listed by the following parametric values in Table 1.

Table 1 Different parametric values

Value of the parameter	Item 1	Item 2
Demand elasticity parameter	$a_1 = 120; b_1 = 3.0;$ $c_1 = 0.005;$ $\lambda_1 = 0.1; \alpha_1 = 1$	$a_2 = 140; b_2 = 3.2;$ $c_2 = 0.005;$ $\lambda_2 = 0.15; \alpha_2 = 1.5$
Manufacturing cost ($)	$M_1 = 10$	$M_2 = 12$
Ordering cost ($)	$O_1 = 70$	$O_2 = 80$
Holding cost ($)	$h_1 = 3$	$h_2 = 4$
Deterioration rate	$\theta_1 = 0.08$	$\theta_2 = 0.05$
Stock elasticity parameter	$\mu_1 = 0.6$	$\mu_2 = 0.8$
Initial inventory level	160 units	200 units

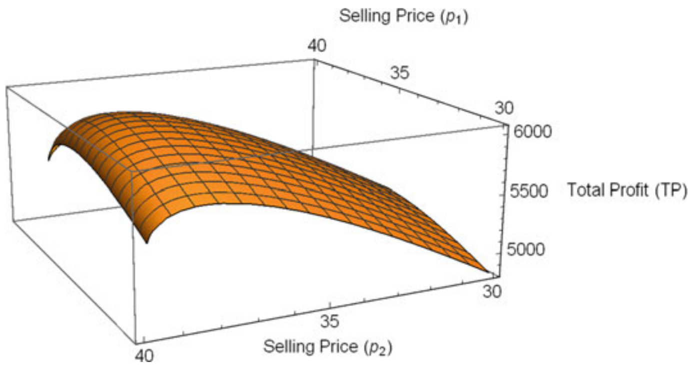

Fig. 1 Variation of optimal profit with respect to selling prices p_1 and p_2

Using Mathematica software, we determine the optimum selling price, time length for the different products, and also the net profit. Unit selling price of first item (p_1^*) = \$35.284, Unit selling price of second item (p_2^*) = \$38.292, Cycle length for first item (T_1^*) = 4.3837 days, Cycle length for second item (T_2^*) = 3.3807 days, and the total profit (TP^*) = \$6049.46. We get $\Delta_1 = -40.55 \leq 0$ and $\Delta_2 = 3174.3878 \geq 0$. The sign of principle minor of the hessian matrix alternate starting with a negative sign. So, the condition of sufficiency is also satisfied.

The following figure (Fig. 1) represents the nature of concavity of the profit function (10). For selling price $p_1^* = \$35.284$ and $p_2^* = \$38.292$, the corresponding total profit $TP^* = \$6049.46$ gives the global maximum in the concave Fig. 1.

5 Sensitivity Analysis

A post-optimality analysis is carried out to analyze the outcome of the changes of different parameters on the optimal solutions. The results of the post-optimality analysis are listed in Table 2 and Table 3, respectively. The changes to the optimum values p_1^*, p_2^*, T_1^*, $T_2,^*$ TP^* have been done by decreasing/increasing the values of the major parameters $a_i, b_i, c_i, \mu_i, \alpha_i, \theta_i, \lambda_i$ for $i = 1, 2...n$. The post-optimality analysis is accomplished from *−20%* to *20%* by changing one parameter at a time and keeping remain parametric values unchanged.

The sensitive analysis which is explored in Tables 2 and 3 indicates the following observations:

- It is visible in Table 2 that, with the increase of purchasing cost (M_i), the selling price (p_i) for both the products as well as the time interval (T_i) will decrease. So, the total profit (*TC*) will decrease with increasing the purchasing cost (M_i).
- It also observed that with the increasing of holding cost (h_i), the selling price (p_i), time length (T_i), and the overall profit (*TC*) decrease rapidly. From a financial

