Dynamic Modeling and Econometrics in Economics and Finance 29

Giuseppe Orlando
Alexander N. Pisarchik
Ruedi Stoop *Editors*

Nonlinearities in Economics

An Interdisciplinary Approach to Economic Dynamics, Growth and Cycles

Springer

Dynamic Modeling and Econometrics in Economics and Finance

Volume 29

In recent years there has been a rapidly growing interest in the study of dynamic nonlinear phenomena in economic and financial theory, while at the same time econometricians and statisticians have been developing methods for modeling such phenomena. Despite the common focus of theorists and econometricians, both lines of research have had their own publication outlets. The new book series is designed to further the understanding of dynamic phenomena in economics and finance by bridging the gap between dynamic theory and empirics and to provide cross-fertilization between the two strands. The series will place particular focus on monographs, surveys, edited volumes, conference proceedings and handbooks on:

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Reviews

Finally, a volume is coming out in our Springer Series *Dynamic Modeling and Econometrics in Economics and Finance* that will be of great service to the community of nonlinear modelers in economics—with very comprehensive surveys on history, theory, and econometrics of complex systems.
Arnhold Professor of International Cooperation and Development at The New School for Social Research

New York, NY, USA — Willi Semmler
May 2021

Handling nonlinear dynamic phenomena has always been a difficult challenge in economics. The contributions in this volume present new perspectives and views—also from disciplines outside of economics—helping to tackle these challenges. The volume thus provides valuable impulses for advancing economic theory as well as empirical research.
Chair of Financial Econometrics - LMU Munich

Munich, Germany — Stefan Mittnik
May 2021

This highly-valuable book is a great entry-point for understanding the economy as a self-organizing non-linear dynamical system. This book not only introduces the reader to advanced techniques but also applies them to modern economic growth and business cycle models.

Princeton, New Jersey, USA — Markus Brunnermeier
May 2021

Preface

Dear Reader,

Rather unanticipated, this book became a demonstration of how fruitful scientific meetings can be if held among open-minded participants. During a recent Nonlinear Dynamics of Electronic Systems (NDES) conference, an inquisitive-minded economist from Bari (Giuseppe Orlando, G.O.) sparked the idea that economics should be treated along the guidelines that we use for the analysis of systems composed of strongly nonlinear subsystems, such as magnets, neurons, and more. Pretty much all of the people agreed that it was a pity that a book leading students of economics or similarly interested readers into such a direction seemed to be missing. Usually, things end after having arrived at such an agreement, in particular if you are not an economist. However, never-tired, incredibly patient, and gentle insistence by G.O. pushed a bunch of us into preparing for such a journey with him. This book is the result of the journey, where we have been strongly supported by former teachers and academic friends of G.O. Together, we hope that the collected volume will prove helpful in leading a young generation of scholars to new pathways of understanding economics. Specifically, the text aims at providing a bridge between nonlinear dynamics (with chaotic behaviour lurking in the background) towards opening new horizons of insight into economics, by the tools and concepts provided. Therefore, the book starts with a mathematical introduction into nonlinear dynamics, followed by a layout of signal analysis tools and their application to some well-known abstract models of economics generating chaotic response, before we present new insights into how we see economic dynamics. Finally, we present economics models that emerge in this perspective and demonstrate that real-world data supports their validity and usefulness. Since students are the strongest motivation and a main privilege of our profession, I take this opportunity to dedicate my part in this book to my late student and friend Clemens Wagner. Had not his heart suddenly stopped beating before we started writing this book, my contribution would have been with him. I would also like to strongly thank Celso Grebogi for his never-fading generous and unselfish support for everything that advances Nonlinear Science.

Without him, many exciting developments and events (not least the mentioned NDES conferences) would not have been achieved.

We hope that you will find the text inspiring and useful. Let even the parts that you might like less sharpen your view and become beneficial to you in this way as well.

Bari, Italy — Giuseppe Orlando
Pozuelo de Alarcón, Madrid, Spain — Alexander N. Pisarchik
Zürich, Switzerland — Ruedi Stoop
May 2021

Acknowledgments

This book was a long trip with great companions that strengthened both academic and human ties between authors and co-editors. To some extent, this was at the expense of our families, and of friends not engaged in this endeavour.

Giuseppe Orlando, in particular, would like to ask the indulgence of his children Niccolò Libero and Gaia Carmela Francesca who have many good reasons to complain about their father during this period. He would also like to express his sincere gratitude and appreciation to Carlo Lucheroni at the University of Camerino for numerous helpful comments, to Nicola Basile at the University of Bari for valuable suggestions provided, to Luigi Fortuna and his colleagues at the University of Catania for having organized the NDES2019—Nonlinear Dynamics of Electronic Systems Conference and to Edward Bace at the Middlesex University for his help in polishing some of the chapters. Sincere special thanks go to Michele Mininni at the University of Bari who has been a firm point of reference from G.O.'s early study days on. Last but not least, he expresses his profound obligation to his parents, as without support and love by the parents, not much can be accomplished during one's lifetime.

