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Geometric Aspects of Harmonic Analysis

Springer

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Volume 45

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Geometric Aspects of Harmonic Analysis

Springer

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ISSN 2281-518X ISSN 2281-5198 (electronic)
Springer INdAM Series
ISBN 978-3-030-72057-5 ISBN 978-3-030-72058-2 (eBook)
https://doi.org/10.1007/978-3-030-72058-2

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To Fulvio Ricci, with gratitude.

Preface

On 25–29 June 2018 the INdAM Meeting "Geometric Aspects of Harmonic Analysis" took place in Cortona. This conference, which saw the participation of over 120 mathematicians from around the world, was organised on the occasion of Fulvio Ricci's 70th birthday.

This short introduction is not meant to discuss the interest and relevance of Fulvio Ricci's mathematical contributions, which are witnessed by his bright career, the quality of his scientific production, the awards he received and the level of the scholars who participated in the conference. Some words in that direction can be found in the letter by Elias Stein included in this volume. Instead, we would like to express our appreciation of Fulvio and our gratitude to him for the humanity, the rigour, the fairness he has always shown in mathematics and life and, last but not least, his great openness to interact and collaborate with mathematicians of all ages and from all over the world.

This volume originated in talks given in Cortona and presents timely syntheses of several major fields of mathematics as well as original research articles contributed by some of the finest mathematicians working in these areas.

It is our pleasure to thank all the organisations that contributed generously to the conference with their financial support: the Istituto Nazionale di Alta Matematica-INdAM, the Clay Mathematics Institute, the US National Science Foundation, the Scuola Normale Superiore di Pisa, the Università degli Studi di Milano Bicocca, and the Università degli Studi di Padova. Special thanks are due to the University of Wisconsin–Madison, which kindly hosted the website of the conference.

On behalf of the entire organising committee of the conference we would like to acknowledge our great appreciation to the director of INdAM, Professor Giorgio Patrizio, and to the former director of SNS, Professor Vincenzo Barone. Their efforts and suggestions helped to make this a most fruitful and enjoyable meeting.

We are also pleased to thank all the speakers for the distinguished and outstanding lectures they gave.

It is our pleasure to thank all people working at the Centro Convegni Sant'Agostino and the Palazzone, which were the meeting's venues, for their friendliness, kindness and effectiveness; special thanks are particularly due to Mrs Rita Santiccioli and Mrs Benedetta Biagiotti.

We owe a debt of gratitude to all the other organisers of the conference: Luigi Ambrosio, Gian Maria Dall'Ara, Bianca Di Blasio, and especially the US organisers Loredana Lanzani, Betsy Stovall and Brian Street, who applied for the NSF funding.

Finally, we would like to take this opportunity to thank all the participants in the conference. We hope that the warm atmosphere of those days in Cortona will be a nice memory for all of them.

Padova, Italy — Paolo Ciatti
Birmingham, UK — Alessio Martini
October 2020

At the Occasion of Fulvio's Conference

Elias M. Stein could not attend the June 2018 conference in Cortona. Instead, he sent a letter, which was read during the conference and is reproduced below.

Dear Fulvio and friends,

I'm sorry that I'm missing this wonderful celebration in your honor, Fulvio—I can only blame my overly cautious doctor for this. But I want to take this opportunity to say a few words of appreciation of your many remarkable achievements, and then indulge in a few reminiscences.

First, we all know and recognize that your work continues to have broad impact and wide influence—indeed your efforts have played a major role in transforming a number of diverse area in analysis. Your constant urge to try to look at things differently, your deep insights, great energy, and your keen appreciation for what is really important, has made all of this possible. In working with others (you've had at least 20 collaborators), your wisdom and warmth have brought out the best in your coworkers, and in many cases made them even better than they thought possible, as I can readily attest.

I will indicate the sweep of your interests and contributions by sketching only a partial list of the main areas of your work.

- Harmonic analysis of singular integrals of Radon-type on nilpotent groups.
- Geometry and analysis of non-symmetric harmonic spaces, and the study of their boundary groups.
- The theory of solvability of invariant differential operators on the Heisenberg group.
- The study of maximal functions and singular integrals associated to polynomial maps.
- Spectral multipliers on the Heisenberg group, their connection with the Hodge-Laplacian, and the origin of flag kernels.
- The general theory of operators with flag kernels on nilpotent groups, and most recently, the theory of singular integrals controlled by multiple norms.

Fulvio—allow me now to come to some personal recollections. I'm not sure when we first met. It might have been before 1980, but we really got to know each other a few years later when you came to the Institute for the whole academic year with Sandra and Alberto. We began working together then, and wrote a nice (but forgettable) paper. However what was important is that we learned to appreciate each other, that we had mathematical empathy, and that we could easily talk together in that common language we both loved.

There followed a series of visits by you in Princeton, and by me in Torino. Besides all the mathematics we did together—which I will always treasure—I remember with nostalgia the hotel Bologna near the train station, the cafes in the elegant Piazza San Carlo, and the pleasant walks to the Politecnico where we worked all day, interrupted only by lunch (not at a mensa!), but with paninis in the nice cafes in the area.

We also had the good fortune to twice spend one-week stays during the summer (with our families and a few friends) at the Villa Ronconi, right on the shore of Lake Como, with its marvelous grounds and stunning views. However, soon thereafter my university, in its wisdom, decided to dispose of this unique holding, and we were thus expelled from our own private paradise. Nevertheless, a few years later we had the lucky chance to spend (again with our families and some good friends) a summer month in Berkeley. While not paradise, Berkeley and its surroundings were the next best thing on earth! It was there that Alex Nagel joined our collaboration, and a few years later we also attracted Steve Wainger to our common effort.

And now, after these few warm recollections of the past, I come to some words about the present and future. Having myself passed this milestone a number of years ago, I can say with some certainty that this is a new beginning—maybe not what one would like as an ideal starting point—but nevertheless bracing, full of interesting challenges to undertake and try to master, and rich in the achievements that can be hoped for, and the joy and satisfaction they entail. So with this in view, I wish you all the best of fortune in your further life and adventures!

Happy birthday!
Eli

Contents

An Extension Problem and Hardy Type Inequalities for the Grushin Operator

Rakesh Balhara, Pradeep Boggarapu, and Sundaram Thangavelu

Dedicated to Professor F. Ricci on his 70th birthday

Abstract In this paper we study the extension problem associated to the Grushin operator $G = -\Delta - |x|^2\partial_w^2$ on $\mathbb{R}^{n+1}$ and use the solutions to prove trace Hardy and Hardy inequalities for fractional powers of G.

Keywords Grushin operator · Extension problem · Hardy and trace Hardy Inequalities

1 Introduction and Main Results

In this article we are interested in proving Hardy type inequalities for fractional powers of the Grushin operator $G = -\Delta - |\xi|^2\partial_w^2$ on $\mathbb{R}^{n+1}$. Recall that in the case of Laplacian Δ on $\mathbb{R}^n$ such inequalities are well known and there is a vast literature on the topic. For $0 < s < 1$, two kinds of Hardy inequalities for the fractional powers $(-\Delta)^{s/2}$ have been studied. The inequality

$$((-\Delta)^{s/2}f, f) \geqslant c_{n,s} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x|^2)^s}\, dx$$

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P. Ciatti, A. Martini (eds.), *Geometric Aspects of Harmonic Analysis*,
Springer INdAM Series 45, https://doi.org/10.1007/978-3-030-72058-2_1

with a sharp explicit constant $c_{n,s}$ is known as Hardy's inequality with non-homogeneous weight function whereas the inequality

$$((-\Delta)^{s/2}f, f) \geqslant C_{n,s} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^s}\,dx$$

is the Hardy inequality with homogeneous weight. The constant $C_{n,s}$ is also known to be sharp and explicit. It is of interest to prove such inequalities when Δ is replaced by more general elliptic/subelliptic operator. A particularly interesting case is the one where we have the sublaplacian $\mathcal{L}$ on Heisenberg groups $\mathbb{H}^n$ in place of the Laplacian Δ. In the articles [13] and [14] the authors have established Hardy type inequalities for (conformally invariant) fractional powers of $\mathcal{L}$.