Table 2 Sensitivity analysis with respect to different parameters

Parameter	Percentage change in parameter	p_1	p_2	T_1	T_2	TP
M_i	−20%	35.367	38.336	4.4410	3.4156	6916.08
	−10%	35.326	38.314	4.4121	3.3981	6482.71
	$M_1 = 10; M_2 = 12$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.242	38.269	4.3558	3.3636	5616.31
	20%	35.200	38.247	4.3284	3.3468	5183.28
O_i	−20%	35.284	38.292	4.3837	3.3807	6079.46
	−10%	35.284	38.292	4.3837	3.3807	6064.46
	O_i	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.284	38.292	4.3837	3.3807	6034.46
	20%	35.284	38.292	4.3837	3.3807	6019.46
h_i	−20%	35.597	38.584	4.6106	3.6346	6384.55
	−10%	35.441	38.439	4.4930	3.5012	6215.58
	h_i	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.127	38.143	4.282	3.2713	5886.05
	20%	34.969	37.994	4.1863	3.1714	5725.22
Q_i	−20%	35.036	38.096	3.8780	2.9830	4998.59
	−10%	35.160	38.195	4.1369	3.1864	5561.73
	$Q_1 = 160; Q_2 = 200$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.405	38.384	4.6199	3.5669	6678.74
	20%	35.522	38.472	4.8470	3.7459	7234.29

lookout, it is clear that increasing inventory holding cost directly affects the total profit.

- From Table 2, it is clear that ordering cost (O_i) does not affect the selling price (p_i). So, the change of overall profit (TC) is negligible for that case.
- From Table 2, it is found that increasing with initial inventory stock level (Q_i) the selling price and time interval for both the items will increase. For this case, the net profit will also increase rapidly.
- When the parametric value a_1, a_2 increase the value of selling price (p_i) for both the product will increase but the time length for each item (T_i) of the inventory cycle will decrease. In this case, the overall profit (TC) also increases very rapidly which is shown in Table 3.
- With increasing the value of the other parameter b_i, c_i the value of the selling price (p_i) decreases. So, the value of the total profit (TC) also decreases. For that case, b_i is more effective other than the parameter c_i.
- The effect of change in overall profit (TC) with respect to the change in parameters λ_i and α_i are shown in Table 3 separately. It is observed that, as λ_i or α_i increases, the increment of total profit (TC) is not significant.

Table 3 Sensitivity analysis with respect to different parameters

Parameter	Percentage change in parameter	p_1	p_2	T_1	T_2	TP
a_i	−20%	28.425	31.045	4.6555	3.5545	3630.42
	−10%	31.873	34.697	4.5140	3.4647	4847.11
	$a_1 = 120; a_2 = 140$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	38.658	41.832	4.2626	3.3018	7238.05
	20%	41.998	45.3201	4.1501	3.2276	8413.38
b_i	−20%	43.192	46.108	4.5789	3.5519	8548.57
	−10%	38.885	41.887	4.4817	3.4666	7191.25
	$b_1 = 3.0; b_2 = 3.2$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	32.237	35.204	4.2869	3.2962	5079.12
	20%	29.630	32.529	4.1929	3.2140	4246.66
c_i	−20%	35.718	38.989	4.4207	3.4289	6222.68
	−10%	35.499	38.635	4.4019	3.4044	6134.79
	$c_1 = 0.005; c_2 = 0.008$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.074	37.959	4.3659	3.3579	5966.55
	20%	34.868	37.638	4.3486	3.336	5885.94
λ_i	−20%	35.275	38.291	4.4768	3.4654	6025.37
	−10%	35.279	38.291	4.4296	3.4224	6037.45
	$\lambda_1 = 0.1; \lambda_2 = 0.15$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.288	38.293	4.3391	3.3405	6061.39
	20%	35.293	38.294	4.2957	3.3015	6073.23
μ_i	−20%	34.323	37.545	4.2037	3.2193	5608.43
	−10%	34.830	37.944	4.2915	3.2984	5835.33
	$\mu_1 = 0.4; \mu_2 = 0.6$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.685	38.590	4.4802	3.4663	6250.93
	20%	36.036	38.843	4.5810	3.5549	6439.94
θ_i	−20%	35.594	38.454	4.6934	3.5441	6362.40
	−10%	35.437	38.373	4.5311	3.4597	6239.93
	$\theta_1 = 0.08; \theta_2 = 0.05$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.134	38.212	4.2491	3.3066	6006.71
	20%	34.987	38.131	4.1255	3.2369	5895.45
α_i	−20%	35.579	38.514	4.9575	3.8329	5921.67
	−10%	35.418	38.394	4.6456	3.5871	5986.68
	$\alpha_1 = 1; \alpha_2 = 2$	**35.284**	**38.292**	**4.3837**	**3.3807**	**6049.46**
	10%	35.172	38.204	4.1598	3.2045	6109.82
	20%	35.077	38.129	3.9658	3.0518	6167.71

- When the value of parameter μ_i increases then the selling price (p_i), time length (T_i), and the total profit (TC) also increase. The physical phenomena of this parameter suggest that demand proportional to the inventory of the firm.
- Increasing of parameter θ_i the overall profit (TC) will decrease. The economic viewpoint of this observation shows that as increasing the deterioration rate the profit will be minimized.