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Chapter 1
Introduction

Giuseppe Orlando, Alexander N. Pisarchik, and Ruedi Stoop

> *Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.*
> *Galileo Galilei*
> *The Assayer*

1.1 Nonlinearity and Unpredictability in Economics

Until the twentieth century, many mathematical economists, such as Francis Edgeworth, William Jevons, Alfred Marshall, Leon Walras, and Vilfredo Pareto, developed economic models based on principles of Newtonian mechanics by focusing exclusively on static states or equilibrium points. Later, their ideas culminated

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in a general equilibrium theory (see, for example, Debreu [6], Arrow and Hann [1], etc.). Only at the beginning of the twentieth century, to describe market dynamics in economic models of business cycles, economists did start using difference and differential equations. Ragnar Frisch [10], for example, suggested to study the evolution of economics in time from a dynamical systems viewpoint. The future state of such a system depends only on its past and present. Such a slush in economic theory is largely due to outstanding discoveries of mathematicians in the field of nonlinear dynamics made at the end of the nineteenth century. The most prominent achievements were the development of stability theory by Aleksandr Lyapunov and the solution of three-body problem by Henri Poincaré [32].

Three centuries later, Birkhoff showed that Poincaré's geometric intuition can be cast within a mathematically precise description. This could have been the early official birthday of chaos theory, if aspects of irregular, unpredictable dynamics were recognised as relevant for real-world experimental systems as well. If appropriately perceived, Van der Pol, who presented almost simultaneously with Birkhoff his electronic heart pacemaker circuit (showing, in addition to period "heartbeats" also chaotic behaviour), would have provided this experimental connection.

Originally, the term "chaos" was generally used for dynamical processes that lacked aspects of order that would render their dynamics easily seizable in the sense of predictability. As a consequence, the main understanding was, for a long time, that we would never be able to arrive at a thorough understanding of such phenomena. A chaotic system has specific properties that differ from a stochastic or noisy system. First of all, a chaotic motion is deterministic (in the sense that every event is physically determined by an unbroken chain of earlier events) and crucially depends on initial conditions.

The reason why this more precise notion of chaos took so long to penetrate into the standard science was twofold. First, apart from the mentioned more global approaches, the preferred mathematical analysis methods were linearisation around fixed points of the motion, from which the asymptotic dynamics were extracted. This local approach, however, is sufficient only for linear systems. Nonlinearity provides fixed-objects that are not present in linear systems, such as limit cycles, that are not captured by linearization methods.

Edward Norton Lorenz was among the first to discover in the 1960s a so-called strange attractor in a three-dimensional continuous-time dynamical system, when carrying out numerical experiments on convection flows [15]. Indeed, the advent of direct numerical simulations of differential equations on computers made it finally possible to access and to explore the chaos phenomenology, to anyone for whom computational power was available. Initially, Lorenz was able to publish his work in marginal journals only. Until well into the late 1970s, several physics scholars in the western world, in particular, Mitchell Feigenbaum [2, 7] established the breakthrough of the new view on dynamics. He drew the attention of the scientific society to the ubiquitousness of this phenomenon, and offered new pathways of how to characterise and to predict such phenomena. It would, however, be unfair to withhold the great contributions to this area by scientists of the former Soviet Union, such as Andrey Kolmogorov, Vladimir Arnold, Aleksandr Andronov, Oleksandr

Sharkovsky, and others. In fact, Feigenbaum's findings aroused interest and finally got accepted only after a presentation of Feigenbaum given to Soviet scientists. Through the research conducted by many scholars not following the scientific mainstream, and their collaborative efforts, it was finally shown that, seen on a higher level, chaotic dynamics is not only ordered [9] and therefore intelligible but also that it is deterministic and controllable [30, 33, 34].

It took some time to work out the fundamentals of chaotic dynamics and then, the analysis proceeded towards the description and the exploitation of chaos for applications. A field that seems particularly attractive and suitable for this is economics that, from its statistical description, can be expected to be organised along the fundamental principles of symmetry breaking and self-organisation as well, and manifests a behaviour that could, with an appropriate permitting analysis, be associated with chaotic processes. Economic dynamics is obviously nonlinear. In particular, it is characterised by cyclical fluctuations called "business cycles". Burns and Mitchell [4] define business cycles as a type of fluctuation which "consists of expansions occurring at about the same time in many economic activities, followed by similarly general recessions, contractions, and revivals which merge into the expansion phase of the next cycle" (where a recession is a negative variation of the economy for two consecutive quarters cf. Fig. 1.1).