In this work we are mainly interested in proving Hardy type inequalities for fractional powers of G. There are several ways of proving Hardy inequalities for the Laplacian, see [2, 10] and [20]. For the case of sublaplacian $\mathcal{L}$ the authors in [13] have used the method of ground state representation developed by Frank, Lieb and Seiringer [9] in proving a version of Hardy inequality for the sublaplacian with non-homogeneous weight. Later, in [14] the same authors have used a different method in proving analogues of both inequalities making use of solutions of the so called extension problem for the sublaplacian. The extension problem for the Laplacian studied by Caffarelli and Silvestre [4] deals with the initial value problem

$$(\Delta + \partial_\rho^2 + \frac{1-s}{\rho}\partial_\rho)u(x,\rho) = 0, \quad u(x,0) = f(x), x \in \mathbb{R}^n, \rho > 0.$$

The solutions of this problem can be written down explicitly and using them one proves the following inequality known as trace Hardy inequality: for reasonable real valued functions φ from the domain of $(-\Delta)^{s/2}$ one has

$$\int_0^\infty \int_{\mathbb{R}^n} |\nabla_{x,\rho}u(x,\rho)|^2 \rho^{1-s}\,dx\,d\rho \geq c_s \int_{\mathbb{R}^n} u(x,0)^2 \frac{(-\Delta)^{s/2}\varphi(x)}{\varphi(x)}dx$$

valid for all real valued functions $f \in C_0^\infty(\mathbb{R}^{n+1})$. When u is a solution of the extension problem with initial condition f, the left hand side of the above reduces to a constant multiple of $((-\Delta)^{s/2}f, f)$. Further, the choice $\varphi(x) = (1+|x|^2)^{-(n-s)/2}$ allows us to simplify the right hand side and we obtain the Hardy inequality

$$((-\Delta)^{s/2}f, f) \geq c_{n,s} \int_{\mathbb{R}^n} \frac{|f(x)|^2}{(1+|x|^2)^s}\,dx.$$

When $f(x) = (1+|x|^2)^{-(n-s)/2}$ both sides of the above inequality are equal with $c_{n,s} = 2^s \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n-s}{2})}$.

All of these are well known in the case of the Laplacian on $\mathbb{R}^n$. Recently, in [14] the authors have carried out similar analysis for the sublaplacian $\mathcal{L}$ on the

Heisenberg group $\mathbb{H}^n$. Our aim in this article is to show that the same analysis can be done also for the case of the Grushin operator. Thus we will be studying the extension problem for the Grushin operator and use the solutions to prove trace Hardy and Hardy inequalities for fractional powers of the Grushin operator.

In the Euclidean case, an important role is played by the identity

$$(-\Delta)^{s/2}(1+|x|^2)^{-(n-s)/2} = 2^s \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n-s}{2})}(1+|x|^2)^{-(n+s)/2}$$

which follows from the transformation property of the Macdonald function. This can be easily proved by taking the Fourier transform: writing

$$\varphi_s(x) = (1+|x|^2)^{-(n+s)/2} = \frac{1}{\Gamma(\frac{n+s}{2})}\int_0^\infty e^{-t(1+|x|^2)} t^{\frac{n+s}{2}-1} dt$$

and taking the Fourier transform we see that

$$\widehat{\varphi_s}(\xi) = \frac{(4\pi)^{-n/2}}{\Gamma(\frac{n+s}{2})}\int_0^\infty e^{-t} e^{-\frac{1}{4t}|\xi|^2} t^{\frac{s}{2}-1} dt.$$

The integral on the right hand side is given in terms of the Macdonald function $K_{-s/2}$, see [12], page 407):

$$K_{-s/2}(|\xi|^2) = 2^{s/2-1}|\xi|^{-s}\int_0^\infty e^{-t} e^{-\frac{1}{4t}|\xi|^2} t^{\frac{s}{2}-1} dt.$$

The change of variables $u = \frac{|\xi|^2}{4t}$ proves that

$$|\xi|^s \widehat{\varphi_{-s}}(\xi) = 2^s \frac{\Gamma(\frac{n+s}{2})}{\Gamma(\frac{n-s}{2})}\widehat{\varphi_s}(\xi)$$

as desired. The corresponding identity used in the case of the sublaplacian $\mathcal{L}$ is the Cowling-Haagerup [6] formula

$$\mathcal{L}_s((1+|z|^2)^2+16t^2)^{-(n+1-s)/2} = 4^{2s}\frac{\Gamma(\frac{n+1+s}{2})^2}{\Gamma(\frac{n+1-s}{2})^2}((1+|z|^2)^2+16t^2)^{-(n+1+s)/2}.$$

This identity is a consequence of certain transformation property of the Kummer's function. In our case we make use of the following relation which is the analogue of the above identity:

$$\tilde{G}_s((1+|\xi|^2)^2+w^2)^{-(n+2-2s)/4} = 2^{2s}\frac{\Gamma(\frac{n+2+2s}{4})^2}{\Gamma(\frac{n+2-2s}{4})^2}((1+|\xi|^2)^2+w^2)^{-(n+2+2s)/4}.$$

Here $\tilde{G}_s$ stands for the conformally invariant fractional power of the Grushin operator and the above identity can be proved by expanding the functions involved in terms of Laguerre functions and making use of an identity proved for Laguerre operators in [5].

2 Preliminaries on the Grushin Operator

By the Grushin operator we mean the degenerate elliptic operator $G = -\Delta - |\xi|^2\partial_w^2,\ \xi \in \mathbb{R}^n,\ w \in \mathbb{R}$ on $\mathbb{R}^{n+1}$. Here Δ stands for the standard Laplacian on $\mathbb{R}^n$. When f is an integrable function on $\mathbb{R}^{n+1}$ let

$$f^\lambda(\xi) = \int_{-\infty}^{\infty} f(\xi, w)e^{i\lambda w}dw$$

stand for the inverse Fourier transform of f in the last variable. Then it follows that $(Gf)^\lambda(\xi) = H(\lambda)f^\lambda(\xi)$ where $H(\lambda) = -\Delta + \lambda^2|x|^2$ is the scaled Hermite operator on $\mathbb{R}^n$. The spectral decomposition of G can be written in terms of Hermite expansions. Let $P_k(\lambda)$ stand for the projections of $L^2(\mathbb{R}^n)$ onto the $k-$th eigenspace of $H(\lambda)$ with eigenvalue $(2k+n)|\lambda|$ so that

$$H(\lambda) = \sum_{k=0}^{\infty}((2k+n)|\lambda|)P_k(\lambda).$$

Then the spectral decomposition of G is given by

$$Gf(\xi, w) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-i\lambda w}\Big(\sum_{k=0}^{\infty}((2k+n)|\lambda|)P_k(\lambda)f^\lambda(\xi)\Big)d\lambda.$$

For a bounded function m defined on the spectrum of G, viz. $\mathbb{R}^+$ we can define the operator $m(G)$ by

$$m(G)f(\xi, w) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-i\lambda w}\Big(\sum_{k=0}^{\infty}m((2k+n)|\lambda|)P_k(\lambda)f^\lambda(\xi)\Big)d\lambda$$

which is clearly a bounded linear operator on $L^2(\mathbb{R}^{n+1})$. The choice $m(a) = e^{-ta}, t > 0$ leads to the heat semigroup e^{-tG} generated by the Grushin operator. For information on the spectral theory of the Hermite operator we refer to [17].

