

MATHEMATICAL
FOUNDATION OF
**RAILROAD VEHICLE
SYSTEMS**

GEOMETRY AND MECHANICS

AHMED A. SHABANA

WILEY

Mathematical Foundation of Railroad Vehicle Systems

Geometry and Mechanics

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Ahmed A. Shabana

Richard and Loan Hill Professor of Engineering
University of Illinois at Chicago
Chicago, Illinois, USA

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Preface

The design of passenger and freight railroad vehicle systems, which are complex transportation systems, involves many areas of science and engineering, including mechanical, civil, electrical, and electronic engineering. Because a comprehensive treatment of such complex systems cannot be covered in a single book, the focus of this book is on developing the mathematical foundation of railroad vehicle systems with an emphasis on the integration of *geometry* and *mechanics*. Such a geometry/mechanics integration is necessary for developing a sound mathematical foundation, accurate formulation of nonlinear dynamic equations, and general computational algorithms that can be used effectively in the virtual prototyping, analysis, design, and performance evaluation of railroad vehicle systems. Geometry is particularly important in the formulation of the railroad system dynamic equations of the motion generated by contact between the wheel and rail surfaces. The surface geometry, therefore, plays a fundamental role in formulating the kinematics, forces, and equations of motion of such systems. The theory of curves is equally important, and it is central in the description of the track geometry. For this reason, the subject of differential geometry plays a fundamental role in formulating the equations that govern the motion of railroad vehicle systems.

In addition to basic geometry concepts, principles of mechanics are required in order to formulate railroad kinematic relationships and dynamic equations of motion. These mechanics principles allow for systematically modeling motion constraints resulting from mechanical joints and specified motion trajectories. Railroad vehicles have components that experience large displacements, including finite rotations; therefore, it is necessary to avoid linearization of the kinematic and dynamic equations when developing general computational algorithms. In the mechanics approaches used in this book, a fully nonlinear motion description is used in the formulation of dynamic equations of motion. As demonstrated in this book, applying mechanics principles can lead to different forms of dynamic equations of motion. Regardless of the form of the equations of motion obtained, the integration of geometry and mechanics is necessary.

This book is written to complement a previous book, *Railroad Vehicle Dynamics: A Computational Approach*. Both books are designed to be self contained, and therefore, overlap of some topics cannot be avoided. This book is designed for an introductory course on railroad vehicle system dynamics suitable for senior undergraduate and first-year graduate students. It can also be used as a reference book by researchers and practicing engineers. The book consists of seven chapters that are organized to introduce the reader to the

basic concepts, formulations, and computational algorithms used in railroad vehicle system dynamics. Unlike other texts in the area of railroad vehicle dynamics, this book emphasizes geometry/mechanics integration throughout. It is shown how new mechanics-based approaches, such as the absolute nodal coordinate formulation (ANCF), can be used to achieve the geometry/mechanics integration necessary for developing accurate virtual prototyping algorithms.

Chapter 1 introduces the reader to several topics that are covered in subsequent chapters. Basic geometry concepts are discussed, and the need to integrate geometry and mechanics in the formulation of the railroad vehicle system equations is explained. The hunting oscillations that characterize the dynamic behavior of railroad vehicles and can lead to instabilities and derailments are discussed, and the methods used to describe wheel and track geometries are introduced. During curve negotiations, the effect of centrifugal forces must be taken into account to define the allowable balance speed that ensures the safe operation of the rail vehicle. This important topic, as well as the wheel/rail contact formulations and their integration with multibody system (MBS) computational algorithms, are introduced in Chapter 1. The chapter also discusses other important topics, including derailment criteria, high-speed rail, pantograph/catenary systems, and linear algebra, which can be conveniently used to formulate the kinematic and dynamic equations of railroad vehicle systems.

Chapter 2 covers topics in differential geometry that are fundamental in developing the nonlinear equations that govern the motion of the vehicle. Curve and surface geometries and their application to railroad vehicle systems are discussed. It is shown how the principal curvatures and principal directions of surfaces can be determined using the coefficients of the first and second fundamental forms of surfaces. The numerical representation of wheel and rail profile geometries using spline functions is discussed. The use of ANCF finite elements to describe the rail surface geometry is explained. It is shown that arbitrary surface geometry for the rail can be systematically described using the displacement field of fully parameterized ANCF finite elements.

Railroad vehicle motion and geometry descriptions are the subject of Chapter 3, which demonstrates how body motion equations and geometry are integrated. The coordinate systems used to describe rigid-body kinematics are first introduced, and different parameters for describing body orientation are presented. Among the orientation parameters discussed in this chapter are direction cosines, Euler angles, and Euler parameters. These orientation parameters are used to define the position, velocity, and acceleration equations in terms of the body generalized coordinates. The chapter explains the use of Euler angles in railroad vehicle system mechanics for two fundamentally different purposes: motion description and track geometry description. Using Euler angles as field variables to describe track geometry is explained. In Chapter 3, geometric motion constraints that result from mechanical joints used to connect vehicle components are introduced, and the trajectory coordinates used to develop specialized railroad vehicle system computer programs are defined.

While Chapters 2 and 3 deal, for the most part, with basic geometry and motion kinematic approaches and concepts, the focus of Chapter 4 is on railroad geometry. Wheel geometry equations are first defined, and it is shown how these geometry equations are used to define curvatures at an arbitrary point on the wheel surface. Two methods are introduced for describing rail geometry: semi-analytical and ANCF. Limitations of the semi-analytical

approach are discussed to provide a justification for using the more general ANCF approach to describe rail geometry. Using the ANCF interpolation to define the tangents and normal at an arbitrary point on the rail surface is explained. As discussed in this chapter, such an ANCF approach, which employs position gradients, allows using higher-order interpolation for the position coordinates and avoids the interpolation of rotations. Fully parametrized ANCF finite elements can be used to systematically define surfaces by introducing a functional relationship between the element spatial coordinates. The structure of the track data used in computer simulations of railroad vehicle systems is discussed, and methods for describing track deviations that represent track irregularities are presented. The chapter concludes by providing a comparison between the semi-analytical and ANCF approaches used to describe rail geometry.