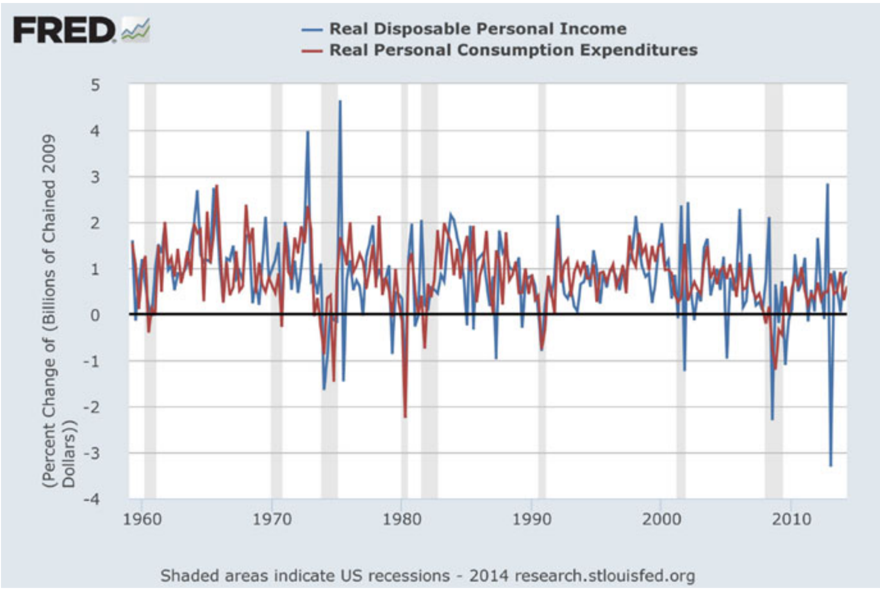

Fig. 1.1 Changes in US real disposable personal income (i.e., the personal income net of income taxes) (blue—DSPIC96) and Real personal consumption expenditures (red—PCECC96) 1959 (Q1)—2014 (Q2). Source: Federal Reserve Economic Data (FRED), St. Louis Fed. Greyed vertical areas correspond to periods of economic recessions (i.e., economic contraction) as reckoned by FRED (Table A.1)

The emergence of these very robust cycles is one strong motivation for our proposed change of paradigm. In contrast to the stochastic models focusing exclusively on external and random shocks (like the so-called real business cycle (RBC)), we propose to—alternatively or synergetically—consider structural system characteristics that are endogenous (in contrast to the exogenous influences considered in the traditional approaches). In that regard, by comparing an Ornstein–Uhlenbeck stochastic process [22, 23] with a Kaldor–Kalecki [17, 18] deterministic chaotic model, Orlando et al. [28] exhibited that nonlinearity in the latter model permits to represent reality at least as well as the stochastic model. Furthermore, the Kaldor–Kalecki model was able to reproduce an extreme event (black swan [28, 35]). A further confirmation can be found in Orlando et al. [29], where it was shown that real and simulated business cycle dynamics have similar characteristics, thus validating the chaotic model as a suitable tool to simulate reality. Notwithstanding economic dynamics is not purely deterministic, a strong stochastic component always exists and should be included in economic models. Therefore, statistical analysis of economic data is important to reveal the presence of determinism in the behaviour and to forecast future evolution. However, the effect of random fluctuations is unpredictable although probabilistic analysis in financial modelling can provide us with some information about possible scenarios. For example, increasing noise could indicate an impending financial crisis, because the dynamical systems are known [8, 14] to amplify random fluctuations when approaching a critical point. Another promising application of nonlinear analysis methods to economy would be the possibility to predict sudden jumps or drops in stock prices that occur for no apparent reason. By considering these jumps as extreme events, the same approach as for forecasting sudden weather changes [3], giant laser pulses [31], or forthcoming epileptic seizures [11] could be used. Such a research (not included in this book) is still under development.

1.2 Scheme of This Book

The whole endeavour taken by this work embraces (i) finding suitable models of business cycles, (ii) searching for indicators for structural changes in economic signals, and (iii) comparing time series generated by the studied models versus real-world time series [27–29]. Our book consists of four parts. Focusing on a particularly attractive class of economic nonlinear models related to growth and business cycles, we explore chaotic behaviour of these models by means of numerical analysis, recurrence quantification, and statistical techniques, and demonstrate that the results are consistent with those obtained by applying the same techniques to time series of real-world macroeconomic data. This implies that with the help of these methods we are able to (i) identify common features between data and model whenever they exist, (ii) to discover features that guide economic dynamics, and (iii) to extract indicators of structural changes in the signal (e.g., precursors of crashes and more general catastrophes). We emphasise the importance of nonlinear analysis and outline methodological prospects of dynamical approaches which can help to solve fundamental economic problems such as “Is the economy growing?”, “How efficiently are the resources being utilised?”, “Is economy stochastic or deterministic?”, and “Is it possible to predict changes in economy?” These problems cannot be solved by traditional methods of stochastic and linear analyses.

