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Bruce Hunt

Locally Mixed Symmetric Spaces

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Meiner Frau Monika gewidmet, die selbstlos mir all dies ermöglicht hat.

Introduction

In mathematics one generally studies objects and morphisms, using mathematical machinery to obtain information on the objects (or morphisms). One can equally consider the set of all objects, the "geographical approach". If one considers for example the set of smooth manifolds, this is comparable to a visit to the zoo; here the different exhibitions are for the visitor more or less interesting. Among the main attractions for all are the following "species of manifolds", which is what this book is about: *symmetric* spaces, *locally* symmetric spaces and locally *mixed* symmetric spaces.

Why study symmetric spaces? Because they are important in so many areas of mathematics (topology, differential geometry, representation theory and harmonic analysis, algebraic geometry, theory of moduli, arithmetic geometry and number theory, among others), because their structures can be described explicitly and because there is a satisfactory classification—one can really list *all* individuals. Why study locally symmetric spaces? Because they add to the structure of an underlying symmetric space so many new and exciting features: global functions living on the locally symmetric space are a kind of grand generalization of periodic functions (which correspond to the flat case), many invariants are described by number-theoretic quantities, starting with the volume and lengths of geodesics, continuing with (finite) numbers of "ends" and numbers of totally geodesic submanifolds, extending to topological or analytic invariants like the signature, the Euler-Poincaré characteristic and the arithmetic genus; also they come in families, the members of which are related by finite maps. Finally, especially in the algebraic case the varieties are "beautiful" in the sense of algebraic geometry—one can't help falling in love with them. Why study locally mixed symmetric spaces? Because of the fascinating properties, arising from a combination of $\mathbb{Q}$-group and a representation ρ of that group in a vector space defining them. These spaces seem to be a kind of universal bundle of a subtle kind: traditionally for a Lie group G the universal bundle E_G is a G-bundle over the classifying space B_G of G; E_G is contractible and the homotopy groups of the base $\pi_i(B_G) = \pi_{i-1}(G)$ describe the homotopy of G. For a locally mixed symmetric space $S_{\Gamma,\rho} \to X_\Gamma$, the base X_Γ is a $K(\Gamma, 1)$-space (so in a sense a classifying space) and both the fibers and the total space are quotients of contractible spaces; the total space is a universal Λ-bundle, where Λ is a lattice preserved by $\rho(\Gamma)$. Traditionally the set of homotopy equivalence classes $[X, B_G]$ represents the functor of equivalence classes of G-principal bundles over X; for locally mixed symmetric spaces $S_{\Gamma,\rho} \to X_\Gamma$ the motivating picture is that for some appropriately defined notion of equivalence, the set $[[X, X_\Gamma]]$ of these equivalence classes represents the functor of torus-principal bundles over X of the same type as $S_{\rho,\Gamma} \to X_\Gamma$, a statement which needs to be clarified in more detail. What we can show is that X_Γ is the space of appropriately defined integral equivalence classes of geometric forms and the fiber of $S_{\Gamma,\rho} \to X_\Gamma$ at a point x is a geometric realization of the equivalence class defined by x. Depending on the structure of the fiber, real, complex or quaternionic or even algebraic more or less can be said; when the fibers are Abelian varieties, $S_{\Gamma,\rho}$ is a Kuga fiber variety and one has a complete verification of the motivating picture, provided by the notion of *moduli space* in algebraic geometry.

A symmetric space is a homogeneous space $X = G/K$ with a maximal amount of symmetry, any two points being geodesically equivalent. Starting with a symmetric space, which may be compact or non-compact, one obtains further interesting spaces by dividing by the action of a discrete group acting properly discontinuously, leading to the notion of *locally symmetric space.* In the compact case a discrete subgroup is finite, and one obtains in this manner only finitely many other spaces (like the 2-sphere and the real projective plane), but in the non-compact case infinite families of interesting discrete subgroups exist, and one obtains an infinity of new spaces in this manner. Via a duality between the compact and non-compact symmetric spaces (assumed here to be Riemannian) one obtains a general proportionality principle relating numerical invariants of the compact space and the locally symmetric spaces arising from the non-compact duals. The locally symmetric spaces are in general not compact, and compactifications can be considered: one has both open quotients as well as compactifications, and the structure of these can be incredibly rich. This kind of space can even bridge the gap between configurations in classical algebraic geometry and much more modern considerations, a sample of which can be found in [254, 253] and [257], which established the author's interest in them. In particular, the discrete subgroups, especially when they are *arithmetic*, the case of interest in this book, lead by their very definition to arithmetic results.