The contact problem is covered in Chapter 5, starting with a discussion of the wheel/rail contact mechanism in order to provide a better understanding of the need to account for the creep phenomenon in railroad vehicle system applications. The constraint contact formulation (CCF) and the elastic contact formulation (ECF), which are used to determine the locations of wheel/rail contact points online, are introduced. As explained in this chapter, the CCF approach does not allow wheel/rail separation, but the ECF approach does allow this separation. Therefore, when the ECF approach is used, no degrees of freedom are eliminated. Determining the normal contact force in both the CCF and ECF approaches is discussed in this chapter, and it is explained how the surface geometry and Hertz contact theory can be used to determine the dimensions of the contact area. To determine the tangential creep forces, velocity creepages used in wheel/rail creep force formulations are defined. Several creep force formulations are discussed in Chapter 5, and the assumptions used in these formulations are highlighted. The chapter concludes with a discussion of magnetically levitated (maglev) trains.

Methods for developing the nonlinear dynamic equations of motion of railroad vehicle systems are introduced in Chapter 6. The Newtonian and Lagrangian approaches are compared to highlight the basic concepts used in the two approaches. As discussed in this chapter, the Lagrangian approach does not require the use of free-body diagrams or making cuts at the joints because this approach is based on using connectivity conditions. The constraint forces can be defined systematically using algebraic constraint equations, and therefore, the Lagrangian approach lends itself to developing general computational algorithms for railroad vehicle systems. Using generalized coordinates, the inertia and applied and contact forces can be formulated and used to develop equations of motion. The form of the equations of motion can be defined using the augmented formulation or the embedding technique. The augmented formulation leads to a large, sparse system expressed in terms of redundant coordinates and constraint forces, while the embedding technique leads to a minimum number of equations of motion expressed in terms of the system degrees of freedom; therefore, these equations do not include any constraint forces. The chapter also discusses the formulation of other force elements used in railroad vehicle dynamics and explains using trajectory coordinates to develop longitudinal train dynamics (LTD) algorithms. Simple models that can be used to study hunting oscillations are also developed in Chapter 6. The chapter concludes with a discussion of MBS modeling of electromechanical systems.

Pantograph/catenary systems that provide the power supply for high-speed trains are the subject of Chapter 7. The chapter discusses in detail their design and the formulation of catenary equations of motion using ANCF finite elements. The formulation of pantograph/catenary contact forces is presented using the constraint and elastic contact formulations. It is shown how to account for lateral relative sliding between the pan-head and catenary when using the constraint and elastic contact formulations. Chapter 6 also discusses pantograph/catenary force control, the effect of aerodynamic forces, and the wear problem when such current-collection systems are used for high-speed trains.

The development of the materials presented in this book is based on research collaboration between the author and many colleagues and students. The author would like to acknowledge the contributions of his colleagues and students including UIC current and former students and visiting scholars Zhengfeng Bai, Issac Banes, Dario Bettamin, Mahmoud Elbakly, Ahmed Eldeek, Sibi Kandasamy, Hao Ling, Ramon Martinez, Huailong Shi, and Dayu Zhang for their help in the preparation and proofreading of the book manuscript. The author would like to thank Drs. Khaled Zaazaa and Hiroyuki Sugiyama for past collaboration on writing the book *Railroad Vehicle Dynamics: A Computational Approach*. The editorial and production staff of Wiley & Sons deserve special thanks for their cooperation and professional work in producing this book. Finally, the author would like to thank his family for their patience and understanding during the time spent preparing this book.

Ahmed A. Shabana
Chicago, Illinois, USA

Chapter 1

INTRODUCTION

Passenger and freight railroad vehicles are complex transportation systems whose design involves many areas of science and engineering, including mechanical, civil, electrical, and electronic engineering. Because, a comprehensive treatment of such railroad vehicle systems cannot be covered in a single volume book, the focus of this book is on developing the mathematical foundation with the emphasis placed on the integration of *geometry* and *mechanics*.

Integration of Geometry and Analysis As will be explained in this book, a sound mathematical foundation, the formulation of nonlinear dynamic equations, and the design of general computational algorithms of such complex systems require the integration of basic concepts of geometry and mechanics. Geometry is particularly important in formulating the railroad system dynamic equations that govern the vehicle motion produced by contact between the wheel and rail surfaces. The surface geometry, therefore, plays a fundamental role in formulating the kinematics, forces, and equations of motion of such transportation systems. Without an accurate description of the surface geometry, it is not possible to develop accurate formulations and computational algorithms that account for the nonlinear dynamic behavior of railroad vehicle systems. The theory of curves is equally important, and it is central to the description of track geometry. High-speed trains are electrically powered using pantograph/catenary systems. As discussed in this book, the catenary system consists of wires whose geometry and deformations are critical in ensuring high-quality electric current collection. These catenary wires also define space curves whose geometry must be accurately described in order to better understand their dynamic behavior and their response to contact and aerodynamic forces.

While geometry enters into formulating railroad kinematic relationships and dynamic equations of motion, these equations are formulated using the principles of mechanics, which allow for systematically modeling motion constraints resulting from mechanical joints and specified motion trajectories. Because railroad vehicles have components that experience large displacements, including finite rotations, it is important to avoid linearization of the kinematic and dynamic equations when developing general computational algorithms. In the mechanics approaches used in this book, fully nonlinear motion descriptions are adopted in formulating the dynamic equations of motion. Applying the principles of mechanics can lead to different forms of the dynamic equations of motion, as discussed in this book. Regardless of the form of the equations of motion obtained, the

integration of geometry and mechanics is necessary in order to establish the foundation of those equations.