Part I is formal-methodological, providing the mathematical background for the remainder. In the aforementioned part we introduce the reader to the complex system theory and provide the theoretical basis for the rest of the book. This part is divided into Chaps. 2–7. Chapter 2 starts with basic definitions widely accepted in Nonlinear Dynamics, such as types of dynamical systems and attractors, and we derive Schwarz and Sarkovsky theorems that are fundamental theorems in chaos theory.

Then, using the generic example of the Logistic map, in Chap. 3 we demonstrate how the route to chaos proceeds if nonlinearity is increased via a cascade of period-doubling bifurcations when a control parameter is increased. Finally, we describe the main properties of chaos, using measures like information on the corresponding attractors. In Chap. 4 we provide the definition of homoclinic and heteroclinic orbits and we provide a summary on local bifurcations for both the continuous case and the discrete-time case. In Chap. 5, we pay a special tribute to the Hopf bifurcation from the applicative point of view. In fact, the Hopf theorem was one of the most powerful tools to prove the existence of closed orbits for systems of ordinary differential equations. Using two popular dynamical system models, i.e., the Lorenz and Rössler ones, we describe different types of bifurcations and chaotic attractors. Then, we discuss Shilnikov’s chaos through homoclinic orbits and emphasise the importance of the concept of delay-differential equations in economics. Special attention is given to delay-differential equations due to the fact that changes in economy are not immediate. Finally, brief comments on some applications of economic models are added at the end of each section. Chapter 6 first explains the concept of an embedding dimension. Then we discuss the deeper meaning of the term “chaos” and highlight its relation to the sensitive dependence on initial condition, stretching and folding of the phase-space. Statistical measurement characterising these features such as the Lyapunov exponents and measures of attractors are introduced. Finally, the last Chap. 7 is dedicated to the embedding dimension and the mutual information which are relevant for dealing with measured time series and experimental data.

Part II is divided into Chaps. 8–10. Chapter 8 presents a specific view of signal processing. After providing some basic definitions of signal processing, we explain the topic in more detail by a selection of relevant algorithms and approaches. In a nutshell, the idea is that signal processing is a process of computation, that computation describes the process of information destruction, and that the efficacy of this process can be measured. Chapter 9 explains what a signal in economy looks like and what physical properties it has, such as frequency, spectral energy, phase, power, etc. Furthermore, some examples are provided on how to analyse a signal. Chapter 10 is devoted to the so-called recurrence quantification analysis (RQA). We describe mathematical basis and requirements for the recurrent plot analysis. We explain how this method can be applied to detect spatio-temporal recurrent patterns underlying different dynamical regimes in economic time series. Such analysis allows us to reveal the nature of business cycles and corresponding macroeconomic variables, i.e., whether their character is deterministic or stochastic. Furthermore we show how RQA provides an indicator of structural changes in chaotic time series [25, 26].

As a book on economic dynamics, Part III focuses more on economics itself, providing more specifically economics-related background and literature. While in economics the phenomenon of chaos may be expected to contribute many fascinating aspects, to go beyond purely intellectual constructions with limited real-world explanatory strength, we provide the reader the background information regarding theories on growth and business cycles. In particular, Chap. 11 focuses on the ties between real-world economics and nonlinear dynamics. Chapter 12 provides a sketch of the Keynesian multiplier and of the multiplier-accelerator model by Hansen and Samuelson, and of the Kaldor model. Chapter 13 explains Domar's and the Harrod's model separately. In contrast to standard economic literature that glues those models together, in our view Harrod instability (tested in Chap. 18) has nothing to do with the mathematical notion of the instability characterising the Domar model. Chapter 14 is about the interpretation of cycles as a struggle between capitalists and workers. This is introduced by the Phillips curve (which statistically relates unemployment with the rate of change of nominal wages), followed by the Lotka–Volterra model which is the basic framework of the Goodwin model. The latter reinterprets, in economic terms, the dynamics of prey-predator of biological systems. Chapter 15 explains how control over a system that becomes unstable or highly noisy can be achieved. The objective is to calm down and optimise the system's behaviour. We show that such measures lead to stable cycles in any generic nonlinear system and that hard-limiter control follows a nongeneric dynamical system behaviour with a bifurcation cascade to chaos that is fundamentally distinct from the Feigenbaum case.