We make use of the following representation of the Heisenberg group $\mathbb{H}^n$ in order to transfer operators in the Heisenberg setting into the setting of Grushin. On $L^2(\mathbb{R}^{n+1})$ we define the representation π by

$$\pi(z,t)f(\xi, w) = f(\xi - y, w - t - \xi\cdot x + \frac{1}{2}x\cdot y), \quad f \in L^2(\mathbb{R}^{n+1}),\ z = x + iy.$$

It is easy to see that π is a strongly continuous unitary representation of $\mathbb{H}^n$ on $L^2(\mathbb{R}^{n+1})$. More generally, for any $f \in L^p(\mathbb{R}^{n+1})$, $1 \le p \le \infty$, we can check that $\pi(z,t)f$ converges to f in $L^p(\mathbb{R}^{n+1})$ as (z,t) goes to 0. The connection between the sublaplacian $\mathcal{L}$ and the Grushin operator G arises from the following. We can easily check that $\pi(X_j) = -\xi_j \frac{\partial}{\partial w}$ and $\pi(Y_j) = -\frac{\partial}{\partial \xi_j}$ where $X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t}$ and $Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t}$ are the vector fields on $\mathbb{H}^n$ which along with $T = \frac{\partial}{\partial t}$ form a basis for the Heisenberg Lie algebra. Thus we see that $\pi(\mathcal{L}) = G$ and this allows us to express certain functions of G in terms of operators related to $\mathcal{L}$. For example, the heat semigroup e^{-tG} generated by the Grushin operator can be written as

$$e^{-tG} f(\xi, w) = \int_{\mathbb{H}^n} q_t(z,a) \pi(z,a) f(\xi, w) dz da \tag{1}$$

where $q_t(z,a)$ stands for the heat kernel associated to $\mathcal{L}$. A simple proof of this goes as follows.

$$\int_{\mathbb{H}^n} q_t(z,a)\pi(z,a) f(\xi,w) dz da = \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} q_t(z,a) f(\xi - y, w - a - \xi \cdot x + \frac{1}{2} x \cdot y) dz da$$

which by Plancherel theorem for the Euclidean Fourier transform simplifies to

$$\int_{\mathbb{C}^n} \int_{-\infty}^{\infty} e^{-i\lambda w} e^{i\lambda(-x\cdot\xi + \frac{1}{2} x \cdot y)} q_t^\lambda(z) f^\lambda(\xi - y) d\lambda dz.$$

Recalling the definition of the Schrödinger representation $\pi_\lambda(z,a)$ of $\mathbb{H}^n$ (see [18]) and using the fact that $q_t(x+iy, a)$ is even in y we get

$$\int_{\mathbb{H}^n} q_t(z,a)\pi(z,a) f(\xi,w) dz da = \int_{-\infty}^{\infty} \int_{\mathbb{C}^n} e^{-i\lambda w} q_t^\lambda(z) \pi_\lambda(z,0) f^\lambda(\xi) dz d\lambda.$$

But it is well known that

$$\int_{\mathbb{C}^n} q_t^\lambda(z) \pi_\lambda(z,0) dz = \int_{\mathbb{H}^n} q_t(z,a) \pi_\lambda(z,a) dz da = e^{-tH(\lambda)}$$

(see Sections 2.8, 2.9 in [18]). In view of the spectral resolution of G we obtain the desired representation:

$$\int_{\mathbb{H}^n} q_t(z,a)\pi(z,a) f(\xi,w) dz da = e^{-tG} f(\xi,w).$$

The above representation gives us an easy proof of the fact that $e^{-tG} f$ converges to f in $L^p(\mathbb{R}^{n+1})$ for all $1 \le p < \infty$. This can be seen as follows. It is well known that q_t is a Schwartz class function $\mathbb{H}^n$ with $\int_{\mathbb{H}^n} q_t(z,a) dz da = 1$ and it satisfies $q_t(z,a) = t^{-n-1} q_1(t^{-1/2} z, t^{-1} a)$. Therefore, making a change of variables in (1) we get

$$e^{-tG}f - f = \int_{\mathbb{H}^n} q_1(z,a)\big(\pi(t^{1/2}z, ta)f - f\big)dz da \tag{2}$$

from which our claim is immediate. More generally, the following is true and we will make use of it later.

Lemma 1 *Suppose* $\varphi \in L^1(\mathbb{H}^n)$ *with* $\int_{\mathbb{H}^n} \varphi(z,a)dz da = c$ *and for* $t > 0$ *define* $\varphi_t(z,a) = t^{-n-1}\varphi(t^{-1/2}z, t^{-1}a)$. *Then for any* $f \in L^p(\mathbb{R}^{n+1}), 1 \le p < \infty, \pi(\varphi_t)f$ *converges to* cf *in the norm as* $t \to 0$.

3 An Extension Problem for the Grushin Operator

In this section we study the following extension problem for the Grushin operator G. Given $f \in L^p(\mathbb{R}^{n+1})$ we are interested in finding solutions $u(\xi, w, \rho)$ of the equation

$$\big(-G + \partial_\rho^2 + \frac{1-2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_w^2\big)u(\xi,w,\rho) = 0, \quad u(\xi,w,0) = f(\xi,w). \tag{3}$$

It might appear to be natural to study the extension problem

$$\big(-G + \partial_\rho^2 + \frac{1-2s}{\rho}\partial_\rho\big)u(\xi,w,\rho) = 0, \quad u(\xi,w,0) = f(\xi,w) \tag{4}$$

instead of the above. However, the problem (3) is more suitable for the study of trace Hardy and Hardy inequalities. The solutions of (4) are related to pure powers G^s of the Grushin operator whereas those of (3) are related to the conformally invariant fractional powers G_s (see Section 4 for the definitions of G^s and G_s). A solution of (4) is given by the following formula of Stinga–Torrea [15]:

$$u(\xi,w,\rho) = \frac{\rho^{2s}}{\Gamma(s)}\int_0^\infty e^{-\frac{1}{4t}\rho^2}e^{-tG}f(\xi,w)t^{-s-1}dt \tag{5}$$

where e^{-tG} is the heat semigroup generated by G. Then it is not difficult to see that $u(\xi,w,\rho)$ solves (4) and $u(\xi,w,\rho)$ converges to f in $L^p(\mathbb{R}^{n+1})$ as $\rho \to 0$ for $1 \le p < \infty$. It is also known, see [15], that $\rho^{1-2s}\partial_\rho u(\xi,w,\rho)$ converges to a constant multiple of $G^s f$ as $\rho \to 0$.

By modifying the Stinga–Torrea formula (5) we can also write down a solution of the extension problem (3). Let $p_{t,s}(\rho, w)$ be the heat kernel associated to the generalised sublaplacian (see [1])

$$\mathcal{L}(s) = \partial_\rho^2 + \frac{1+2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_w^2$$

on $\mathbb{R}^+ \times \mathbb{R}$. Then the solution of the above extension problem can be written down explicitly in terms of the function $e^{-tG}f$. Indeed, we have the following analogue of the Stinga–Torrea formula.

Theorem 2 *For $f \in L^p(\mathbb{R}^{n+1})$, $1 \leq p \leq \infty$ a solution of the extension problem (3) is given by*

$$u(\xi, w, \rho) = \rho^{2s} \int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w') e^{-tG} f(\xi, w - w') dw' dt. \tag{6}$$

As ρ tends to zero, the solution $u(\xi, w, \rho)$ converges to $C_s f$ in $L^p(\mathbb{R}^{n+1})$ for $1 \leq p < \infty$ where $C_s = \frac{1}{4}\Gamma(s)\pi^{-s-1}$.