Passenger and Freight Trains The methods developed in this book are applicable to both passenger and freight railroad vehicle systems. These two types of systems can operate at significantly different speeds, have significantly different axle loads, and require the use of different track quality. Figure 1 shows a high-speed passenger train and a freight train that operate under different rules and guidelines and require different track-quality



(a)



(b)

Figure 1.1 Passenger and freight trains. Sources: (a) WANG SHIH-WEI/123 RF (2016); (b) Mike Danneman/Moment/Getty Images.

standards developed based on safety considerations. For both systems, however, regardless of the power supply needed for their operation, motion is generated by the force of interaction of wheels and rails like the ones shown in Figure 2. For the most part, freight trains use diesel engines and often run on lower-quality tracks, so their speeds cannot exceed a certain limit. High-speed trains, on the other hand, are electrically powered and can operate at a much higher speed as compared to freight trains. While the mathematical foundations developed in this book can be applied to both cases, it is important to recognize that practical issues must be taken into consideration when modeling these two different systems. While the basic geometric and mechanical formulations are the same, the analysis models, including dimensions, system configurations, number of rail cars, suspension characteristics, ride-comfort criteria, noise level, power supply, operating speeds, etc., can be significantly different. Developing accurate virtual prototyping models for both systems, based on proper integration of geometry and mechanics, is necessary in order to avoid deadly, costly, and environmentally damaging accidents. Passenger trains transport a large number of people around the world, while freight trains transport goods and hazardous materials. As the demand increases for higher-speed, heavier-axle loads; stricter operational and safety guidelines; less noise; a greater degree of comfort; and more robust transportation systems; more accurate virtual models with significant details are needed.



Figure 1.2 Wheel/rail contact.

Organization and Scope of This Chapter This chapter introduces the basic topics discussed in this book, starting with the *differential geometry* that covers the theory of curves and surfaces. The integration of *geometry* and *motion description* plays an important role in formulating the railroad vehicle equations of motion, as discussed in Section 1.1. In Section 1.2, the important topic of the *integration of geometry and mechanics* is discussed in more detail. Railroad wheelsets exhibit a type of vibration known as *hunting oscillations* during which the wheelset lateral displacement and yaw angle are related because of wheelset conicity. Hunting oscillations are discussed in Section 1.3 based on pure geometric considerations. Using a basic description of differential geometry and motion, *track and wheel geometries* can be defined, as explained in Section 1.4. Section 1.5 explains the effect of *centrifugal forces* during curve negotiations and presents a simple analysis for defining the *balance speed* that must not be exceeded by the vehicle when it travels on a curved section of track, to avoid derailment. Section 1.6 introduces the wheel/rail contact problem and the

forces that influence vehicle stability. While the main focus in this book is on trains driven by the wheel/rail contact forces, maglev trains are also briefly discussed in Section 1.6. In Section 1.7, the multibody system (MBS) approach for formulating governing dynamic equations using the principles of mechanics is discussed. Section 1.8 discusses existing *derailment criteria* and the role of three-dimensional computational dynamics in developing more accurate and general derailment criteria. Section 1.9 reviews topics related to the operation of high-speed trains, including the *pantograph/catenary systems*. Section 1.10 discusses some mathematical preliminaries that are used throughout the chapters and the notations adopted in this book.

1.1 DIFFERENTIAL GEOMETRY

Figure 3 shows a train negotiating a curved track, the wheel and rail that come into contact, and the pantograph/catenary system used to supply the electric power necessary for the operation of high-speed trains. The description of the motion of the vehicle on the track, the layout of the track, the mathematical definition of the surfaces of the wheel and rail that come into contact to produce the train motion, and the geometry and deformations of the catenary cables that carry the electric current necessary for the operation of the high-speed rail system are examples that demonstrate the importance of geometry in railroad vehicle system dynamics. For this reason, understanding the differential-geometry theories of curves and surfaces is the first step in correctly formulating the dynamic equations that govern the motion of these complex systems (Do Carmo 1976; Goetz 1970; Kreyszig 1991).

For example, the track and wheel geometries must be accurately described in order to correctly predict the wheel/rail contact forces. Using both the differential-geometry *theory of curves* and the *theory of surfaces* is necessary in order to be able to formulate the wheel and rail kinematic and force equations. These theories are used to define nonlinear geometric equations that can be solved for the locations of the wheel/rail contact points, as discussed in this book. Furthermore, the spatial wheel and rail geometric representation is crucial in the study of railroad vehicle nonlinear dynamics in different motion scenarios, including curve negotiations and travel on tracks with irregularities and worn profiles that influence ride comfort, vehicle stability, and safe operation. Therefore, geometric concepts are an integral part of formulating nonlinear dynamic equations and computational algorithms that can be used effectively to develop credible virtual prototyping models that contribute to developing operational and safety guidelines and to reducing the possibility of train derailments and accidents.

Theory of Curves While a distinction is made between curve and surface geometry and dynamic motion descriptions, simple motion examples can be used to explain some concepts that are used in this book. In this section, the motion of a particle is used to explain basic concepts related to the geometry of curves and surfaces. In general, three coordinates (parameters) are required in order to define the position of a particle in space. These three coordinates can be selected to describe the location of the particle with respect to the origin of a coordinate system formed by three orthogonal axes, as shown in Figure 4. In this case, the position of the particle is defined by the vector $\mathbf{r} = [x \ y \ z]^T$, where x , y , and z are



(a)



(b)

Figure 1.3 Geometry of railroad vehicle systems. Sources: (a) serjib74/Adobe Stock; (b) Susan Isakson/Alamy Stock Photo.

three independent parameters that define the position coordinates of the particle along the three orthogonal axes of the coordinate system XYZ . If the particle is constrained to trace a space curve like the one shown in Figure 4, the particle has the freedom to move only along the curve; the particle freedom to move along two directions perpendicular to the curve is eliminated. That is, the curve equation is completely defined in terms of one parameter, and the three coordinates of the particle are no longer independent.