Part IV consists of Chaps. 16–20 that are devoted to new perspectives in understanding economics. A reality check on the theories discussed in earlier parts by the means described in Parts I–III is provided. Chapter 16 introduces the experimental Part of the book, following Goodwin's opinion that nonlinearities are the origin of oscillations in economics [12]. We use this approach to study cycles with Kaldor's framework, as is detailed in Chap. 12. The model presented is an alternative to the usual models available in literature [17, 18]. Additional features, such as a full set of parameters and the ability to embed randomness, open the way to a mixture of stochastic and deterministic chaos. Chapter 17 describes an indicator that exhibits structural changes in a signal/time series related to chaos [25]. To achieve that, RQA and statistical techniques are applied, to both real time series and model-generated time series [26]. Our aim is to (i) find common properties if and where they do exist, (ii) discover some hidden features of economic dynamics and (iii) highlight some indicators of structural changes in the signal (i.e., in our case to look for precursors of a crash). In Chap. 18 we focus on the Harrod's model detailed in Chap. 13. We present a specification of Harrod's model where chaos is a consequence of the gaps between actual, warranted and natural rates of growth. For this model, we prove that real-world economic dynamics can be replicated by a suitable calibration of the parameters of the model. Moreover, we prove that opening the economy to foreign trade can lead to reducing cyclical instability, thus confirming Harrod's conjecture [19]. Chapter 19 presents an extension of the Goodwin growth-cycle model that considers the rate of capacity utilisation as a new variable in an adapted Lotka–Volterra system of differential equations, where capacity utilisation is proportional to the difference between the output expansion function and capital accumulation. With this approach, connections between demand, labour market, and capital accumulation are established in a model that generates a cyclical pattern amongst the employment rate, the profit share, and capacity utilisation. The model is then tested against the US economy, using quarterly data from 1970 to 2019 and the Vector Auto-Regression (VAR). The latter is a stochastic process model widely used in econometrics to capture the linear interdependencies between time series. The conclusion is that positive profit share innovation affects positively both the employment rate and the rate of capacity utilisation, suggesting a profit-led pattern supporting the theoretical model presented (especially to the profit-squeeze mechanism).

Lastly, Chap. 20 summarises advances in nonlinear model predictive control (NMPC) through multi-regime cointegrated VAR (MRCIVAR). The study exhibits the impact of financial stress shocks and monetary policy at macroeconomic level in different countries. The chapter illustrates the vector error correction model (VECM), that is commonly used to model macroeconomic time series, because VECM is able to connect the economic theory around equilibrium and the dynamic process towards the equilibrium into a set of empirically testable relations. The said feature has been exploited to study business cycles during different phases by many (e.g., see Mittnik and Semmler [16], Chen et al. [5] and Hamilton [13]). MRCIVAR is used to examine the impact of real activities on the financial stress. The outcome is that financial shocks have asymmetric effects on the short term interest rate, depending on the regime the economy is in. More precisely, in the rate-cut regime a financial stress shock will decrease the short term rate while, in the non-rate-cut regime, the shock will increase the short term rate (even though in some cases the effects are not statistically significant).

We hope that, with this book we provide some food for thought to a wide audience and stimulate curiosity in approaching economics unconventionally by hybridisation with physics, engineering, and economics. We have left out our research on financial mathematics [20, 21, 23, 24] intentionally, as this matter runs parallel to the presented material as long as market stability, solvency, and resilience of financial institutions are concerned.

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Part I
Mathematical Background

Chapter 2
Dynamical Systems

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2.1 Dynamical Systems and Their Classification

The concept of dynamical system that we will use here is taken from R.E. Kalman [1] who introduced it in the 1960s while studying the problem of linear filtering and prediction.

Roughly speaking, a system consists of a set of the so-called states (generally vectors of real numbers), where the adjective *dynamics* emphasizes the fact that these states vary in time according to a suitable dynamical law. This concept of dynamical system is cast in the following definition.

Definition 2.1 (Dynamical System) A *dynamical system* is an entity defined by the following axioms:

1. There exist an ordered set T of times, a set X of states and a function ϕ from $T \times T \times X$ to X. ϕ is called a *state transition function*.
2. For all $t, \tau \in T$ and for all $x \in X$ one has that $\phi(t, \tau, x)$ represents the state at time t of a system whose initial state at time τ is x.
3. The function ϕ satisfies the following properties:

   Consistency: $\phi(\tau, \tau, x) = x$ for all $\tau \in T$, and for all $x \in X$.

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Composition: $\phi(t_3, t_1, x) = \phi\big(t_3, t_2, \phi(t_2, t_1, x)\big)$, for all $x \in X$ and for all $t_1, t_2, t_3 \in T$ with $t_1 < t_2 < t_3$.

In the following we always consider $X = \mathbb{R}^n$.

Definition 2.2 (Reversibility) If the state transition function ϕ defined for any (t, τ) in $T \times T$, once assigned the initial time τ and the initial state x, the state of the system is uniquely determined for the future (i.e. for all $t > \tau$), as well as for the past (i.e. for $t < \tau$), the system is said to be *reversible*.

If the state transition function ϕ is defined only for $t \geq \tau$, then the system is said to be *irreversible*.