Proof Applying G to the function u and noting that $e^{-tG}f(\xi, w)$ satisfies the heat equation $-Gu_t(\xi, w) = \partial_t u_t(\xi, w)$ we see that

$$Gu(\xi, w, \rho) = -\rho^{2s} \int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w') \partial_t e^{-tG} f(\xi, w - w') dw' dt.$$

Integrating by parts in the t variable we can transfer the t derivative to $p_{t,s}(\rho, w)$ and since it satisfies the heat equation associated to $\mathcal{L}(s)$ we obtain

$$Gu(\xi, w, \rho) = \rho^{2s} \big(\partial_\rho^2 + \frac{1 + 2s}{\rho} \partial_\rho + \frac{1}{4}\rho^2 \partial_w^2\big) \times \int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w') \partial_t e^{-tG} f(\xi, w - w') dw' dt.$$

A simple calculation shows that

$$\rho^{2s} \big(\partial_\rho^2 + \frac{1 + 2s}{\rho} \partial_\rho + \frac{1}{4}\rho^2 \partial_w^2\big) v(\xi, w, \rho) = \big(\partial_\rho^2 + \frac{1 - 2s}{\rho} \partial_\rho + \frac{1}{4}\rho^2 \partial_w^2\big) (\rho^{2s} v(\xi, w, \rho))$$

for any function $v(\xi, w, \rho)$. This proves that u satisfies the extension problem.

Since the heat semigroup e^{-tG} is contractive on L^p spaces it follows that

$$\|u(\cdot, \rho)\|_p \leq \rho^{2s} \Big(\int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w') dw' dt \Big) \|f\|_p.$$

We also know that (see [1])

$$\int_{-\infty}^\infty p_{t,s}(\rho, w) e^{i\lambda w} dw = (4\pi)^{-s-1} \Big(\frac{\lambda}{\sinh(t\lambda)} \Big)^{s+1} e^{-\frac{1}{4}\lambda \coth(t\lambda)\rho^2}.$$

In view of this we obtain

$$\|u(\cdot,\rho)\|_p \leq C\rho^{2s}\Big(\int_0^\infty e^{-\frac{1}{4t}\rho^2}t^{-s-1}dt\Big)\|f\|_p \leq C\|f\|_p.$$

In order to prove that $u(\cdot,\rho)$ converges to f we make use of the fact that $e^{-tG}f$ converges to f in $L^p(\mathbb{R}^{n+1})$ as t tends to zero for $1 \leq p < \infty$. From the explicit form of $p_{t,s}(\rho,w)$ we note that $p_{\rho^2 t,s}(\rho,w) = \rho^{-2s-4}p_{t,s}(1,w/\rho^2)$. Thus the solution u of the extension problem is given by the integral

$$u(\xi,w,\rho) = \rho^{-2}\int_0^\infty\int_{-\infty}^\infty p_{t,s}(1,w'/\rho^2)e^{-t\rho^2G}f(\xi,w-w')dw'dt.$$

Letting

$$\begin{aligned}C_s = \rho^{-2}&\int_0^\infty\int_{-\infty}^\infty p_{t,s}(1,w'/\rho^2)dw'dt\\ &= (4\pi)^{-s-1}\int_0^\infty t^{-s-1}e^{-\frac{1}{4t}}dt = \frac{1}{4}\Gamma(s)\pi^{-s-1}\end{aligned}$$

we write $u(\xi,w,\rho) - C_s f(\xi,w)$ as the sum of the following two terms:

$$I_1(\xi,w,\rho)=\rho^{-2}\int_0^\infty\int_{-\infty}^\infty p_{t,s}(1,w'/\rho^2)\big(e^{-t\rho^2G}f(\xi,w-w')-f(\xi,w-w')\big)dw'dt$$

and

$$I_2(\xi,w,\rho) = \rho^{-2}\int_0^\infty\int_{-\infty}^\infty p_{t,s}(1,w'/\rho^2)\big(f(\xi,w-w') - f(\xi,w)\big)dw'dt.$$

Clearly,

$$\|I_1\|_p \leq \rho^{-2}\int_0^\infty\int_{-\infty}^\infty p_{t,s}(1,w'/\rho^2)\|e^{-t\rho^2G}f - f\|_p dw'dt$$

and hence converges to zero as ρ goes to zero. On the other hand $\|I_2\|_p$ also converges to zero as translation is continuous on L^p and $p_{t,s}(\rho,w)$ satisfies the estimate $p_{t,s}(\rho,w) \leq Ct^{-s-1}e^{-\frac{c}{t}(\rho^2+|w|)}$ for some constants C and c. □

Remark 3 The solution of the extension problem for the Grushin operator given in (6) can be written as $\rho^{2s}\pi(\Phi_{s,\rho})$ for a suitable function $\Phi_{s,\rho}$ on the Heisenberg group. In fact let us define

$$\Phi_{s,\rho}(z,w) = \int_0^\infty\Big(\int_{-\infty}^\infty p_{t,s}(\rho,w')p_t(z,w-w')dw'\Big)dt.$$

Using the fact that $\pi(p_t) = e^{-tG}$ it can be easily shown that

$$\rho^{2s}\pi(\Phi_{s,\rho}) = \rho^{2s}\int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w')\pi(0, w')e^{-tG}dt.$$

Since $\pi(0, w')f(\xi, w) = f(\xi, w - w')$ it follows that

$$\rho^{2s}\pi(\Phi_{s,\rho})f(\xi, w) = \rho^{2s}\int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w')e^{-tG}f(\xi, w - w')dt \tag{7}$$

is the solution defined in (6). Using the homogeneity properties of the heat kernels $p_{t,s}$ and p_t we can check that $\rho^{2s}\Phi_{s,\rho}(z, w) = \rho^{-2n-2}\Phi_{s,1}(\rho^{-1}z, \rho^{-2}w)$ and $\|\Phi_{s,1}\|_{L^1(\mathbb{H}^n)} = \frac{1}{4}\Gamma(s)\pi^{-1-s}$. Thus the solution of the extension problem is given by

$$u(\xi, w, \rho) = \rho^{-2n-2}\int_{\mathbb{H}^n} \Phi_{s,1}(\rho^{-1}z, \rho^{-2}a)\pi(z, a)f(\xi, w)dzda \tag{8}$$

which gives, in view of Lemma 1, another proof that $u(\xi, w, \rho)$ converges to $\frac{1}{4}\Gamma(s)\pi^{-1-s}f(\xi, w)$ as ρ goes to 0.

Remark 4 We can also rewrite the solution in the form

$$u(\xi, w, \rho) = \int_{\mathbb{R}^{n+1}} K_\rho(\xi, y, w - w')f(y, w')dydw'$$

where the kernel K_ρ satisfies the homogeneity condition

$$K_\rho(x, y, w) = \rho^{-n-2}K_1(\rho^{-1}x, \rho^{-1}y, \rho^{-2}w).$$

The kernel K_ρ is expressible in terms of $\Phi_{s,\rho}$. We also remark that the functions $\Phi_{s,\rho}$ are known explicitly (see [13]). By using explicit formulas for the kernels $p_{t,s}$ and p_t we can calculate the above integral obtaining

$$\Phi_{s,\rho}(z, w) = C_1(n, s)\big((\rho^2 + |z|^2)^2 + w^2\big)^{-(n+1+s)/2},$$

where $C_1(n, s) = 2^{n+s-1}\pi^{-n-s-2}\Gamma\left(\frac{n+1+s}{2}\right)^2$.

In the above theorem we have shown that the solution defined by (6) satisfies the uniform estimates $\|u(\cdot, \rho)\|_p \leq C\|f\|_p$. It is therefore natural to ask if all the solutions of (3) satisfying the uniform estimates $\|u(\cdot, \rho)\|_p \leq C, \rho > 0$ are given by the formula (6) for some $f \in L^p(\mathbb{R}^{n+1})$.

Theorem 5 *Assume* $1 \leq p < \infty$ *and let* $u(\xi, w, \rho)$ *be a solution of the extension problem* (3) *which satisfies the uniform estimates* $\|u(\cdot, \rho)\|_p \leq C, \rho > 0$. *Then there exists a unique* $f \in L^p(\mathbb{R}^{n+1})$ *such that* u *can be expressed as in* (6).