To demonstrate this simple concept, consider a particle that traces the circle shown in Figure 5. The circle represents a curve with constant curvature defined by the radius of curvature a . If the motion of the particle is restricted to trace this circular curve, two constraints

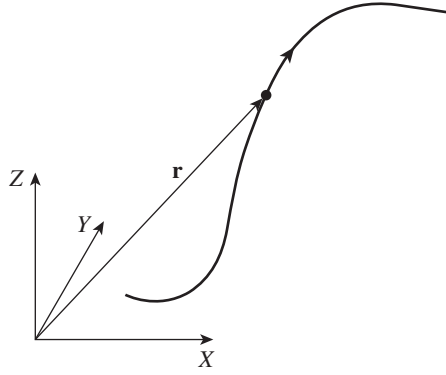


Figure 1.4 Curve geometry.

are imposed on that motion. These constraints, which relate the particle coordinates, can be described using the algebraic equations

$$(x)^2 + (y)^2 = (a)^2, \quad z = 0 \quad (1.1)$$

The first equation relates the coordinates x and y , while the second equation implies that the motion on the circle is planar. Using the first equation, the coordinate y can be written in terms of the coordinate x as $y = \pm\sqrt{(a)^2 - (x)^2}$. Therefore, the particle position, or the position of an arbitrary point on the curve $\mathbf{r} = [x \ y \ z]^T$, can be written as

$$\mathbf{r} = [x \ y \ z]^T = [x \ \pm\sqrt{(a)^2 - (x)^2} \ 0]^T \quad (1.2)$$

This equation shows that points on the curve can be traced if the parameter (coordinate) x is given, and the curve equation can be written in terms of one parameter, which is x in this example. Given the parameter x , the location of an arbitrary point on the curve can be completely determined.

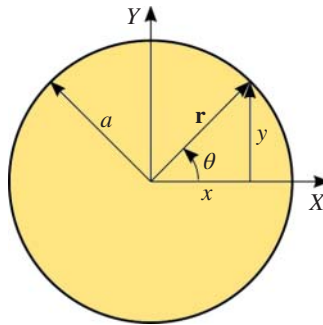


Figure 1.5 Circle geometry.

Curve Parameterization The curve parameter is not unique: that is, other parameters can be used to develop the curve equation. For example, the angle θ shown in Figure 5 can be used as the curve parameter. In this case, one must be able to write the curve Cartesian coordinates x , y , and z in terms of the parameter θ . It is clear from Figure 5 that the coordinates x and y can be written, respectively, in terms of the parameter θ as $x = a \cos \theta$

and $y = a \sin \theta$. Therefore, the curve equation can be written in terms of the parameter θ as

$$\mathbf{r} = [x \ y \ z]^T = [a \cos \theta \ a \sin \theta \ 0]^T \quad (1.3)$$

Equations 2 and 3 clearly show that the same curve can have different parameterizations. The different parameters, however, are related by algebraic equations, as previously explained using the simple example considered in this section. Equation 2 or 3 is called the curve equation in its *parametric form*. The equation $(x)^2 + (y)^2 = (a)^2$, which defines the circle equation, is called the curve equation in its *implicit form*.

Arc Length It is clear from the simple analysis presented in this section that different parameters can be used to define the parametric form of the curve equation. As discussed earlier, these parameters are related, as evidenced by the fact that one can write the parameter x in terms of the parameter θ as $x = a \cos \theta$, and the parameter θ can be written in terms of the parameter x as $\theta = \cos^{-1}(x/a)$. Similarly, a curve can be parameterized by its *arc length* s . This can be easily demonstrated by writing $\theta = s/a$. This equation can be substituted into Eq. 3 to obtain

$$\mathbf{r} = [x \ y \ z]^T = [a \cos(s/a) \ a \sin(s/a) \ 0]^T \quad (1.4)$$

Therefore, given any space curve, the curve parametric equation can be written in terms of one parameter as (Do Carmo, 1976; Goetz, 1970; Kreyszig, 1991)

$$\mathbf{r}(t) = [x(t) \ y(t) \ z(t)]^T \quad (1.5)$$

where t is the curve parameter. While a curve can be parameterized using any scalar variable, the curve *arc length* that measures the distance from the curve starting point to an arbitrary point on the curve is often used as the curve parameter. Using the parametric form of the curve expressed in terms of its arc length, a spatial curve can be uniquely defined in terms of geometric invariants called the *curvature* and *torsion*, which appear in the curve *Serret-Frenet equations* presented in Chapter 2. A curve can also be defined in its *implicit form* by eliminating the parameter to obtain a single equation expressed in terms of the curve coordinates, as previously demonstrated by the circular curve example.

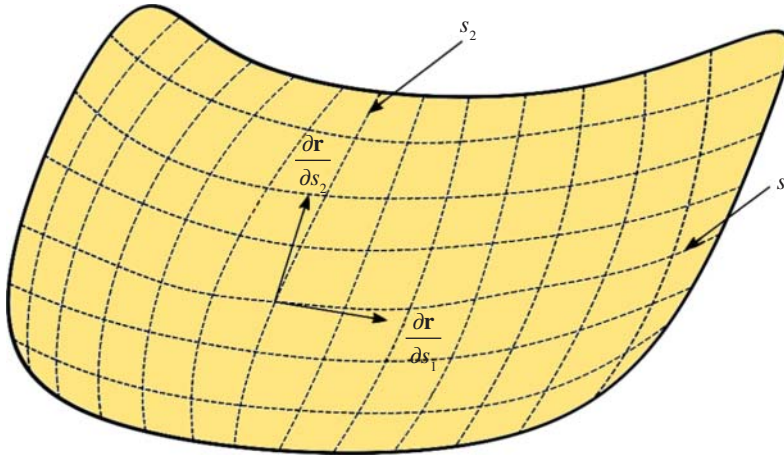


Figure 1.6 Surface geometry.