6 Conclusion

In this paper, we explored a short life period multi-item EOQ model where deterioration is considered. A stock level-dependent, time-varying, and price-sensitive deterministic demand have been considered to develop the model under a known primary stock. To design the model, the effect of nonlinear selling price and nonlinear time-varying demand functions has been estimated. Our model is demonstrated and illustrated with one numerical example with a graphical explanation. Sensitivity analysis is shown to see the changes in overall profit with respect to the variant of several parameters involved in this model. The contribution of this paper helps decision-makers to increase the overall profit by understanding the market demand situation. As a result, retailers may change their earlier selling price of the items to earn the maximum profit.

This paper can be extended by incorporating various other concepts like inflation, reliability, or some other fuzzy environments.

Acknowledgements First, the author is gratefully thankful to MHRD, Govt. of India for giving support for carrying out research at the National Institute of Technology Silchar and also thankful to TEQIP III for supporting financially to present this paper in AMSE-2019 at Bhubaneswar.

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On Estimation of Reliability Following Selection from Pareto Populations

Ajaya Kumar Mahapatra, Brijesh Kumar Jha, and Chiranjibi Mahapatra

Abstract Let $\prod_1, \prod_2, \ldots, \prod_k$ be k populations, where $\prod_i$ follows a Pareto distribution with unknown scale parameters α_i and common known shape parameters $\beta_i; i = 1, \ldots, k$. Independent random samples are drawn from each of these populations. Let T_i be the smallest observation in the ith sample. The natural selection rule is to select the population with the largest T_i. Then, we consider the estimation of the reliability function of the selected population. The uniform minimum variance unbiased estimator is derived. A class of scale equivariant estimators have been proposed. An inadmissibility result in regards to the class of scale equivariant estimators is established generally

Keywords Selection rule · UMVUE · MLE · Scale equivariant estimators · Brewster–Zidek technique

1 Introduction

Let $\prod_1, \prod_2, \ldots, \prod_k$ be as defined above with each of them corresponding to a probability density function/ probability mass function $f(x|\theta_i), i = 1, \ldots, k$. A common problem is to choose the population or a subset of populations having the best. The population may be regarded as the best according to some attributes such as the largest mean, smallest variance, etc. An important practical problem is to estimate the parameters of the selected population or an attribute of the selected subset. These problems are in general mentioned as "Estimation after selection". Such problems have been at first constructed and explored by Rubinstein [16]. Estimation of the quantile of a

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S. R. Mishra et al. (eds.), *Recent Trends in Applied Mathematics*, Lecture Notes in Mechanical Engineering, https://doi.org/10.1007/978-981-15-9817-3_2

selected population has been considered by Sharma and Vellaisamy [17], Kumar and Kar ([10–12]) and Vellaisamy [18]. Mishra and van der Meulen [15] have studied the estimation after selection in general truncation distributions. The estimation of the reliability function of a selected subset was studied by Kumar et al. [13] for the case of two-parameter exponential distribution. It was assumed that the scale parameters are known and the location parameters are unknown and unequal. They derived the Uniform Minimum Variance Unbiased Estimator(UMVUE) for the survival function and proposed some natural estimators. These estimators are compared in terms of their risks using Brewster and Zidek technique. Further, this estimator is also improved by solving a differential inequality in the light of Vellaisamy and Punen [19]. They have considered the estimation of the location parameter from a selected subset of exponential distribution.