The (non-compact) symmetric space has an underlying Lie group, the automorphism group, and considering finite-dimensional representations of this group, a new object can be constructed which fibers over a locally symmetric space. In a natural manner, the discrete group giving rise to the locally symmetric space (now being assumed to be arithmetic) defines a lattice in the representation space, and the semi-direct product of the discrete group with this lattice defines an object which is perfectly natural but seems to have escaped adequate attention previously. It seems only the specific case arising from *hermitian* symmetric spaces has been previously explicitly studied (the *Kuga varieties*), starting with Kuga's work in the 1960s, presented in [315] and [316], and continuing to the present day (Shimura varieties). The more general notion, i.e., not assuming hermitian symmetry, defines the spaces giving rise to the title of this book, the *locally mixed symmetric spaces*. The fibers are simply tori, defining in fact a principal bundle over the locally symmetric space. This more general point of view is legitimized by the main result on sections of such fiber spaces, Theorem 3.5.13 in the text, which not only gives a new proof of the known finiteness in the case of Kuga varieties, but more importantly shows that finiteness does *not* arise from the algebraicity of the Kuga varieties; it results rather from a classical finiteness result in the context of algebraic groups—valid for any locally symmetric space (arithmetic quotient).

These three types of spaces are introduced and studied in the first three chapters of the book. Throughout, the author pursues a rather elementary point of view: all (non-exceptional) structures arise upon consideration of *geometric forms*, objects of linear algebra, the symmetric, skew-symmetric, hermitian (over $\mathbb{C}$ or $\mathbb{H}$) or skew-hermitian forms. In general, symmetric spaces are studied in the context of differential geometry, and there are many excellent texts from this point of view. Locally symmetric spaces on the other hand have been treated in book form mainly for very specific cases (Siegel modular varieties, hyperbolic three-folds, Hilbert modular varieties). The book [96] by Borel and Ji considers both symmetric and locally symmetric spaces, but more from the point of view of compactifications. Locally mixed symmetric spaces have only been considered previously in the specific case mentioned above (Kuga varieties), which dives deep into the realm of algebraic geometry; the book [316] gives an idea of this. In the hermitian symmetric case there is an established generalization of the Kuga fiber varieties, called *mixed Shimura varieties*, the definition of which can be found in [412], Definition 2.1 and [355], VI, 1.1. This notion is much more sophisticated and even more demanding than the material of Kuga's book, but very powerful; see the remark on page 422 for more on this. Shimura varieties (and even more so mixed Shimura varieties) are not defined in this book, the presentation here being rather more elementary.

The notion of Kodaira dimension κ in algebraic geometry transmits the picture that varieties (provided $\kappa \geq 0$) are either of "general type" (maximal Kodaira dimension) or possess a fibration whose fibers have Kodaira dimension 0, among which the Abelian varieties are predominant. One may posit the point of view that the Kuga varieties are classifying spaces of such fibrations, i.e., very general varieties not of general type are "derived" by pulling back a Kuga variety via a classifying map as in the case of elliptic surfaces. Do the more general locally mixed symmetric spaces play a similar role for the set of real analytic manifolds under appropriate circumstances? Is there a notion, analogous to Kodaira dimension, in the real analytic category which similarly describes the existence of torus fibrations?

In addition to the first three chapters mentioned above introducing the symmetric, locally symmetric and locally mixed symmetric spaces, the book contains two further chapters and an appendix. Chapter 4 considers the *specific case*, mentioned above, when the symmetric space is hermitian; various points of view are considered all of which have a bearing on the structures of interest. Chapter 5, on the other hand, considers a *generalization* of the notion of locally mixed symmetric spaces, in the following sense. Of all elliptic surfaces, those which are locally mixed symmetric spaces are just Shioda's elliptic modular surfaces, which are very special. In that chapter it is described how a general elliptic surface is in a sense a pull-back of one of the elliptic modular surfaces.