Theory of Surfaces Unlike curves, surface equations are defined in terms of two parameters, because on a surface, one has the freedom to move in two independent directions, as shown in Figure 6. For example, in the case of a rail surface, a wheel can roll and/or slide on the surface longitudinally or laterally. If the particle considered previously in this section is not constrained to move in the plane on the circular curve, the coordinate z is no longer constant, and such a coordinate can vary. In this case, there is only one constraint equation on the motion of the particle. This constraint equation is defined as $(x)^2 + (y)^2 = (a)^2$, and therefore the vector $\mathbf{r}$ can be written in terms of two independent coordinates x and z as

$$\mathbf{r}(x, z) = [x \ y \ z]^T = \left[x \ \pm \sqrt{(a)^2 - (x)^2} \ z \right]^T \quad (1.6)$$

It is clear that this equation, which defines the cylindrical surface shown in Figure 7, depends on the two independent *surface parameters* (coordinates) x and z . Once these two parameters are known, the coordinates of an arbitrary point on the surface can be determined. As in the case of curves, the surface parameterization is not unique. That is, other parameters such as θ and z can be used. In this case, the surface parametric equation can be written as

$$\mathbf{r}(\theta, z) = [x \ y \ z]^T = [a \cos \theta \ a \sin \theta \ z]^T \quad (1.7)$$

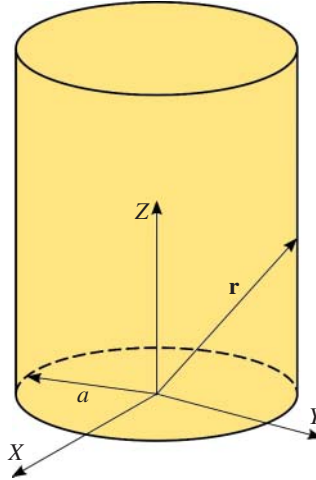


Figure 1.7 Cylindrical geometry.

In general, the surface equation can be written in its parametric form in terms of two independent parameters s_1 and s_2 as (Do Carmo, 1976; Goetz, 1970; Kreyszig, 1991)

$$\mathbf{r}(s_1, s_2) = [x(s_1, s_2) \ y(s_1, s_2) \ z(s_1, s_2)]^T \quad (1.8)$$

Therefore, for a surface, one can define two independent tangent vectors $\partial \mathbf{r} / \partial s_1$ and $\partial \mathbf{r} / \partial s_2$ which define what is called a *tangent plane*. The geometric properties of a surface are defined using the *first* and *second fundamental forms* of surfaces, which are presented in

Chapter 2. The coefficients of these fundamental forms are used to define the *principal curvatures* and *principal directions* that enter into formulating the wheel/rail contact force equations.

Computational Approach for Geometric Representations To use the theory of differential geometry in practical applications, polynomial approximations are used to describe curves and surfaces. In railroad vehicle dynamics, using polynomials allows for defining arbitrary rail and wheel profile geometries. In Chapter 2, a finite element (FE) approach called the *absolute nodal coordinate formulation* (ANCF) is used to define the rail surface geometry. Starting with the polynomial interpolation, the ANCF finite elements are developed by replacing the polynomial coefficients with position and position gradient coordinates. This allows for describing the position of arbitrary points on a continuum using the equation $\mathbf{r}(x, y, z, t) = \mathbf{S}(x, y, z)\mathbf{e}(t)$, where $\mathbf{S}$ is a shape function matrix, t is time, and $\mathbf{e}$ is the vector of position and position gradient coordinates. If ANCF finite elements are used to describe the geometry of fixed rigid rail, one has $\mathbf{r}(x, y, z) = \mathbf{S}(x, y, z)\mathbf{e}$. Using this approach, which enables integrating geometry and analysis, allows for defining a surface systematically by writing an algebraic equation in which one coordinate (parameter) can be expressed in terms of the other two coordinates. For example, one can write $z = f(x, y)$ and use this functional relationship to define the rail surface equation as

$$\mathbf{r}(x, y) = [x \ y \ f(x, y)]^T \quad (1.9)$$

In this surface equation, which can be conveniently defined using ANCF finite elements, only two parameters can be varied. Therefore, ANCF elements can be used to describe the geometries of curves and surfaces in their most general forms based on polynomial interpolations. The use of the ANCF position-gradient coordinates allows for conveniently describing complex shapes as well as the deformations in the case of flexible rails. The approach described in Chapter 2 also allows for using numerical or tabulated data to describe the surface geometry. The fact that one method can be used to define the geometry correctly and to accurately predict the deformation of the rail in the case of flexible rails allows for the systematic integration of geometry and the analysis of complex railroad vehicle systems, as discussed in more detail in the following section.

1.2 INTEGRATION OF GEOMETRY AND MECHANICS

The integration of geometry and mechanics represents the foundation for formulating the railroad vehicle system nonlinear dynamic equations of motion. The dynamic behavior and stability of the rail vehicle depend on the wheel/rail contact forces. These forces are functions of the geometry of the wheel and rail surfaces, which can be described using the techniques of differential geometry as well as computational geometric methods based on polynomial interpolations, as discussed in the preceding section. The track geometry also has a significant impact on rail-vehicle motion and stability. *Track irregularities* can influence vehicle dynamics and be a source of derailments and serious accidents when the vehicle negotiates curved and straight tracks. Therefore, the geometries of these irregularities need to be accurately represented in the simulation models in order to be able to predict their effect on overall vehicle behavior and nonlinear dynamics.

When the vehicle negotiates a curved track, the effect of *centrifugal forces* must be taken into account. To avoid derailments as the result of high centrifugal forces, the geometry of the track is altered by providing a track elevation that results in a lateral gravity-force component that opposes and balances the centrifugal forces, as discussed in this chapter. Accurate prediction of the effect of the centrifugal forces requires an accurate representation of the track geometry. Curved track sections can consist of curves with constant curvatures, and *spirals* that have curvatures that vary along the track. In the case of spirals, the radius of curvature is not constant, and consequently, the centrifugal force does not remain constant. Therefore, in railroad vehicle dynamics, geometry, motion descriptions, and force formulations are interrelated and cannot be separated.

General Displacement In the general case of unconstrained motion, the displacement of a rigid body in space can be described using six independent coordinates. Three coordinates define the global position of a point on the body, called the *body reference point*, and three coordinates define the orientation of the body with respect to the global coordinate system. The global position of the body reference point can be defined using three Cartesian coordinates. The orientation coordinates, on the other hand, can be introduced using three independent parameters that can represent angles or can be parameters that do not have an obvious physical meaning. Therefore, in spatial analysis, the orientation parameters are not unique, and different sets of parameters have been used in the literature and in developing computational *multibody system* (MBS) algorithms.