Proof Under the hypothesis on u it follows that there is a subsequence ρ_k tending to 0 and an $f \in L^p(\mathbb{R}^{n+1})$ such that $u(\xi, w, \rho_k)$ converges to f weakly. With this f let us define

$$v(\xi, w, \rho) = \rho^{2s} \int_0^\infty \int_{-\infty}^\infty p_{t,s}(\rho, w')e^{-tG} f(\xi, w - w')dw'dt.$$

The theorem will follow once we show that $u = v$. In order to prove this we make use of the uniqueness theorem for solutions of the extension problem for the sublaplacian proved in [14]. This theorem for the sublaplacian was proved as an easy consequence of results from [3] and [7].

We make use of the fact that $\mathcal{L}(\pi(z, t)\varphi, \psi)) = (\pi(z, t)G\varphi, \psi)$ for any two functions φ, ψ on $\mathbb{R}^{n+1}$. Therefore, if $u(\xi, w, \rho)$ is a solution of the extension problem for the Grushin operator with initial value 0 then for any $\varphi \in L^{p'}(\mathbb{R}^{n+1})$

$$\big(- \mathcal{L} + \partial_\rho^2 + \frac{1 - 2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_t^2\big)\big(\pi(z, t)u(\cdot, \rho), \varphi\big)$$
$$= \big(\pi(z, t)(-G + \partial_\rho^2 + \frac{1 - 2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_w^2)u(\cdot, \rho), \varphi\big) = 0.$$

Hence the hypothesis on u shows that $\|(\pi(\cdot, \cdot)u(\cdot, \rho), \varphi)\|_\infty \le C$ and so by the uniqueness theorem for the sublaplacian (see Theorem 1.1 in [14]) we conclude that $(\pi(z, t)u(\cdot, \rho), \varphi) = 0$ for all φ and hence $u = 0$. □

4 Fractional Powers of the Grushin Operator

Given a bounded function m on the spectrum of G one can define the operator $m(G)$ via spectral theorem by

$$m(G)f(\xi, w) = (2\pi)^{-1} \int_{-\infty}^\infty e^{-i\lambda w} m(H(\lambda)) f^\lambda(x)d\lambda.$$

Thus we can think of $m(G)$ as an operator valued multiplier for the Euclidean Fourier transform on $\mathbb{R}$. Indeed, by identifying $L^2(\mathbb{R}^{n+1})$ with $L^2(\mathbb{R}, X)$ where $X = L^2(\mathbb{R}^n)$ the above can be rewritten as

$$m(G)F(w) = (2\pi)^{-1} \int_{-\infty}^\infty e^{-i\lambda w} M(\lambda)\hat{F}(\lambda)\, d\lambda$$

where $F(w)(\xi) = f(\xi, w)$ and $M(\lambda) = m(H(\lambda))$. Assuming that $m(H(\lambda))$ is a bounded linear operator on $X = L^2(\mathbb{R}^n)$ the above is precisely the definition of operator valued Fourier multipliers studied by L. Weis in [19]. The operator valued function $M(\lambda)$ is known as the multiplier corresponding to $m(G)$. With this terminology, the fractional powers G_s, $0 < s < 1$ are defined via the multiplier

$$M_s(\lambda) = (2|\lambda|)^s \sum_{k=0}^{\infty} \frac{\Gamma(\frac{2k+n+1+s}{2})}{\Gamma(\frac{2k+n+1-s}{2})} P_k(\lambda).$$

More explicitly,

$$G_s f(\xi, w) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\lambda w} (2|\lambda|)^s \Big(\sum_{k=0}^{\infty} \frac{\Gamma(\frac{2k+n+1+s}{2})}{\Gamma(\frac{2k+n+1-s}{2})} P_k(\lambda) \Big) f^\lambda(\xi) d\lambda.$$

Observe that $G_s f$ is well defined as an L^2 function under the assumption that $M_s(\lambda) f^\lambda(\xi)$ is an L^2 function of (ξ, λ) on $\mathbb{R}^{n+1}$. By Stirling's formula, $\frac{\Gamma(\frac{2k+n+1+s}{2})}{\Gamma(\frac{2k+n+1-s}{2})}$ behaves like $(2k+n)^s$ and hence $G_s f$ will be in $L^2(\mathbb{R}^{n+1})$ if for every λ, $H(\lambda)^s f^\lambda \in L^2(\mathbb{R}^n)$ and

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |H(\lambda)^s f^\lambda(\xi)|^2 d\xi d\lambda < \infty.$$

The domain of G_s consists precisely of those $f \in L^2(\mathbb{R}^{n+1})$ for which the above condition is satisfied. It is clear that all Schwartz functions are in the domain and therefore G_s is densely defined.

Returning to the solution of the extension problem (3) we can now prove the following result which is the analogue of the result proved in [4] (see equation (3.1)) for the Laplacian on $\mathbb{R}^n$.

Theorem 6 *Assume $0 < s < 1$ and let $f \in L^p \cap L^2(\mathbb{R}^{n+1})$ be such that $G_s f$ also belongs to $L^p \cap L^2(\mathbb{R}^{n+1})$. Let $u(\xi, w, \rho)$ be the solution of the extension problem (3) defined by (6). Then $-\rho^{1-2s} \partial_\rho u(\xi, w, \rho)$ converges to $B_s G_s f$ in $L^p \cap L^2(\mathbb{R}^{n+1})$ as ρ goes to 0, where $B_s = 2^{-1-2s} \pi^{-1-s} \Gamma(1-s)$.*

Proof In order to prove this theorem we make use of the formula (7) for the solution of the extension problem. We claim that there is an explicit constant $C_{n,s}$ such that $-\rho^{1-2s} \partial_\rho \big(\rho^{2s} \pi(\Phi_{s,\rho})\big) = C_{n,s} \pi(\psi_{s,\rho}) G_s$ as operators where $\psi_{s,\rho}(z, w) = \rho^{-2n-2} \psi_{s,1}(\rho^{-1} z, \rho^{-2} w)$ with $\|\psi_{s,1}\|_{L^1(\mathbb{H}^n)} = C_2(n, s)$, where

$$C_2(n, s) = 2^{-n+s} \pi^{n+1} \Gamma(1-s) / \Gamma\Big(\frac{n+1-s}{2}\Big)^2.$$

Once we have this claim, it follows from Remark 3 that $-\rho^{1-2s} \partial_\rho u(\xi, w, \rho) = C_{n,s} \pi(\psi_{s,\rho}) G_s f(\xi, w)$ and hence the theorem follows from Lemma 1 with $B_s = C_{n,s} C_2(n, s)$. In order to prove the claim, we make use of the formula

$$\mathcal{L}_s \Phi_{-s,\rho}(z, w) = (2\pi)^{2s} \rho^{2s} \Phi_{s,\rho}(z, w) \tag{9}$$

which has been proved in Cowling–Haagerup [6] (see also [5]). Here $\mathcal{L}_s$ is the conformally invariant fractional power of the sublaplacian which is defined by the relation $\widehat{\mathcal{L}_s f}(\lambda) = \hat{f}(\lambda) M_s(\lambda)$ where $M_s(\lambda)$ is the same family of operators used in the definition of G_s and $\hat{f}$ stands for the operator valued group Fourier transform of f on $\mathbb{H}^n$. From (9) we obtain

$$\pi(\Phi_{-s,\rho}) G_s f(\xi, w) = (2\pi)^{2s} \rho^{2s} \pi(\Phi_{s,\rho}) f(\xi, w) = (2\pi)^{2s} u(\xi, w, \rho). \tag{10}$$