In spite of the elementary point of view taken here, the background material required in the presentation is considerable; to aid the reader in this, the appendix contains some definitions and notations used throughout, together with detailed guides to literature. This appendix also contains a lot of tables where the information is gathered in a convenient form. In addition there is a references section at the end of each chapter (with the exception of Chap. 3—there is nothing to reference here) with pointers to relevant sources in the literature. The bibliography contains not only the immediately referenced material, but also many of the original sources, more than sufficient for all background material.

The text contains in addition to the basic properties of the three kinds of spaces mentioned above many results not currently available in book form, strewn throughout the journal literature, and touching on more specialized topics. In particular, the rather technical and difficult theory of *compactifications* of non-compact locally symmetric spaces and their relation to *degenerations* of various kinds is given ample room for development. Not only the species are interesting, even more so the individuals: examples are the test of any theory, and a large number of such examples, for all the three kinds of spaces, in varying amounts of detail are considered; much of this is also not available in book form. It is known that there are only two hermitian symmetric structures arising from exceptional groups and there are no Kuga fiber varieties over these; the more general point of view presented here makes a consideration of non-trivial examples arising from exceptional groups possible, even the notion of octonionic structure or Jordan algebra structure can be made sense of. Individual chapters could serve as the basis for one-semester lectures, the given examples providing ample material for exercises.

At this point I should point out what this book does not contain. First and foremost, the vast theory of automorphic forms is not considered—this requires a book by itself. Accordingly various beautiful relations to arithmetic questions arising from automorphic forms are not developed, but rather just the basic geometric structures are considered. The general theory of arithmetic groups and related notions are not even mentioned in the text; we use little more than what was already contained in the original work [94]. Also, the important and acclaimed notion of Shimura variety is not even defined. Nevertheless the descriptions of the examples should give most readers sufficient background to turn to those arithmetic questions without difficulty.

I could not have written this book without the understanding and support of my wife and family, to which I am most thankful. It is a tribute to the modern age and its digital possibilities that I could write it at all, being outside of academics for decades, with no direct access to a mathematical library. Many colleagues were very supportive with their comments and kind responses to inquiries of various kinds, of which I would in particular like to mention Bert van Geemen, Masaaki Yoshida, Michael Kapovich, Jürgen Wolfart, and Amir Dzembic.

It remains to hope the reader may enjoy reading the book as much as the author enjoyed writing it.

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Chapter 1
Symmetric Spaces

The notion of symmetric space is a very classical topic in differential geometry, originally created by E. Cartan at the turn of the nineteenth century, and is fundamental to all that follows; this chapter introduces this notion with a certain amount of detail with special emphasis on examples. Symmetric spaces are special cases of a more general class of manifolds, the homogeneous spaces; it is therefore instructive to begin with a survey of homogeneous spaces, presented in the first section. Homogeneous spaces are characterized by a transitively acting group of automorphisms, i.e., are of the form G/H for a closed subgroup $H \subset G$ in a real Lie group G which is the stabilizer of a point. Symmetric spaces are homogeneous spaces with, as the name suggests, a high degree of symmetry, which by definition means the existence of a global symmetry (automorphism of order 2) at each point, i.e., having the given point as isolated fixed-point. In terms of the description G/H this is relatively easily seen to mean that the subgroup H is fixed by an automorphism of order 2. A symmetric space comes equipped with a G-invariant metric, and this metric is Riemannian exactly when H is compact (it is then of finite index in a maximal compact subgroup). Many results are valid for arbitrary symmetric spaces, while some are valid only for the Riemannian symmetric spaces; these matters are clarified in Sect. 1.2, and in Sect. 1.3 the classification is explained (but not proved in all details). Section 1.4 is concerned with "inherited traits", i.e., given a symmetric space and a subspace, what are conditions on the subspace implying that it is itself symmetric; the necessary and sufficient condition is that the subspace is totally geodesic with respect to the G-invariant metric, which is to be expected as the notion of totally geodesic means that the curvature and metric of the ambient space restricts to the curvature and metric of the subspace. Of particular importance are Riemannian symmetric spaces which have a G-invariant complex structure which is compatible with the Riemannian structure: these are the hermitian symmetric spaces, treated in some detail in Sect. 1.5. Section 1.6 presents many examples, the heart of the topic; the presentation given here is based on the notion of geometric forms and describes many spaces in terms of geometric forms. The last two sections treat somewhat more specialized topics;

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Sect. 1.7 deals with Satake compactifications of non-compact Riemannian symmetric spaces, while Sect. 1.8 deals with compact Riemannian symmetric spaces and describes Bott's method of studying geodesics on these spaces to prove the famous Bott periodicity theorem. The following notations are used: M is a smooth manifold, X, Y denote vector fields or vectors, up to Sect. 1.7. In Sect. 1.6 (examples of symmetric spaces derived from geometric forms) M will also denote a general matrix.