To define the configuration of a component (body) i in a vehicle system, two coordinate systems are first introduced, as shown in Figure 8. The first coordinate system is the global XYZ coordinate system, which is assumed fixed in time, while the second coordinate system $X^iY^iZ^i$ is the body coordinate system, which is assumed to be rigidly attached to the body *reference point* O^i . Using these two coordinate systems, the global position vector $\mathbf{r}^i$ of an arbitrary point on the rigid body i in the vehicle system can be written as

$$\mathbf{r}^i = \mathbf{R}^i + \mathbf{u}^i \quad (1.10)$$

where $\mathbf{R}^i = [R_x^i \ R_y^i \ R_z^i]^T$ is the global position vector of the body reference point O^i , and $\mathbf{u}^i = [u_x^i \ u_y^i \ u_z^i]^T$ defines the location of the arbitrary point with respect to the origin of the body coordinate system $X^iY^iZ^i$ in the global system: that is,

$$\mathbf{u}^i = [u_x^i \ u_y^i \ u_z^i]^T = u_x^i \mathbf{i} + u_y^i \mathbf{j} + u_z^i \mathbf{k} \quad (1.11)$$

In this equation, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are, respectively, unit vectors along the global axes X , Y , and Z . As discussed in Chapter 3, the vector $\mathbf{u}^i$ can be written in terms of constant components defined in the body coordinate system $X^iY^iZ^i$. This can be achieved by developing a transformation matrix that defines the body orientation. The columns of the transformation matrix define orthogonal unit vectors along the axes of the body coordinate system. While the body transformation matrix can be expressed in terms of three independent orientation parameters such as Euler angles or any other sets of parameters, the elements of the transformation matrix must assume the same numerical values regardless of the orientation parameters used. These elements of the transformation matrix, as discussed in Chapter 3, are the direction cosines of unit vectors along the axes of the body coordinate system $X^iY^iZ^i$.

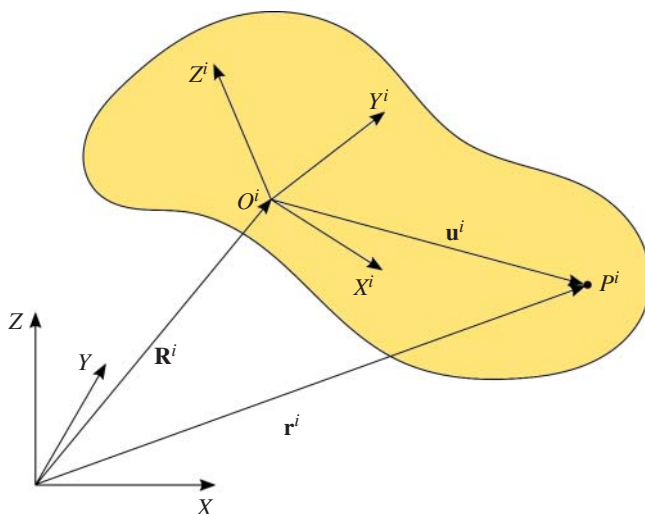


Figure 1.8 Coordinate systems.

Angular Velocity and Orientation Parameters By differentiating Eq. 10 once and twice with respect to time, the absolute velocity and acceleration vectors of the arbitrary point on the body can be defined. The derivative of the transformation matrix with respect to time can be used to define the angular velocity vector, as discussed in Chapter 3. In spatial analysis, the angular velocities are not *exact differentials*, and therefore they are not the time derivatives of orientation parameters. That is, the angular velocities cannot be directly integrated to determine the orientation parameters. Nonetheless, the angular velocities can always be written as linear functions of the time derivatives of the orientation parameters using a velocity transformation matrix. This velocity transformation matrix plays a fundamental role in determining the *generalized forces* associated with the orientation parameters since these orientation parameters serve as *generalized coordinates* and are not directly associated with the Cartesian moments applied to the bodies, as will be discussed in Chapter 6.

The kinematic description that will be used in this book to develop the equations of motion of the components of railroad vehicle systems is introduced in Chapter 3. It is shown in Chapter 3 that the use of three parameters, such as *Euler angles* (Greenwood, 1988; Huston, 1990; Roberson and Schwertassek, 1988; Rosenberg, 1977), to define the body orientation in space leads to kinematic singularities. Such singularities, however, can be avoided by using the four *Euler parameters* at the expense of adding an algebraic constraint equation that relates the four Euler parameters. Euler parameters, which are becoming more popular in developing general MBS algorithms, have many identities that can be used to simplify the kinematic and dynamic equations of the railroad vehicle system.

Euler Angles and Track Geometry In addition to using Euler angles to describe time-dependent motion by defining the orientation of bodies in space, these angles have also been used in railroad vehicle dynamics to define the geometry of the track based on given simple industry inputs. For the most part, track is constructed using three main segments: *tangent* (straight), *curve*, and *spiral*, as shown in Figure 9. The tangent segment has zero curvature,

the curve segment has constant curvature, and the spiral segment, used to connect two segments with different curvature values, has a curvature that varies linearly along the track to ensure a smooth transition between the two segments connected by the spiral. The track geometry is often described using three inputs at points along the track at which the geometry changes. These three inputs are the *horizontal curvature*, *superelevation*, and *grade*, and they can be expressed in terms of three Euler angles that are used to construct the track and rail space curves. To this end, Euler angles are converted to *field variables* and used systematically to construct a curve with well-defined geometry based on the given simple track inputs. Unlike the three Euler angles used to describe the time-dependent motion of an



(a)



(b)

Figure 1.9 Track segments. Sources: (a) Dinodia Photos/Alamy Stock Photo. (b) Jens Teichmann/Adobe Stock.

unconstrained body in space, the three Euler angles used to describe the geometry of a curve are written in terms of one parameter that can be the arc length. Therefore, when Euler angles are used to describe curve geometry, these angles are converted to field variables expressed in terms of the curve arc length to ensure a unique definition of the geometry.