In [14] the authors have calculated that $-\rho^{1-2s} \partial_\rho \varphi_{-s,\rho} = \rho^{-2n-2} \psi_{s,1}(\rho^{-1} z, \rho^{-2} w)$ for an explicit function $\psi_{s,1}$ and constant $C_2(n,s)$, where $\varphi_{s,\rho}(z,w) = ((\rho^2 + |z|^2)^2 + 16 w^2)^{-\frac{n+s+1}{2}}$. Thus differentiating both sides of (10) by ρ and multiplying by $-\rho^{1-2s}$, we obtain

$$C_1(n, -s) \pi(\psi_{s,p}) G_s f(\xi, w) = -(2\pi)^{2s} \rho^{1-2s} \partial_\rho u(\xi, w, \rho).$$

This proves our claim with $C_{n,s} = (2\pi)^{-2s} C_1(n, -s) = 2^{n-3s-1} \pi^{-n-s-2} \Gamma(\frac{n+1-s}{2})^2$. Finally we calculate B_s using the value of $C_2(n,s)$ calculated in [14]:

$$B_s = \left(2^{n-3s-1} \pi^{-n-s-2} \Gamma(\frac{n+1-s}{2})^2\right) \times \left(2^{-n+s} \pi^{n+1} \Gamma(1-s) / \Gamma(\frac{n+1-s}{2})^2\right)$$
$$= 2^{-1-2s} \pi^{-1-s} \Gamma(1-s).$$

The proof is complete. □

5 Trace Hardy and Hardy Inequalities

Consider the vector fields $X_j = \xi_j \frac{\partial}{\partial w}$, $Y_j = \frac{\partial}{\partial \xi_j}$ and $T = \frac{1}{2} \rho \frac{\partial}{\partial w}$ on $\mathbb{R}^{n+1} \times \mathbb{R}^+$. Let X be one of these vector fields. For real valued functions u, v defined on $\mathbb{R}^{n+1} \times \mathbb{R}^+$ consider

$$\int_0^\infty \int_{\mathbb{R}^{n+1}} \left(Xu - \frac{u}{v} Xv\right)^2 \rho^{1-2s} \, d\xi \, dw d\rho.$$

Using integration by parts and assuming that u and v are such that $\frac{u^2}{v} Xv$ vanishes at infinity, we have

$$\int_{\mathbb{R}^{n+1}} \frac{u}{v} Xu \, Xv \, d\xi dw = -\int_{\mathbb{R}^{n+1}} \frac{u}{v} Xu \, Xv \, d\xi dw - \int_{\mathbb{R}^{n+1}} u^2 X(\frac{1}{v} Xv) \, d\xi dw.$$

Simplifying, we get

$$\int_{\mathbb{R}^{n+1}} \frac{u^2}{v^2}(Xv)^2\, d\xi - 2\int_{\mathbb{R}^{n+1}} \frac{u}{v} Xu\, Xv\, d\xi = \int_{\mathbb{R}^{n+1}} \frac{u^2}{v} X^2 v\, d\xi. \tag{11}$$

Consequently,

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^{n+1}} \left(Xu - \frac{u}{v}Xv\right)^2 \rho^{1-2s}\, d\xi\, dw d\rho \\ &\quad = \int_0^\infty \int_{\mathbb{R}^{n+1}} (Xu)^2 \rho^{1-2s}\, d\xi\, dw d\rho + \int_0^\infty \int_{\mathbb{R}^{n+1}} \frac{u^2}{v}(X^2 v)\rho^{1-2s}\, d\xi\, dw d\rho.\end{aligned}$$

In a similar way, using integration by parts, we can check that

$$\begin{aligned}&\int_0^\infty \frac{u^2}{v^2}(\partial_\rho v)^2 \rho^{1-2s}\, d\rho - 2\int_0^\infty \frac{u}{v}\partial_\rho u\, \partial_\rho v\, \rho^{1-2s}\, d\rho \\ &\quad = \int_0^\infty \frac{u^2}{v}\partial_\rho(\rho^{1-2s}\partial_\rho v)\, d\rho + \lim_{\rho\to 0}\left(\frac{u^2}{v}\rho^{1-2s}\partial_\rho v\right)\end{aligned}$$

which leads to the equation

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^{n+1}} \left(\partial_\rho u - \frac{u}{v}\partial_\rho v\right)^2 \rho^{1-2s}\, d\xi\, d\rho \\ &\quad = \int_0^\infty \int_{\mathbb{R}^{n+1}} (\partial_\rho u)^2 \rho^{1-2s}\, d\xi\, d\rho + \int_0^\infty \int_{\mathbb{R}^{n+1}} \frac{u^2}{v}\partial_\rho(\rho^{1-2s}\partial_\rho v)\rho^{1-2s}\, d\xi\, d\rho \\ &\qquad + \int_{\mathbb{R}^{n+1}} \frac{u^2(\xi,0)}{v(\xi,0)} \lim_{\rho\to 0}(\rho^{1-2s}\partial_\rho v)(\xi,\rho)\, d\xi.\end{aligned}$$

Let us now consider the gradient

$$\nabla u = (X_1 u, \cdots, X_n u, Y_1 u, \cdots, Y_n u, \frac{1}{2}\rho\, \partial_w u, \partial_\rho u). \tag{12}$$

Adding the above equations we obtain the identity

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^{n+1}} |\nabla u - \frac{u}{v}\nabla v|^2 \rho^{1-2s}\, d\xi\, dw d\rho &= \int_0^\infty \int_{\mathbb{R}^{n+1}} |\nabla u|^2 \rho^{1-2s}\, d\xi\, dw d\rho \\ &+ \int_{\mathbb{R}^{n+1}} \frac{u^2(\xi,0)}{v(\xi,0)} \lim_{\rho\to 0}(\rho^{1-2s}\partial_\rho v)(\xi,\rho)\, d\xi\, dw d\rho \\ &+ \int_0^\infty \int_{\mathbb{R}^{n+1}} \frac{u^2}{v}(-G + \frac{1}{4}\rho^2\partial_w^2 + \partial_\rho)(\rho^{1-2s}\partial_\rho v) d\xi\, dw d\rho.\end{aligned}$$

If v solves the extension problem

$$\left(-G + \partial_\rho^2 + \frac{1-2s}{\rho}\partial_\rho + \frac{1}{4}\rho^2\partial_w^2\right) v(\xi,\rho) = 0,$$

then the last term in the above vanishes. As the left hand side is non-negative, this leads to the following inequality known as trace Hardy inequality in the literature.

Proposition 7 *Let u be a real valued compactly supported continuous function on $\mathbb{R}^{n+1} \times [0,\infty)$ which is smooth for $\rho > 0$. Let v be a real valued function which solves the extension problem for the Grushin operator. Then*

$$\int_0^\infty \int_{\mathbb{R}^{n+1}} |\nabla u(\xi,w,\rho)|^2 \rho^{1-2s}\, d\xi\, dw d\rho \geqslant -\int_{\mathbb{R}^{n+1}} \frac{u^2(\xi,w,0)}{v(\xi,w,0)} \lim_{\rho\to 0} (\rho^{1-2s}\partial_\rho v)(\xi,w,\rho)\, d\xi dw.$$

When v is the solution of the extension problem for the Grushin operator G with initial condition $\varphi \in L^2(\mathbb{R}^{n+1})$ defined by (6) then we have proved that $\lim_{\rho\to 0} \rho^{1-2s}\partial_\rho v = -B_s G_s \varphi$ where B_s is given in Theorem 4.1. Thus the inequality in the above proposition takes the form

$$\int_0^\infty \int_{\mathbb{R}^{n+1}} |\nabla u(\xi,w,\rho)|^2 \rho^{1-2s}\, d\xi\, dw d\rho \geqslant c_s \int_{\mathbb{R}^{n+1}} \frac{u^2(\xi,w,0)}{\varphi(\xi,w)} G_s\varphi(\xi,w,\rho)\, d\xi dw$$

where $c_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$. In the case of the sublaplacian on H-type groups, there are explicit functions $\varphi_{s,\delta}$ such that $\mathcal{L}_s\varphi_{-s,\delta} = \delta^s C_{n,s}\varphi_{s,\delta}$ with explicit constant $C_{n,s}$ which have allowed the authors in [14] to simplify the quotient $\frac{\mathcal{L}_s\varphi_{-s,\delta}}{\varphi_{-s,\delta}}$ to get a sharp trace Hardy inequality. Unfortunately in our context, though we can find analogues of $\varphi_{s,\delta}$ the quotient $\frac{G_s\varphi_{-s,\delta}}{\varphi_{-s,\delta}}$ does not seem to simplify. But things are not so bad if we slightly modify the definition of the fractional power G_s.