Income distributions were studied initially using Pareto distribution. Later on, it was applied to reliability and life testing, industrial and engineering studies. Johnson and Katz [8], Harris [7], Davis and Feldstein [4], Freiling [6], Berger and Mandelbrot [2], etc., have described several situations where the Pareto model is very useful. This model has been found suitable to describe the allotment of service times in regard to city maintenance, allocation of fallout of nuclear particles, etc. Kumar and Gangopadhyaya [9] have taken up the case to estimate the scale parameter of the chosen Pareto population. In this paper, we study the estimation of the reliability function in the following selection. In Sect. 4, we have derived the UMVUE for the reliability function of the selected Pareto population. In Sect. 5, an inadmissibility result has been established generally for the estimators in the scale equivariant class.

2 Deriving the UMVUE

Independent random samples $X_{i1}, X_{i2}, \ldots, X_{in}$, $i = 1, \ldots, k$ are drawn from k populations $\prod_1, \prod_2, \ldots, \prod_k$, respectively. Let these observations from the respective populations have an associated probability density function $f_i(.)$, given by the Pareto model.

$$f_i(x) = \begin{cases} \frac{\beta\alpha_i^\beta}{x^{\beta+1}}, & \text{if } \alpha_i \le x < \infty, \alpha_i > 0, \beta > 0, \\ 0, & \text{elsewhere, for } i = 1, 2, \ldots, k. \end{cases} \quad (1)$$

Let us assume overall that the scale parameters $\alpha_1, \alpha_2, \ldots, \alpha_k$ are completely unknown and the common shape parameter β is known. Let $T_i = \min(X_{i1}, X_{i2}, \ldots, X_{in})$. Then the statistic $\underline{T} = (T_1, T_2, \ldots, T_k)$ is complete and sufficient. It is also the maximum likelihood estimator (MLE) of $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $i = 1, \ldots, k$. It may be seen that T_i follows a Pareto distribution with shape parameter $n\beta$ and scale parameter α_i, $i = 1, \ldots, k$. It is given by

$$g_i(y) = \begin{cases} \frac{n\beta\alpha_i^{n\beta}}{y^{n\beta+1}}, & \text{if } \alpha_i \le y < \infty, \\ 0, & \text{elsewhere, for } i = 1, 2, \ldots, k. \end{cases} \tag{2}$$

The reliability function of the population $\prod_i$ is given by

$$R_i(t) = P(X_{ij} > t) = \left(\frac{\alpha_i}{t}\right)^{\beta}, \ \alpha_i < t. \tag{3}$$

Our goal is to choose the population associated with the highest reliability $R_i(t)$, $i = 1, \ldots, k$. The probability density of T_i has monotone likelihood ratio property in (α_i, T_i), $i = 1, \ldots, k$. A logical selection rule is to select the population $\prod_i$ if $T_i = max(T_1, \ldots, T_k)$, $i = 1, \ldots, k$. Optimality properties in this regard have been scrutinized by Bahadur and Goodman [1], Lehmann [14] and Eaton [5]. Let $T_{(1)} \le T_{(2)}, \le T_{(k)}$ stand for the ordered values of $T_i's$. We want to estimate

$$R_J(t) = \sum_{j=1}^{k} \left(\frac{\alpha_j}{t}\right)^{\beta} I_j, \tag{4}$$

where

$$I_j = \begin{cases} 1, \text{ if } T_j = T_{(k)}, \ j = 1, \ldots, k; \\ 0, \text{ otherwise.} \end{cases} \tag{5}$$

An unbiased estimator δ of $R_J(t)$ satisfies $E(\delta - R_J(t)) = 0$. To derive the UMVUE of $R_J(t)$, we need the following lemmas.

Lemma 2.1 *Let X be a random variable with pdf $g_i(.)$, given by*

$$g_i(x) = \begin{cases} \frac{n\beta\alpha_i^{n\beta}}{x^{n\beta+1}} & \text{if } \alpha_i \le x < \infty \\ 0 & \text{otherwise, for } i = 1, \ldots, k. \end{cases} \tag{6}$$

Suppose that U(x) be a real-valued function defined on R, such that
(a) $E_\alpha(U(x)) < \infty \quad \forall \alpha \in \Omega$,
(b) The integral $\int_x^\infty U(t)P(t, \beta)dt$ exists and is finite,
where $P(x, \beta) = \frac{n}{\beta^{n-1}x^n\beta+1}$ for $0 < x < \infty$,
(c) $\lim_{x\to\infty}[x^\beta \int U(t)P(t, \beta)dt] = 0$.
Then the function

$$V(x) = x^\beta U(x) - \frac{\beta x^{\beta-1}}{P(x, \beta)} \int_x^\infty U(t)P(t, \beta)dt$$

satisfies

$$E_\alpha V(x) = \alpha^\beta E_\alpha U(x) \tag{7}$$

Proof The proof follows using integration by parts to the second expression. The following lemma is a generalization of the above lemma.