Therefore, it is important to recognize that Euler angles are used in this book for two fundamentally different purposes: (i) as motion-generalized coordinates to describe rigid body kinematics in space; and (ii) as geometric field variables to uniquely define the geometry of the track and rail space curves. The analysis presented in Chapter 3 is used as the basis for a computer procedure for developing the track geometry data required for nonlinear dynamic simulations of railroad vehicle systems. The data can be generated before the dynamic simulation at a preprocessing stage in a *track preprocessor* computer program, as will be explained in Chapter 4. The track preprocessor output file normally has data for three different space curves: the *track centerline space curve*, the *right rail space curve*, and the *left rail space curve*, as shown in Figure 10, in which R_H is the radius of curvature of the track centerline curve. These three curves can have different geometries. The right and left rail space curves are used in formulating the wheel/rail contact conditions, while the track space curve is used in the definition of the distance traveled and in the motion description of the coordinate systems of the vehicle components.

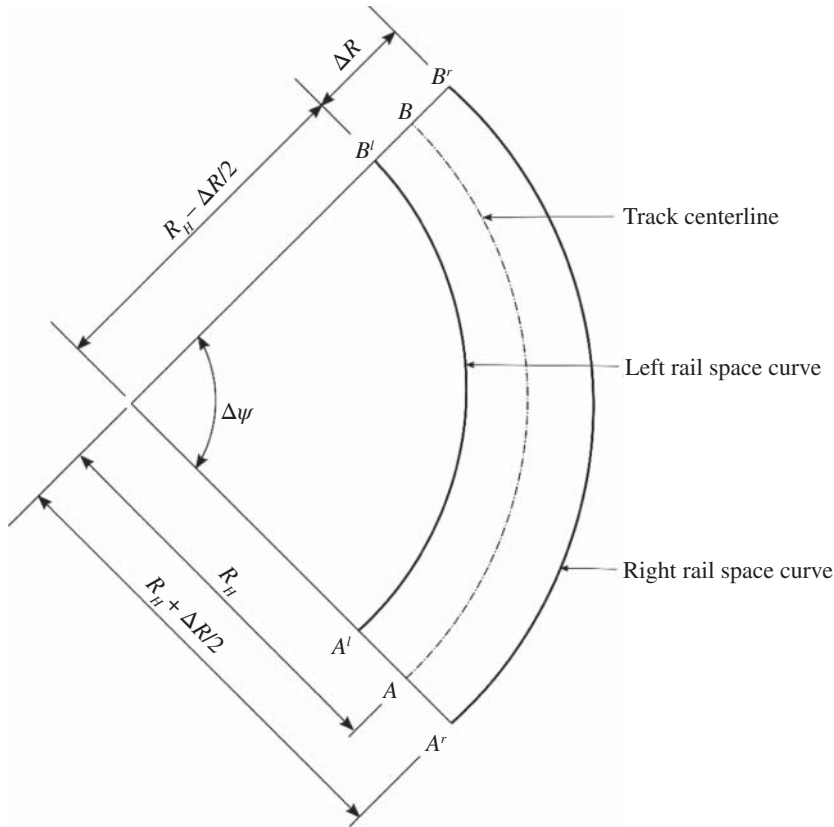


Figure 1.10 Track space curves.

1.3 HUNTING OSCILLATIONS

A simple analysis based on pure kinematics and geometric considerations can be used to shed light on the dynamics of railroad vehicle systems without consideration of the forces. In most railroad vehicle systems, a wheelset consists of two wheels connected by a stiff axle, as shown in Figure 11. The wheels are assumed to have conical profiles with the larger diameter close to the flange in order to achieve self-centering and minimize flange contact (Karnopp, 2004). Lateral wheelset oscillations with respect to the track centerline are referred to as *hunting*. During hunting oscillations, there is a relationship between the wheelset *lateral displacement* and the *yaw angle*, which represents the rotation of the wheelset about an axis normal to the track structure. In this section, a simple analysis based on pure geometry is used to demonstrate the relationship between the lateral displacement and yaw angle of the wheelset when it exhibits hunting oscillations. Such oscillations play a fundamental role in the stability of railroad vehicle systems. As will be demonstrated in this section, the hunting frequency is a function of the forward velocity of the wheelset as well as some other geometric parameters, including the wheel conicity, nominal rolling radius, and distance between the two rails.

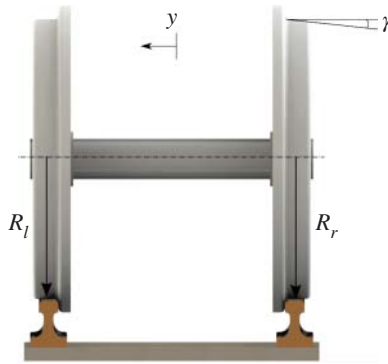


Figure 1.11 Railroad wheelset.

To provide an example of this simple geometric analysis, the wheelset shown in Figure 11 is considered. As shown in the figure, the wheelset conicity is denoted as γ , which defines the slope of the wheel profile curve. The lateral displacement of the wheelset center of mass is denoted as y . Before displacement, the wheelset is assumed to be centered and the displacement y is assumed to be zero: that is, $y = 0$. At this initial configuration, the radii of the two wheels at the points of contact with the rails are equal and denoted as R_o . As a result of disturbances that can be attributed to initial conditions or rail irregularities, the rolling radii of the two wheels will deviate from R_o as the wheelset starts to move forward. These rolling radii are denoted as R_r and R_l for the right and left wheels, respectively. As a result of a lateral displacement y , the rolling radii of the two wheels change, and such a change in the rolling radii is defined by $\Delta R = y\gamma$. It follows from simple geometry that $R_r = R_o - y\gamma$ and $R_l = R_o + y\gamma$. If the wheelset is assumed to rotate with a constant angular velocity ω ,