Recall that G_s is defined in terms of the multiplier

$$M_s(\lambda) = (2|\lambda|)^s \sum_{k=0}^\infty \frac{\Gamma(\frac{2k+n+1+s}{2})}{\Gamma(\frac{2k+n+1-s}{2})} P_k(\lambda).$$

Instead of this we use the slightly different multiplier

$$\tilde{M}_s(\lambda) = (2|\lambda|)^s \sum_{k=0}^\infty \frac{\Gamma(\frac{2k+n+2+2s}{4})}{\Gamma(\frac{2k+n+2-2s}{4})} P_k(\lambda)$$

in defining the modified fractional power $\tilde{G}_s$. Note that $\tilde{G}_s$ is nothing but $(\frac{1}{2}G)_s$. The operators G_s and $\tilde{G}_s$ are comparable. For $0 < s < 1$ let $C_1(s)$ and $C_2(s)$ be defined by

$$C_1(s) = \inf_k \frac{\Gamma(\frac{2k+n+1+s}{2})}{\Gamma(\frac{2k+n+1-s}{2})} \frac{\Gamma(\frac{2k+n+2-2s}{4})}{\Gamma(\frac{2k+n+2+2s}{4})}, \; C_2(s)^{-1} = \inf_k \frac{\Gamma(\frac{2k+n+1-s}{2})}{\Gamma(\frac{2k+n+1+s}{2})} \frac{\Gamma(\frac{2k+n+2+2s}{4})}{\Gamma(\frac{2k+n+2-2s}{4})}.$$

In view of Stirling's formula for the gamma function, these constants are positive and finite. Then it follows that

$$C_1(s)\langle \tilde{G}_s f, f\rangle \leq \langle G_s f, f\rangle \leq C_2(s)\langle \tilde{G}_s f, f\rangle. \tag{13}$$

The operator $\tilde{G}_s$ is better behaved as we can see from the following proposition.

For any $s \in \mathbb{R}$ and $\delta > 0$ let $u_{s,\delta}(x, w) = ((\delta + |\xi|^2)^2 + w^2)^{-\frac{n+2+2s}{4}}$ defined on $\mathbb{R}^{n+1}$. We have an explicit expression for the action of $\tilde{G}_s$ on $u_{-s,\delta}$.

Proposition 8 *For any $\delta > 0$ and $0 < s < 1$, we have $\tilde{G}_s u_{-s,\delta}(\xi, w) = C_{n,s}\delta^s u_{s,\delta}(\xi, w)$, where $C_{n,s} = \frac{2^{2s}\Gamma(\frac{n+2+2s}{4})^2}{\Gamma(\frac{n+2-2s}{4})^2}$.*

Proof Since $u_{-s,\delta}$ is radial in ξ the action of G on $u_{-s,\delta}$ is the same as that of the generalised sublaplacian $\mathcal{L}(n/2-1) = -\partial_r^2 - \frac{(n-1)}{r}\partial_r - r^2\partial_w^2$. Therefore, the result follows from Theorem 3.11 in [5]. This can be proved by expanding the function $u^{\lambda}_{-s,\delta}(r)$ in terms of Laguerre functions of type $(n/2-1)$. We leave the details to the reader. □

Using the above proposition it is easy to prove a Hardy inequality for the modified fractional power $\tilde{G}_s$. If we let T_s to stand for the operator

$$T_s f(\xi, w) = ((\delta + |\xi|^2)^2 + w^2)^{\frac{n+2}{4}} \tilde{G}_s\big(((\delta + |\xi|^2)^2 + w^2)^{-\frac{n+2}{4}} f\big)(\xi, w),$$

then it follows that

$$T_s((\delta + |\xi|^2)^2 + w^2)^{\frac{s}{2}} = C_{n,s}\delta^s((\delta + |\xi|^2)^2 + w^2)^{-\frac{s}{2}}.$$

Therefore, using Schur test we can prove the following inequality (see Section 5.1 in [13]).

Theorem 9 *Let $f \in L^2(\mathbb{R}^{n+1})$ be real valued and assume that $G_s f \in L^2(\mathbb{R}^{n+1})$. Then for any $\delta > 0$ we have the inequality*

$$\langle \tilde{G}_s f, f\rangle \geqslant A_1(n, s)\delta^s \int_{\mathbb{R}^{n+1}} \frac{(f(\xi, w))^2}{((\delta + |\xi|^2)^2 + w^2)^s}\, d\xi\, dw$$

where $A_1(n, s) = 4^s \frac{\Gamma(\frac{n+2+2s}{4})^2}{\Gamma(\frac{n+2-2s}{4})^2}$. The inequality is sharp and equality holds when $f = u_{-s,\delta}$.

Remark 10 In view of (13) we also have the Hardy inequality for G_s, namely

$$\langle G_s f, f\rangle \geqslant C_1(s)A_1(n,s)\delta^s \int_{\mathbb{R}^{n+1}} \frac{(f(\xi,w))^2}{((\delta+|\xi|^2)^2+w^2)^s}\, d\xi\, dw.$$

Finally, using the above Hardy inequality we can prove the following trace Hardy inequality for the Grushin operator G.

Theorem 11 *Let $\delta > 0$ and $0 < s < 1$. For any real valued compactly supported continuous function u on $\mathbb{R}^{n+1} \times [0,\infty)$ which is smooth for $\rho > 0$, we have the inequality*

$$\int_0^\infty \int_{\mathbb{R}^{n+1}} |\nabla u(\xi,w,\rho)|^2 \rho^{1-2s}\, d\xi\, dw d\rho \geqslant C_1(s)B_s A_1(n,s)\delta^s \int_{\mathbb{R}^{n+1}} \frac{u(\xi,w,0)^2}{((\delta+|\xi|^2)^2+w^2)^s}\, d\xi\, dw.$$

In view of Hardy's inequality for G_s all we have to do is to prove the following energy estimate for the Grushin operator. The following result is the analogue of Theorem 1.2 in [8] proved in the context of Heisenberg groups.

Theorem 12 *Let $\delta > 0$ and $0 < s < 1$. For any real valued compactly supported continuous function u on $\mathbb{R}^{n+1} \times [0,\infty)$ which is smooth for $\rho > 0$, we have the inequality*

$$\int_0^\infty \int_{\mathbb{R}^{n+1}} |\nabla u(\xi,w,\rho)|^2 \rho^{1-2s}\, d\xi\, dw d\rho \geqslant B_s \langle G_s f, f\rangle$$

where $f(\xi,w) = u(\xi,w,0)$ and $B_s = 2^{-1-2s}\pi^{-1-s}\Gamma(1-s)$.

Proof In proving this theorem we closely follow the proof of Theorem 1.2 in [8]. We therefore give only a sketch of the proof referring to [8] for details. In what follows we assume that u is smooth. The general case can be dealt with using an approximation argument.