Lemma 2.2 *Let $T_1, T_2, \ldots, T_k$ be k independent random variables with pdf $g_i(.)$ as defined in* (4.2).
Suppose that $U_1(\underline{t}), U_2(\underline{t}), \ldots, U_k(\underline{t})$ be k real-valued function defined on R, such that
(a) $E_{\underline{\alpha}}(U_i(\underline{T})) < \infty \quad \forall \alpha_i > 0,\ i = 1, \ldots, k.$
(b) The integral $\int_{t_i}^{\infty} U(t_1, t_2, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_k) P(x, \beta) dx$ exists and is finite, where $P(x, \beta) = \frac{n}{\beta^{n-1} x^n \beta+1}$ for $0 < t_i < \infty$,
(c) Then the function

$$V_i(\underline{T}) = t_i^{\beta} U_i(\underline{T}) - \frac{\beta t_i^{\beta-1}}{P(t_i, \beta)} \int_{t_i}^{\infty} U(t_1, t_2, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_k) P(x, \beta) dx$$

satisfies

$$E_{\alpha} V_i(\underline{T}) = {\alpha_i}^{\beta} E_{\alpha} U_i(\underline{T})$$

Next, since $R_J(t) = \sum_{j=1}^{k} \frac{{\alpha_j}^{\beta}}{t^{\beta}} I_j$, define

$$U_i(\underline{t}) = \begin{cases} \frac{1}{t^{\alpha}} & \text{if } T_j = T_{(k)} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

then we can write $E(R_J(t)) = \sum_{i=1}^{k} {\alpha_i}^{\beta} E[U_i(\underline{t})]$, from lemma (4.2) we have

$$E[\sum_{i=1}^{k} V_i(\underline{T})] = E[\sum_{i=1}^{k} {\alpha_i}^{\beta} U_i(\underline{T})]$$

which is the unbiased estimator of $R_J(t)$.

Theorem 2.1 *The UMVUE of $R_J(t)$ is given by* $\hat{R_J^U}(t) = \frac{1}{n} \left(\frac{T_{(k)}}{t}\right)^{\beta} \left[n - 1 - \sum_{i=1}^{k-1} \left(\frac{T_i}{T_{(k)}}\right)^{n\beta+\beta}\right].$

Proof We have

$$\begin{aligned} V_i(\underline{t}) &= t_i^{\beta} U_i(\underline{t}) - \frac{\beta t_i^{\beta-1}}{P(t_i, \beta)} \int_{t_i}^{\infty} U_i(t_1, t_2, \ldots, t_{i-1}, x, t_{i+1}, \ldots, t_k) P(x, \beta) dx \\ &= \frac{t_i^{\beta}}{t^{\beta}} I(t_i \geq \max_{i \neq j} t_j) - \frac{\beta t_i^{\beta-1}}{P(t_i, \beta)} \int_{t_i}^{\infty} I(t_i \geq \max_{i \neq j} t_j) \frac{P(x, \beta)}{t^{\beta}} dx \\ &= \left[t_i^{\beta} - \frac{\beta t_i^{\beta-1}}{P(t_i, \beta)} \int_{t_i}^{\infty} P(x, \beta) dx\right] \frac{I(t_i \geq \max_{i \neq j} t_j)}{t^{\beta}} \end{aligned}$$

$$\Rightarrow E\Big(\sum_{i=1}^{k} V_i(\underline{T})\Big) = \frac{1}{t^\beta}\Bigg[T_{(k)}^\beta - \Big(\sum_{i=1}^{k-1}\frac{\beta T_i^\beta}{P(T_i,\beta)}\Big)\int_{T_{(k)}}^{\infty} P(x,\beta)dx\Bigg]$$
$$= \frac{1}{n}\Big(\frac{T_{(k)}}{t}\Big)^\beta\Bigg[n-1-\sum_{i=1}^{k-1}\Big(\frac{T_i}{T_{(k)}}\Big)^{n\beta+\beta}\Bigg],$$

which is the UMVUE of $R_J(t)$.