Definition 2.3 (Event, Orbit and Flow) For all $t \in T$, $x \in X$, the pair (t, x) is called an *event*. Moreover, for τ and x fixed, the function $t \in T \mapsto \phi(t, \tau, x) \in X$ is called a *movement* of the system. The set of all movements is called a *flow*. The image of the movement, i.e. the set

$$\Big\{ \phi(t, \tau, x) \ \Big| \ t \in T \Big\},$$

is called an *orbit* (or a *trajectory*) of the system, i.e. the orbit passing through the state x at time τ.

It is not always possible to find a closed formula for the orbits of dynamical systems, but it is possible to study the behaviour of the orbits for long time nonetheless.

Definition 2.4 (Fixed or Equilibrium Point) A state $x^* \in X$ is called a *fixed point* (or an *equilibrium point*) of the dynamics, if there exist $t_1, t_2 \in T$, with $t_2 > t_1$, such that

$$\phi(t, t_1, x^*) = x^*, \qquad \text{for all } t \in T \cap [t_1, t_2]\,.$$

x^* is said to be a *fixed point in an infinite time* if there exists $t_1 > T$ such that

$$\phi(t, t_1, x^*) = x^* \qquad \text{for all } t \in T \cap [t_1, +\infty[\,.$$

Definition 2.5 (Eventually Fixed Orbit) An orbit is said to be *eventually fixed* if it contains a fixed point.

Definition 2.6 (Eventually Fixed Point) A point is called *eventually fixed* if its orbit is eventually fixed.

Definition 2.7 (Stability) The fixed point x^* is *stable* if for every $\varepsilon > 0$ there exist $\delta > 0$ and $t_0 \in T$ such that for all $x \in X$ with $|x - x^*| \leq \delta$,

$$\big|\phi(t, \tau, x) - x^*\big| \leq \varepsilon \qquad \text{holds for any } t > t_0.$$

The fixed point x^* is *asymptotically stable* if it is stable and there exists a $\delta > 0$ such that for all $x \in X$ with $|x - x^*| \leq \delta$ it holds that

$$\lim_{t\to\infty} \left|\phi(t, \tau, x) - x^*\right| = 0 \qquad \text{holds.}$$

The fixed point x^* is *globally asymptotically stable* if it is stable and

$$\lim_{t\to\infty} \left|\phi(t, \tau, x) - x^*\right| = 0\,, \qquad \text{for any } \tau \in T \text{ and } x \in X\,.$$

Definition 2.8 (Autonomous System) The system is called *autonomous* if

$$\phi(t, \tau, x) = \widetilde{\phi}(t - \tau, x) \tag{2.1}$$

for some suitable function $\widetilde{\phi}$.

That is, an autonomous system does not explicitly depend on the independent variable. If the variable is time (t), the system is called *time-invariant*. For example, the classical harmonic oscillator yields to an autonomous system. A nonautonomous system of n ordinary first order differential equations can be changed into an autonomous system, by enlarging its dimension using a trivial component, often of the form $x_{n+1} = t$.

Definition 2.9 (Discrete and Continuous System) The system is called *discrete*, if the time set T is a subset of the set of the integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The system is called *continuous* if T is an interval of real numbers.

In Sect. 2.2.1 we consider continuous-time dynamical systems, and in Sect. 2.5 we consider discrete-time dynamical systems.

2.2 Continuous-Time Dynamical

2.2.1 Continuous-Time Dynamical Systems from Ordinary Differential Equations

Let $I = [a, b] \subset \mathbb{R}$ and let $f : I \times \mathbb{R} \to \mathbb{R}$.

We recall the following version of the Cauchy–Lipschitz Theorem (see Bonsante and Da Prato[3]).

Theorem 2.1 (Cauchy–Lipschitz) *Assume that there exists $L > 0$ such that*

$$\left|f(t, x_1) - f(t, x_2)\right| \leq L|x_1 - x_2|\,, \tag{2.2}$$

for any $t \in I$ *and* $x_1, x_2 \in \mathbb{R}$. *Then for any* $\tau \in I$, $\xi \in \mathbb{R}$ *the Cauchy problem*

$$\begin{cases} \dot{x}(t) = f\big(t, x(t)\big) & , t \in I \, , \\ x(\tau) = \xi \end{cases} \tag{2.3}$$

has a unique solution in $[a, b]$.

From Theorem 2.1 it follows that the ordinary differential equation

$$\dot{x} = f(t, x) \tag{2.4}$$

defines a continuous reversible dynamic system. In fact the time set is $T = I$, the state set is $X = \mathbb{R}$ and the state transition function ϕ is the function from $I \times I \times \mathbb{R}$ to $\mathbb{R}$ such that for all $t, \tau \in I$, $\xi \in \mathbb{R}$ one has that

$$\phi(t, \tau, \xi) = x(t) \, ,$$

where $x(t)$ is the unique solution of the Cauchy problem (2.3). In this case the movements are the solutions to Eq. (2.4) and, for any solution x, the corresponding orbit is on the interval $\{x(t) \mid t \in I\}$.