Let $H^s(\mathbb{R}^{n+1})$ be the completion of $C_0^\infty(\mathbb{R}^{n+1})$ with respect to the norm $\|f\|_{(s)}^2 = \langle G_s f, f\rangle$. It can be verified that the dual of $H^s(\mathbb{R}^{n+1})$ is $H^{-s}(\mathbb{R}^{n+1})$. If $g \in H^{-s}(\mathbb{R}^{n+1})$, it follows that $h = G_s^{-1} g = G_{-s} g \in H^s(\mathbb{R}^{n+1})$. Let $H(\xi,w,\rho)$ be the solution of extension problem with initial condition h defined as in (6). In view of Theorem 2 and Theorem 6, $H(\xi,w,\rho)$ converges to $C_s h$ with $C_s = \frac{1}{4}\Gamma(s)\pi^{-s-1}$ and $-\rho^{1-2s}\partial_\rho H(\xi,w,\rho)$ converges to $B_s G_s h$ with $B_s = 2^{-1-2s}\pi^{-1-s}\Gamma(1-s)$. If we let $W(\xi,w,q) = H(2^{-1/2}\xi, 2^{-1}w, \rho)$ with $q = \rho^2/2$, then W satisfies the equation

$$2\big(q\partial_q^2 + (1-s)\partial_q + q\partial_w^2 - G\big)W(\xi,w,q) = 0. \tag{14}$$

Moreover, as $q \to 0$, we have $W(\xi, w, q) \to C_s h(\xi, w)$ in $H^s(\mathbb{R}^{n+1})$ and $-q^{1-s}\partial_q W(\xi, w, q) \to 2^{s-1} B_s G_s h(\xi, w)$ in $H^{-s}(\mathbb{R}^{n+1})$. We define $U(\xi, w, q) = u(\xi, w, \rho)$ with $q = \rho^2/2$ and proceed as in the proof of Theorem 1.2 in [8]. We leave the details to the reader. □

6 An Isometry Property of the Solution Operator Associated to the Extension Problem

In this section we will prove an isometry property of the solution operator associated to the extension problem for the Grushin operator. Such a property has been already proved in the context of $\mathbb{R}^n$ and $\mathbb{H}^n$ in [11], see also [14]. Let $u(\xi, w, \rho)$ be the solution of the extension problem given by (6). For $s > 0$, for which G_s makes sense (e.g. $0 < s < (n+1)$), recall that the Sobolev space $H^s(\mathbb{R}^{n+1})$ is defined as the completion of $C_0^\infty(\mathbb{R}^{n+1})$ under the norm

$$\|f\|_{(s)}^2 = \langle G_s f, f\rangle = \|G_s^{1/2} f\|_{L^2(\mathbb{R}^{n+1})}^2.$$

Let Φ_α^λ, $\alpha \in \mathbb{N}^n$ be the scaled Hermite functions which are eigenfunctions of scaled Hermite operator $H(\lambda) = -\Delta + \lambda^2|\xi|^2$. In view of the spectral decomposition of G_s we have that

$$\|f\|_{(s)}^2 = (2\pi)^{-1}\int_{-\infty}^{\infty}(2|\lambda|)^s\left(\sum_{\alpha\in\mathbb{N}^n}\frac{\Gamma(\frac{2|\alpha|+n+1+s}{2})}{\Gamma(\frac{2|\alpha|+n+1-s}{2})}|\langle f^\lambda, \Phi_\alpha^\lambda\rangle|^2\right)d\lambda.$$

We think of the solution $u(\xi, w, \rho)$ as a function on $\mathbb{R}^{n+3}$ which is radial in the third variable. Thus $U(\xi, w, \zeta) = u(\xi, w, |\zeta|)$ is a function on $\mathbb{R}^{n+3}$. For $(\alpha, \beta) \in \mathbb{N}^n \times \mathbb{N}^2$, let $\Phi_{\alpha,\beta}^\lambda(\xi, \zeta) = \Phi_\alpha^\lambda(\xi)\Phi_\beta^{\lambda/2}(\zeta)$, where $\Phi_\alpha^\lambda(\xi)$ and $\Phi_\beta^{\lambda/2}(\zeta)$ are Hermite functions on $\mathbb{R}^n$ and $\mathbb{R}^2$ respectively. Now we define the Sobolev space $\widetilde{H}^{s+1}(\mathbb{R}^{n+3})$ in terms of $\Phi_{\alpha,\beta}^\lambda(\xi, \zeta)$ as the space of all functions $U \in L^2(\mathbb{R}^{n+3})$ for which the following norm is finite.

$$\begin{aligned}&\|U\|_{\widetilde{H}^{s+1}(\mathbb{R}^{n+3})}^2\\ &= (2\pi)^{-1}\int_{-\infty}^{\infty}(2|\lambda|)^{s+1}\left(\sum_{(\alpha,\beta)\in\mathbb{N}^n\times\mathbb{N}^2}\frac{\Gamma(\frac{2|\alpha|+|\beta|+n+1+1+s+1}{2})}{\Gamma(\frac{2|\alpha|+|\beta|+n+1+1-s-1}{2})}|\langle U^\lambda, \Phi_{\alpha,\beta}^\lambda\rangle|^2\right)d\lambda\end{aligned}$$

where U^λ is defined as usual by

$$U^\lambda(\xi; \zeta) = \int_{-\infty}^{\infty} U(\xi, w, \zeta)e^{i\lambda w}d\lambda.$$

We begin with the following expansion of the function $U(\xi, w, \zeta)$ in terms of Hermite functions. We let $L(\lambda, a, b)$ be defined by

$$L(\lambda, a, b) = \int_0^\infty e^{-\lambda(2t+1)} t^{a-1}(1+t)^{-b}\, dt, \qquad \lambda > 0, a > 0, b \in \mathbb{C}.$$

Proposition 13 *If $U(\xi, w, \zeta) = u(\xi, w, |\zeta|)$, where $u(\xi, w, \rho)$ is the solution of the extension problem given in (6), then*

$$U^\lambda(\xi; \zeta) = \sum_\alpha a^\lambda_{\alpha,\zeta}(s)\langle f^\lambda, \Phi^\lambda_\alpha\rangle \Phi^\lambda_\alpha(\xi) \tag{15}$$

where the coefficients are given by

$$a^\lambda_{\alpha,\zeta}(s) = (4\pi)^{-s-1}(2|\lambda|)^s|\zeta|^{2s} L\Big(\frac{|\lambda||\zeta|^2}{4}, \frac{2|\alpha| + n + 1 + s}{2}, \frac{2|\alpha| + n + 1 - s}{2}\Big).$$

Proof It is easy to see that

$$U^\lambda(\xi, \zeta) = |\zeta|^{2s} \int_0^\infty p^\lambda_{t,s}(|\zeta|) e^{-tH(\lambda)} f^\lambda(\xi) dt \tag{16}$$

which follows from the spectral decomposition of G. We also have

$$e^{-tH(\lambda)} f^\lambda(\xi) = \sum_{\alpha\in\mathbb{N}^n} e^{-t(2|\alpha|+n)|\lambda|}\langle f^\lambda, \Phi^\lambda_\alpha\rangle \Phi^\lambda_\alpha(\xi) \tag{17}$$

and the heat kernel for the generalised sublaplacian is given by

$$p^\lambda_{t,s}(|\zeta|) = (4\pi)^{-s-1} \left(\frac{\lambda}{\sinh(t\lambda)}\right)^{s+1} e^{-\frac{1}{4}\lambda \coth(t\lambda)|\zeta|^2}. \tag{18}$$

We substitute (17) and (18) in (16) and we get

$$U^\lambda(\xi, \zeta) = \sum_\alpha a^\lambda_{\alpha,\zeta}(s)\langle f^\lambda, \Phi^\lambda_\alpha\rangle \Phi^\lambda_\alpha(\xi)$$

where the coefficients are given by the integral

$$a^\lambda_{\alpha,\zeta}(s) = (4\pi)^{-s-1}|\zeta|^{2s} \int_0^\infty \left(\frac{\lambda}{\sinh(t\lambda)}\right)^{s+1} e^{-\frac{1}{4}\lambda \coth(t\lambda)|\zeta|^2} e^{-t(2|\alpha|+n)|\lambda|} dt.$$