3 An Inadmissibility Result

In this section, we will try to find out the form of an equivariant estimator of $R_J(t)$. For this, let us consider the scale group of transformations $G = \{g_c : g_c(x) = cx, c > 0\}$. Under this transformation $\alpha \to c\alpha$, $R_J \to c^\beta R_J$. Hence, the decision problem is invariant under this transformation in regards to the quadratic loss, given by

$$L(\hat{R}_J(t), R_J(t)) = \left(\frac{\hat{R}_J(t) - R_J(t)}{R_J(t)}\right)^2,$$

where $\hat{R}_J(t)$ is any estimator of $R_J(t)$. An estimator $h(\underline{T})$ is said to be equivariant if

$$h(cT_1, cT_2, \ldots, cT_k) = c^\beta h(T_1, T_2, \ldots, T_k).$$

Let $c = \frac{1}{T_k}$, we have

$$h\left(\frac{T_1}{T_k}, \frac{T_2}{T_k}, \ldots, \frac{T_{k-1}}{T_k}, 1\right) = \frac{1}{T_k^\beta} h(\underline{T})$$

$$\Rightarrow h(\underline{T}) = T_k^\beta h(\underline{Z}), \tag{9}$$

where $\underline{Z} = (Z_1, Z_2, \ldots, Z_{k-1})$, $Z_i = \frac{T_i}{T_k}$, for $i = 1, \ldots, k-1$ and let $\underline{z} = (z_1, z_2, \ldots, z_{k-1})$ be any observed value of $\underline{Z}$.

It can be easily seen that the UMVUE is a scale equivariant estimator. We now use the Brewster–Zidek technique for improving upon the equivariant estimators.

The risk of $h(\underline{T})$ for estimating $R_i(t)$, for $i = 1, 2, \ldots, k$ is given by

$$R(T_k^\beta h(\underline{Z}), R_i(t)) = E[T_k^\beta h(\underline{Z}) - R_i(t)]^2 = E^{\underline{Z}}[E^{\underline{T}|\underline{Z}}(T_k^\beta h(\underline{Z}) - R_i(t))^2|\underline{Z}]$$

Differentiating the above equation with respect to $h(\underline{Z})$. We see that the inner conditional expectation is minimized by

$$h^*_\beta(\underline{z}) = R_i(t)\frac{E(T_k^\beta|\underline{Z}=\underline{z})}{E(T_k^{2\beta}|\underline{Z}=\underline{z})} \tag{10}$$

In order to find out the expectation above, we need the conditional density of T_k given $\underline{Z}=\underline{z}$. It is given by

$$\frac{kn\beta\ \alpha_i^{kn\beta}}{t_k^{kn\beta+1}{z_i}^{kn\beta}}, \quad \frac{\alpha_i}{z_i} \leq t_k < \infty, \quad \text{and} \quad \frac{\alpha_{i+1}}{\alpha_i} \leq z_i < \infty. \tag{11}$$

(See also Kumar and Kar [9].
Hence

$$h^*_\beta(\underline{Z}) = \frac{kn-2}{kn-1}\left(\frac{\alpha_j}{t\alpha_i}\right)^\beta {z_i}^\beta, \quad i \neq j.$$

If we fix j and vary i such that $i \neq j$, we have

$$\hat{h_\beta}(\underline{z}) = \sup\ h^*_\beta(\underline{Z}) = \frac{kn-2}{kn-1}\left(\frac{\max(z_1,\ldots,z_{k-1},1)}{t}\right)^\beta \quad \text{also}\ \inf h^*_\beta(\underline{z}) = 0. \tag{12}$$

Summarizing the above results the following theorem is concluded immediately.

Theorem 3.1 *Let* $\Psi(\underline{Z})$ *be an estimator of* R *as defined in* (5.1)*, then define an estimator* $\Psi^*(\underline{Z})$ *by*

$$\Psi^*(\underline{Z}) = \begin{cases} \Psi(\underline{Z}), & \text{if}\ \Psi(\underline{Z}) < \hat{h_\beta}(\underline{Z}), \\ \hat{h_\beta}(\underline{Z}), & \text{otherwise.} \end{cases} \tag{13}$$

Then, $\Psi^*(\underline{Z})$ *is an improved estimator of* $\Psi(\underline{Z})$ *provided* $P\{\Psi(\underline{Z}) \geq \hat{h_\beta}(\underline{Z})\} > 0$.