1.1 Homogeneous Spaces

Let G be a Lie group, $H \subset G$ a Lie subgroup, with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively; let $M = G/H$ be the homogeneous space, which is viewed as the base of the principal bundle $\pi_H : G \longrightarrow G/H$ with fiber H. The tangent space of G at the identity $e \in G$, which is naturally identified with the space of left-invariant vector fields on G and hence with the Lie algebra, $T_e(G) \cong \mathfrak{g}$, decomposes into a *vertical part*, which is determined as the kernel of $T_e\pi_H$, isomorphic to $\mathfrak{h}$, and a complement $\mathfrak{m}$; correspondingly there is a decomposition of $\mathfrak{g}$ into $\mathfrak{h}$ and a complement. Each such complement defines a principal connection on G viewed as a H-bundle over M since the decomposition gives a splitting of the sequence $0 \longrightarrow L_H \longrightarrow T(G) \longrightarrow T(M) \longrightarrow$ with fibers $\mathfrak{h}$, $\mathfrak{g}$, $\mathfrak{m}$, respectively, induce by the tangent map $T\pi_H$. The principal connection is *invariant under* G when for all $s \in G$, the left translation preserves the connection form $\boldsymbol{\omega}$, i.e., $\boldsymbol{l}_s(\boldsymbol{\omega}) = \boldsymbol{\omega}$ (for $s, x \in G$ and $\mathsf{t}_x \in T_x(G)$, one has $\boldsymbol{\omega}_x \cdot \mathsf{t}_x = \boldsymbol{\omega}_{sx} \cdot (s\mathsf{t}_x)$). Conjugation by H defines the adjoint representation in $\mathfrak{g}$; its restriction to $\mathfrak{m}$ is in this case called the *isotropy representation*; since $\mathfrak{m}$ is isomorphic to the tangent space of M at x_0, it is naturally a representation of H in $GL(T_{x_0}(M))$. Details now follow.

1.1.1 Invariant Connections

Proposition 1.1.1 *There is a one-to-one correspondence between the following two sets:*

(i) *G-invariant principal connections on the principal bundle G over M and*
(ii) *complementary subspaces* $\mathfrak{m} \subset \mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ *and* $\mathfrak{m}$ *is invariant under the adjoint action of H, i.e.,* $\mathrm{Ad(h)}\mathfrak{m} \subset \mathfrak{m}$ *for all $h \in H$.*

The correspondence is given as follows: since G is the total space of the principal bundle $G \longrightarrow M$, the connection form ω is a one-form on G; at each $g \in G$ it is a linear form on $T_g(G)$, *denoted* ω_g. Similarly, the curvature form Ω is an $\mathrm{ad}(\mathfrak{g})$-valued two-form on G and its value at $g \in G$, denoted Ω_g, is an alternating bilinear form on $T_g(G)$. Finally, we denote the action of G via Ad on $\mathfrak{g}$ by gX. To a connection $\boldsymbol{P}$, the connection one-form $\boldsymbol{\omega}_e$ at the neutral element of G defines a projection onto $\mathfrak{h}$; taking $\mathfrak{m} \subset \mathfrak{g}$ to be the kernel of this projection, there is a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. By the invariance property of the one-form, the relation $\boldsymbol{\omega}_h(Xh) = \mathrm{Ad}(h^{-1})(\boldsymbol{\omega}_e(X))$ for $X \in \mathfrak{g}$ can also be written $\boldsymbol{\omega}_e(X)\mathrm{Ad}(h^{-1}) = \mathrm{Ad}(h^{-1})\boldsymbol{\omega}_e(X)$ which implies $\mathrm{Ad}(h^{-1})\mathfrak{m} \subset \mathfrak{m}$. Conversely, given the subspace $\mathfrak{m}$, define a one-form with values in $\mathfrak{g}$ for which the projection onto $T_{x_0}(M)$ has $\mathfrak{m}$ as its kernel: let $p : \mathfrak{g} \longrightarrow \mathfrak{h}$ be the projection with kernel $\mathfrak{m}$, which is then clearly a $\mathrm{Ad}(H)$-invariant subspace, and define $\boldsymbol{\omega}_s(sX) := p(X)$ for any $X \in \mathfrak{g}$, $s \in G$, which is $\mathfrak{h}$-valued and therefore vertical. □