The system is autonomous if, and only if, the function f does not depend explicitly on t, that is we have

$$\dot{x}(t) = f\big(x(t)\big)$$

(i.e. in the case of a differential equation of the form $\dot{x} = f(x)$, with $f : \mathbb{R} \to \mathbb{R}$ derivable function with continuous and bounded derivative), since in this case one has

$$\phi(t, \tau, x) = \phi(t - \tau, 0, x) \qquad \text{for all} \quad t, \tau, x \in \mathbb{R}. \tag{2.5}$$

An equilibrium point is a solution of the differential equation $\dot{x} = f(x)$, which is constant on the interval $J = [t_1, t_2] \subset I$. Hence, the equilibrium points of the system are the solutions $x^* \in \mathbb{R}$ of the equation $f(x) = 0$.

2.2.2 Continuous-Time Dynamical Systems from Systems of Ordinary Differential Equations

The discussion contained in the previous Sect. 2.2.1 for a single equation can be extended to systems of ordinary differential equations.

In fact, if $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, let $\boldsymbol{f} = \boldsymbol{f}(t, \boldsymbol{x})$ be a vector function from $I \times \mathbb{R}^n$ to $\mathbb{R}^n$, and let $f_1, f_2, \ldots, f_n$ be the components of $\boldsymbol{f}$.

Assume that $f_1, f_2, \ldots, f_n$ are continuous functions in $I \times \mathbb{R}^n$, that the partial derivatives of $f_1, f_2, \ldots, f_n$ with respect to all variables $x_1, x_2, \ldots, x_n$ exist and are continuous in $I \times \mathbb{R}^n$, and that these partial derivatives are bounded in $[a, b] \times \mathbb{R}^n$ for all $[a, b] \subset I$.

Then, for all $t_0 \in I$ and $\boldsymbol{x}_0 \in \mathbb{R}^n$ the Cauchy problem

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(t, \boldsymbol{x}(t)\big), & t \in I, \\ \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \end{cases} \tag{2.6}$$

has one and only one solution on the interval I.

Therefore, the system of ordinary differential equations

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}\big(t, \boldsymbol{x}(t)\big)$$

defines a *reversible* continuous dynamical system.

The time set is $T = I$, the state set is $X = \mathbb{R}^n$ and the state transition function is the mapping ϕ from $I \times I \times \mathbb{R}^n$ to $\mathbb{R}^n$ such that for all $t, \tau \in I$, $\boldsymbol{x} \in \mathbb{R}^n$ one has that $\phi(t, \tau, \boldsymbol{x})$ is the value in t of the unique solution of the Cauchy problem (2.6).

In this case, the movements are solutions of the system (2.6) and, for any solution $\boldsymbol{x}(t)$ of such a system of differential equations, the corresponding orbit is a curve in $\mathbb{R}^n$ of the parametric equation $\boldsymbol{x} = \boldsymbol{x}(t)$, $t \in I$.

As before, the system is autonomous if $\boldsymbol{f}$ is independent of t, i.e. in the case of a system of differential equations of the form $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$. In this case, an equilibrium point is a solution of the system of differential equations $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ that is constant on an interval $J \subset I$. Thus, the equilibrium points of the system are the solutions $\boldsymbol{x}^* \in \mathbb{R}^n$ of the system of equations $\boldsymbol{f}(\boldsymbol{x}^*) = \boldsymbol{0}$.

Remark 2.1 A nonautonomous system of n ordinary first order differential equations can be changed into an autonomous system, by enlarging its dimension using a trivial component, often of the form $x_{n+1} = t$.

Remark 2.2 The notion of dynamical system, as outlined in Definition 2.1, describes the case in which the evolution of the system depends only on internal causes.

However, there are situations where the evolution of the system can be modified through the action of external forces, i.e. by means of a time-dependent input vector function $\boldsymbol{u}$. In this case Definition 2.1 can be generalized in the sense that a dynamic system is characterized by a time set T, a state set X, an input set U with a set Ω of admissible input functions from T to U and a state transition function ϕ from $T \times T \times X \times \Omega$ to X such that for all $t, \tau \in T$, $\boldsymbol{x} \in X$, $\boldsymbol{u} \in \Omega$, $\phi(t, \tau, \boldsymbol{x}, \boldsymbol{u})$ represents the state of the system at time t, if the state is $\boldsymbol{x}$ at time τ with an input function $\boldsymbol{u}$ acting on the system.

Obviously, the state of the system at time t will only depend on the initial time τ, the initial state $\boldsymbol{x}$ and the restriction of the input function $\boldsymbol{u}$ to the interval of extremes t and τ. Hence, we have to assume that the state transition function ϕ satisfies the following properties.