Remark 3.1 It can be seen that Theorem 3.1 will also hold good even for the usual squared error loss function. This is because the proof of Brewster–Zidek [3] technique was established on the orbits of $\underline{Z}=\underline{z}$.

Remark 3.2 For $n > 2$, then the estimator $\hat{R_J^c} = \frac{1}{n}\left(\frac{T_{(k)}}{t}\right)^\beta\left[c - \sum_{i=1}^{k-1}\left(\frac{T_i}{T_{(k)}}\right)^{n\beta+\beta}\right]$ uniformly dominates $\hat{R_J^U}(t) = \frac{1}{n}\left(\frac{T_{(k)}}{t}\right)^\beta\left[n-1-\sum_{i=1}^{k-1}\left(\frac{T_i}{T_{(k)}}\right)^{n\beta+\beta}\right]$ for $\frac{n^2-2n-1}{n-1} \leq c < n-1$.

Proof Consider the risk difference RD_1 of the above two estimators. So,

$$\begin{aligned} RD_1 &= E[(\hat{R_J^U}(t), R_J)^2 - (\hat{R_J^c}(t) - R_J(t))^2] \\ &= E[(\hat{R_J^U}(t) + \hat{R_J^c}(t))((\hat{R_J^U}(t) - \hat{R_J^c}(t)))] \\ &- E[2R_J(t)((\hat{R_J^U}(t) - \hat{R_J^c}(t)))]. \end{aligned}$$

We have $\hat{R_J^U}(t) + \hat{R_J^c}(t) = \frac{1}{n}\Big(\frac{T_{(k)}}{t}\Big)^{\beta}\Big[n - 1 + c - 2\sum_{i=1}^{k-1}\Big(\frac{T_i}{T_{(k)}}\Big)^{n\beta+\beta}\Big],$

$$\hat{R_J^U}(t) - \hat{R_J^c}(t) = \frac{n-1+c}{n}\Big(\frac{T_{(k)}}{t}\Big)^{\beta}$$

and $E[R_J(t)(\hat{R_J}(t) - \hat{R_J^c}(t))] = \frac{1}{t^{\beta}}E\Big[\frac{n-1-c}{n}\sum_{i=1}^{k}\Big(\frac{\alpha_i}{t}\Big)^{\beta}{T_i}^{\beta}I(T_i \geq \max_{i\neq j} T_j)\Big]$

Taking $U_i(\underline{T}) = {T_i}^{\beta}I(T_i \geq \max_{i\neq j} T_j)$, from Lemma 2.2, we have

$$\begin{aligned} E\Big[\sum_{i=1}^{k}\Big(\frac{\alpha_i}{t}\Big)^{\beta}{T_i}^{\beta}I(T_i \geq \max_{i\neq j} T_j)\Big] &= \sum_{i=1}^{k}E\Big[{T_i}^{2\beta}I(T_i \geq \max_{i\neq j} T_j) \\ &- \beta {T_i}^{n\beta+\beta}\int_{T_i}^{\infty}\frac{I(x \geq \max_{i\neq j} T_j)}{x^{n\beta-\beta+1}}dx\Big] \\ &= E\Big[\frac{T_{(k)}^{2\beta}}{n-1}\Big(n - 2 - \sum_{i=1}^{k-1}\Big(\frac{T_i}{T_{(k)}}\Big)^{n\beta+\beta}\Big)\Big]. \end{aligned} \tag{14}$$

With the help of (14), we are in a position to compute RD_1.

$$\begin{aligned} RD_1 &= \frac{1}{t^{\beta}}\Big[\frac{n-1-c}{n^2}E\Big\{T_{(k)}^{2\beta}\Big(n - 1 + c - 2\sum_{i=1}^{k-1}\Big(\frac{T_i}{T_{(k)}}\Big)^{n\beta+\beta}\Big)\Big\} \\ &- 2\frac{n-1-c}{n(n-1)}E\Big\{T_{(k)}^{2\beta}\Big(n - 2 - 2\sum_{i=1}^{k-1}\Big(\frac{T_i}{T_{(k)}}\Big)^{n\beta+\beta}\Big)\Big\}\Big] \\ &= \frac{1}{t^{\beta}}\frac{n-1-c}{n^2(n-1)}E\Big[T_{(k)}^{2\beta}\Big(1 + 2n - n^2 + c(n-1) + 2\sum_{i=1}^{k-1}\Big(\frac{T_i}{T_{(k)}}\Big)^{n\beta+\beta}\Big)\Big]. \end{aligned} \tag{15}$$

We see that for $\frac{n^2-2n-1}{n-1} \leq c < n - 1$ and $n > 2$, then $RD_1 > 0$. Hence the conclusion follows immediately.