Making the identification of $T_e(M)$ with the values of vector fields at $e \in M$, i.e., mapping a vector field X on M to X_e, defines a map of $T_e(M)$ into $\mathfrak{g}$. If, in the correspondence above, $\mathfrak{m}$ is the set of these elements (of the form X_e for vector fields on M), then the unique connection (Proposition 1.1.1) defined in this manner is called the *canonical connection* on the principal bundle $G \mapsto G/H$ with structure group and fiber H. Fixing this specific complementary subspace, denoting it by $\mathfrak{m}_e$, one obtains for an arbitrary complementary subspace $\mathfrak{m}$ a linear map into $\mathfrak{g}$ as the deviation of $\mathfrak{m}$ from $\mathfrak{m}_e$, by mapping each element in $\mathfrak{m}$ to its $\mathfrak{h}$-component with respect to the decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}_e$, that is $\Lambda_{\mathfrak{m}}(X) := \boldsymbol{\omega}_{p_e}(X)$ for $X \in \mathfrak{m}$ and p_e in the fiber of the principal bundle over $e \in G/H$. This is the point of view used in Sects. 1–2 of [291], Chap. X.

The homogeneous space G/H is *reductive* if one has the decomposition with $\mathrm{ad}(H)$-invariant subspace $\mathfrak{m}$:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad \mathfrak{h} \cap \mathfrak{m} = 0; \quad \mathrm{ad}(H)\mathfrak{m} \subset \mathfrak{m} \tag{1.1}$$

If H is *connected*, then the condition that $\mathfrak{m}$ is invariant under the adjoint action of H is equivalent to $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. The following conditions all insure that the homogeneous space is reductive: (1) H is compact; (2) H is connected and $\mathrm{ad}(\mathfrak{h})$ is completely reducible in $\mathfrak{g}$, which holds when H is connected and semisimple; (3) H is discrete in G. For the remainder of this Sect. 1.1, it will always be assumed that a given homogeneous space G/H is reductive, unless the contrary is explicitly stated. The torsion and curvature tensors of a connection $\boldsymbol{P}$ corresponding by Proposition 1.1.1 to a subspace $\mathfrak{m}$ can be expressed in terms of the linear map $\Lambda_{\mathfrak{m}}$ by the formulas ([291], Proposition 2.3 in Chap. X)

$$\begin{aligned} T(X,Y)_e &= \Lambda_{\mathfrak{m}}(X)Y - \Lambda_{\mathfrak{m}}(Y)X - [X,Y]_{\mathfrak{m}}, \ X, Y \in \mathfrak{m} \\ R(X,Y)_e &= [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}) - \mathrm{ad}([X,Y]_{\mathfrak{h}}), X, Y \in \mathfrak{m} \end{aligned} \tag{1.2}$$

(decomposing the bracket of $\mathfrak{g}$ into an $\mathfrak{m}$ and a $\mathfrak{h}$-component, $[X,Y] = [X,Y]_{\mathfrak{m}} + [X,Y]_{\mathfrak{h}}$ with $[X,Y]_{\mathfrak{m}} \in \mathfrak{m}$, $[X,Y]_{\mathfrak{h}} \in \mathfrak{h}$), and for the canonical connection, given by $\Lambda_{\mathfrak{m}_e}(X) = 0$, one has

$$T(X,Y)_e = -[X,Y]_{\mathfrak{m}_e}, \quad (R(X,Y)Z)_e = -[[X,Y]_{\mathfrak{h}}, Z], \tag{1.3}$$

for all $X, Y, Z \in \mathfrak{m}$, and in this case, both T and R are parallel.

The above formula leads easily to the following description of the (Lie algebra of the) holonomy group.