Consistency: $\phi(\tau, \tau, \boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}(t) \quad \forall \quad (\tau, \boldsymbol{x}, \boldsymbol{u}(\cdot)) \in T \times X \times \Omega$.
Composition: $\phi(t_3, t_1, \boldsymbol{x}, \boldsymbol{u}) = \phi\left(t_3, t_2, \phi\left(t_2, t_1, \boldsymbol{x}, \boldsymbol{u}\right), \boldsymbol{u}\right)$ for each $(x, \boldsymbol{u}) \in X \times \Omega$, and for each $t_1 < t_2 < t_3$.
Causality: If $\boldsymbol{u}, \boldsymbol{v} \in \Omega$ and $\boldsymbol{u}|_{[\tau,t]} = \boldsymbol{v}|_{[\tau,t]}$, then $\phi(t, \tau, \boldsymbol{x}, \boldsymbol{u}) = \phi(t, \tau, \boldsymbol{x}, \boldsymbol{v})$.

This framework can be used for theoretical approaches in continuous-time systems.

2.3 Stability of Continuous-Time Systems

We recall some known facts about the *exponential* of square matrix.

Definition 2.10 (Exponential of Square Matrix) Given a square matrix A, the *exponential* of A is defined by

$$\exp(A) = \sum_{j=0}^{+\infty} \frac{1}{j!} A^j = I + A + \frac{1}{2} A^2 + \frac{1}{6} A^3 + \cdots + \frac{1}{j!} A^j + \cdots .$$

The basic properties of the exponential are listed below:

- If 0 is the null matrix, then $\exp(0) = I$.
- If A and B commute, that is $AB = BA$, then $\exp(A)\ \exp(B) = \exp(A + B)$.
  In particular $\exp(\alpha A)\ \exp(\beta A) = \exp\big((\alpha + \beta)A\big)$, for any $\alpha, \beta \in \mathbb{R}$.
- For any square matrix A, $\exp(A)$ is invertible; moreover

$$\big[\exp(A)\big]^{-1} = \exp(-A)\,.$$

- $\exp(A^T) = \exp(A)^T$.
- $\det\big(\exp(A)\big) = e^{\mathrm{tr}(A)}$, where $\mathrm{tr}(A)$ denotes the trace of A.
- If $B = PAP^{-1}$, where P is an invertible matrix, then

$$\exp(B) = P\ \exp(A)\ P^{-1}\,.$$

- If A is diagonal

$$A = \begin{pmatrix} \lambda_1 & 0 & & \dots & 0 \\ 0 & \lambda_2 & 0 & & \\ \vdots & & \ddots & & \vdots \\ 0 & \dots & & 0 & \lambda_n \end{pmatrix},$$

then

$$\exp(A) = \begin{pmatrix} e^{\lambda_1} & 0 & \dots & 0 \\ 0 & e^{\lambda_2} & 0 & \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_n} \end{pmatrix}.$$

When we deal with a linear system of differential equations expressed in matrix form as

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} \,,$$

A being a fixed matrix, the solution for the initial point $\boldsymbol{x}_0$ at $t = 0$ is given by

$$\boldsymbol{x}(t) = \exp(tA)\,\boldsymbol{x}_0 \,.$$

Indeed, as

$$\boldsymbol{x}(t) = \sum_{j=0}^{+\infty} \frac{t^j}{j!} A^j\,\boldsymbol{x}_0$$

we have

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \sum_{j=1}^{+\infty} \frac{t^{j-1}}{(j-1)!} A^j\,\boldsymbol{x}_0 = \sum_{j=0}^{+\infty} \frac{t^j}{j!} A^{j+1}\,\boldsymbol{x}_0 \\ &= A \sum_{j=0}^{+\infty} \frac{t^j}{j!} A^j\,\boldsymbol{x}_0 = A\,\boldsymbol{x}(t)\,. \end{aligned}$$

We can obtain the behaviour of the solution $\boldsymbol{x}(t)$ by studying the eigenvalues of the matrix A. Indeed, assume for example that A is diagonalizable and all the eigenvalues λ_j, $j = 1, \dots, n$, have negative real part. We then have

$$\boldsymbol{x}(t) = \exp(tA)\,\boldsymbol{x}_0 = P \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & e^{\lambda_n t} \end{pmatrix} P^{-1}\boldsymbol{x}_0 \,,$$

for some invertible matrix P. As $e^{\lambda_n t} \to 0$ for $t \to +\infty$, we see that the solution $\boldsymbol{x}(t) = \boldsymbol{0}$ is stable.

The Hartman–Grobman Theorem 2.2 given below will elucidate the behaviour around the fixed points of nonlinear systems by a linearization in a neighbourhood