Remark 3.3 The natural estimator $\hat{R_J^*}(t) = (\frac{T_{(k)}}{t})^{\beta}$ is inadmissible.

Proof Consider the counterpart of the UMVUE for the component problem $\hat{R_J^{AU}}(t) = \frac{n-1}{n}(\frac{T_{(k)}}{t})^{\beta}$. We claim that this estimator dominates uniformly the natural estimator for $n > 2$. The risk difference between these two estimators RD_2 is given by

$$
\begin{aligned}
RD_2 &= E[(\hat{R}^*_J(t), R_J(t))^2 - (\hat{R}^{AU}_J(t) - R_J(t))^2] \\
&= E[(\hat{R}^*_J(t) + \hat{R}^{AU}_J(t))((\hat{R}^*_J(t) - \hat{R}^{AU}_J(t))) \\
&- E[2R_J(t)((\hat{R}^*_J(t) - \hat{R}^{AU}_J(t)))]
\end{aligned}
$$

$$
\text{We have} \quad \hat{R}^*_J(t) + \hat{R}^{AU}_J(t) = \frac{2n-1}{n}\left(\frac{T_{(k)}}{t}\right)^{\beta},
$$

$$
\hat{R}^*_J(t) - \hat{R}^{AU}_J(t) = \frac{1}{n}\left(\frac{T_{(k)}}{t}\right)^{\beta}
$$

$$
\text{and} \quad E[R_J(t)(\hat{R}^*_J(t) - \hat{R}^{AU}_J(t))] = \frac{1}{t^{\beta}} E\left[\frac{1}{n}\sum_{i=1}^{k}\left(\frac{\alpha_i}{t}\right)^{\beta} T_i{}^{\beta} I(T_i \geq \max_{i \neq j} T_j)\right]
$$

Proceeding similarly as above, we have

$$
RD_2 = \frac{1}{t^{\beta}} \frac{2n-1}{n^2(n-1)} E\left[T_{(k)}^{2\beta}\left(n + 1 + 2n\sum_{i=1}^{k-1}\left(\frac{T_i}{T_{(k)}}\right)^{n\beta+\beta}\right)\right] > 0. \qquad (16)
$$

Hence the conclusion follows.

Remark 3.4 The estimator $\hat{R}^{AU}_J(t) = \frac{n-1}{n}(\frac{T_{(k)}}{t})^{\beta}$ is inadmissible.

Proof For the component problem, let us consider the counterpart of the best scale equivariant estimator, given by, $\hat{R}^S_J(t) = \frac{n-2}{n-1}(\frac{T_{(k)}}{t})^{\beta}$. Here also we see that the estimator $\hat{R}^S_J(t)$ dominates uniformly $\hat{R}^{AU}_J$ for $n > 2$. We compute the risk difference RD_3, which is given by

$$
\begin{aligned}
RD_3 &= E[(\hat{R}^U_J(t), R_J(t))^2 - (\hat{R}^S_J(t) - R_J(t))^2] \\
&= \frac{1}{t^{\beta}} \frac{1}{n^2(n-1)^2} E\left[T_{(k)}^{2\beta}\left(1 + 2n(n-1)\sum_{i=1}^{k-1}\left(\frac{T_i}{T_{(k)}}\right)^{n\beta+\beta}\right)\right] > 0. \qquad (17)
\end{aligned}
$$

Hence the conclusion follows.

Conclusion: Under the mean squared error criterion the estimator $\hat{R}^S_J$ dominates $\hat{R}^U_J$, which once again improves upon the natural estimator $\hat{R}^*_J$. Hence $\hat{R}^U_J$ is preferred. One should not prefer $\hat{R}_J$ unless one is not interested in the class of unbiased estimators. Also we have $\hat{R}^c_J$ which dominates $\hat{R}_J$.