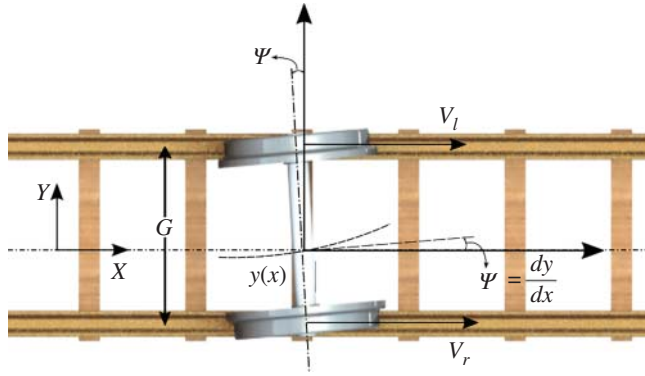


Figure 1.12 Hunting oscillations.

the forward velocities of the right and left wheels can be written, respectively, as

$$\left. \begin{aligned} V_r &= \omega R_r = \omega (R_o - y\gamma) \\ V_l &= \omega R_l = \omega (R_o + y\gamma) \end{aligned} \right\} \quad (1.12)$$

This equation shows that, during hunting oscillations, the two wheels have different forward velocities, and this gives rise to a yaw angle ψ , as shown in Figure 12. Nonetheless, the forward velocity of the wheelset center of mass remains constant and is always defined by the following equation:

$$V = (V_r + V_l) / 2 = \omega R_o \quad (1.13)$$

Using the small oscillation assumption, one can write $\tan \psi = dy/dx \approx \psi$. Because one can write, using Eq. 12, $V_r - V_l = -2y\omega\gamma$, it follows that

$$\left. \begin{aligned} \dot{\psi} &= (V_r - V_l) / G = -2y\omega\gamma / G, \\ \ddot{\psi} &= -2\dot{y}\omega\gamma / G \end{aligned} \right\} \quad (1.14)$$

where G is the distance between the two rails. Furthermore, one can write, using the assumption of constant wheelset forward velocity V ,

$$\left. \begin{aligned} \dot{y} &= \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \psi V = \psi R_o \omega, \\ \ddot{y} &= \dot{\psi} V = \dot{\psi} R_o \omega \end{aligned} \right\} \quad (1.15)$$

Substituting the first equation of Eq. 14 in the second equation of Eq. 15; and substituting the first equation of Eq. 15 in the second equation of Eq. 14, one obtains, respectively, the following second-order homogeneous ordinary differential equations for the lateral displacement and yaw angle, respectively:

$$\ddot{y} + (\omega_h)^2 y = 0, \quad \ddot{\psi} + (\omega_h)^2 \psi = 0 \quad (1.16)$$

where

$$\omega_h = \omega \sqrt{2R_o\gamma / G} \quad (1.17)$$

is the hunting frequency that can be defined only in the case of positive conicity. In the case of positive conicity, solutions of the preceding equations can be assumed in the forms $y = A_y \sin(\omega_h t + \phi_y)$ and $\psi = A_\psi \sin(\omega_h t + \phi_\psi)$, where A_y and A_ψ are the amplitudes, and ϕ_y and ϕ_ψ are phase angles that can be determined using the initial conditions. These solutions for the lateral displacement and yaw angle show that the frequencies of oscillation of the lateral and angular yaw displacements are the same and are defined by $\omega_h = \omega \sqrt{2R_o \gamma / G}$. Furthermore, by using these solutions, the first equation of Eq. 15, $\dot{y} = \psi R_o \omega$, can be used to prove that the amplitudes of the lateral displacement and yaw angles are related by the equation $A_y = R_o \omega A_\psi / \omega_h = V A_\psi / \omega_h$, and there is a phase angle $\pi/2$ between the lateral displacement y and the yaw angle ψ : that is, $\phi_y - \phi_\psi = \pi/2$. This difference in the phase angle and the relationship $\dot{y} = \psi R_o \omega$ show that the maximum and minimum values of the yaw angle ψ occur when the lateral displacement y is zero, and the maximum and minimum y occur when $\psi = 0$. The hunting oscillations in the case of positive conicity are shown in Figure 13a.

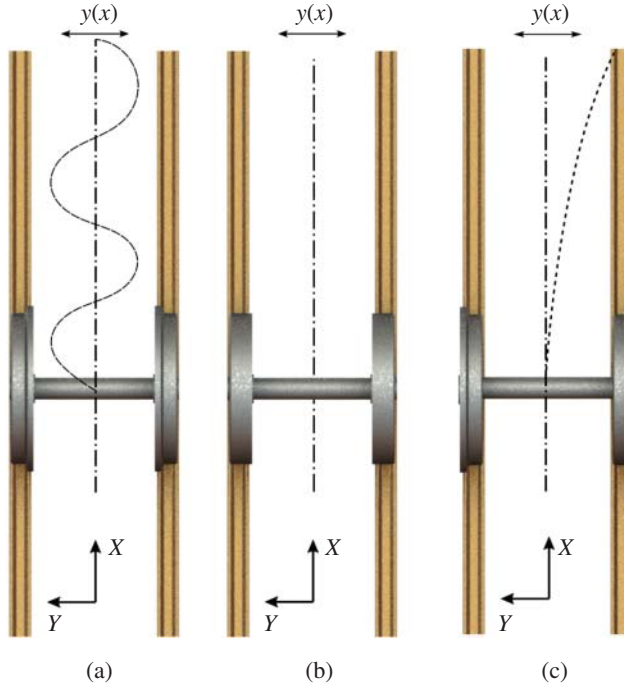


Figure 1.13 Conicity effect.

If, on the other hand, the conicity is equal to zero, $\gamma = 0$, which is the case of a cylindrical wheel, one has, from Eq. 15, $\dot{y} = 0$ and $\dot{\psi} = 0$. Integrating these two equations with respect to time shows that the solution is represented by straight lines and the motion is not oscillatory. If the initial conditions are different from zero, the solution will increase with time, leading to an unstable solution. In this case of a cylindrical wheel, the wheelset does not tend to self-center, as shown in Figure 13b.