Proposition 1.1.2 *Let G/H be reductive with* $\mathrm{Ad}(H)$*-invariant decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. *Then the Lie algebra of the holonomy group (at the origin) of the canonical connection is spanned by* $\{i_0([X, Y]_{\mathfrak{h}}) \mid X, Y \in \mathfrak{m}\}$, *where* $i_0 : H \longrightarrow \mathrm{Aut}(\mathfrak{m})$ *is the isotropy representation.*

Proof This follows from the description of the Lie algebra of the holonomy group in terms of the curvature form and (1.2) since for the canonical connection, $\Lambda_{\mathfrak{m}} = 0$. The formula (1.2) in turn follows essentially from (6.28), which needs to be conjugated by i_0, leading to the third term. □

Proposition 1.1.3 *Let G/H be a reductive homogeneous space; there is a unique G-invariant torsion-free connection which has the same geodesics as the canonical connection.*

Proof This follows from the relations (1.2) and (1.3) by defining the connection, say $\boldsymbol{P}^{\times}$, as the connection corresponding to the subspace $\mathfrak{m}^{\times}$ for which $\Lambda_{\mathfrak{m}^{\times}}(X)Y = -\frac{1}{2}[X, Y]_{\mathfrak{m}_e}$. Both connections are G-invariant and therefore have the same geodesics. □

The torsion-free connection of the proposition is the *Levi-Cevita connection*. An important corollary of this is

Corollary 1.1.4 *Let G/H be a reductive homogeneous space. Then the canonical G-invariant connection on G/H is complete, hence by Proposition 1.1.3 this holds also for the Levi-Cevita connection.*

Proof For $X \in \mathfrak{m}$ let $f_X(t) = \exp tX \in G$, defining a one-parameter subgroup of G, and consider its image $\gamma_X(t)$ in G/H. Then $\gamma_X(t)$ is a geodesic: the one-parameter group $f_X(t)$ may be viewed as acting on the principal bundle, so for p_e in the fiber over $e \in G/H$, the orbit of p_e under the one parameter subgroup is defined, let this be denoted by $\widetilde{f_X}(t)(p_e)$, and let v denote the vector field on G (viewed as a principal bundle over G/H) induced by the one-parameter subgroup acting on the principal bundle. Then at the base point $e \in G/H$ $\mathsf{v} = X$ by definition and since $\Lambda_{\mathfrak{m}_e}(X) = \boldsymbol{\omega}_{p_e}(X)$ the relation $\Lambda_{\mathfrak{m}_e}(X) = 0$ is equivalent to v is horizontal at p_e. By transport of structure this holds at all points, so the orbit $\widetilde{f_X}(t)(p_e)$ is horizontal, and projects to the curve γ_X on G/H, hence γ_X is a geodesic; conversely any geodesic is of this form (i.e., the horizontal lift of γ_X defines a one-parameter subgroup). Since it is clearly defined for all t, this gives completeness. □

Let c_u be a compact involution on a semisimple complex Lie algebra $\mathfrak{g}$ with compact Lie algebra $\mathfrak{g}_u$, c_0 another involution defining a real form $\mathfrak{g}_0$; then c_u induces an involution on $\mathfrak{g}_0$ (called a *Cartan involution*) which is also denoted by c_u, such that $\mathfrak{g}_u$ and $\mathfrak{g}_0$ decompose as $\mathfrak{g}_u = \mathfrak{k}_0 + i\mathfrak{p}_0$ and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$. A subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ is c_u*-stable*, if $c_u(\mathfrak{h}_0) \subset \mathfrak{h}_0$, and in this case one has also a decomposition

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{q}_0, \quad [\mathfrak{h}_0, \mathfrak{h}_0] \text{ semisimple}, \quad \mathrm{ad}_{\mathfrak{h}_0} : \mathfrak{h}_0 \longrightarrow \mathfrak{gl}(\mathfrak{g}) \text{ semisimple}, \tag{1.4}$$

and in particular $\mathfrak{h}_0$ is *reductive*, the sum of its center and the semisimple part. Examples of such subalgebras are

(1) if $\mathfrak{p} \subset \mathfrak{g}_0$ is any c_u-stable subalgebra, then the centralizer and normalizer of $\mathfrak{p}$ are c_u-stable;
(2) any subalgebra of $\mathfrak{g}_0$ which is fixed by a linear automorphism of finite order;
(3) any semisimple subalgebra.

Let G_0 be a Lie group with Lie algebra $\mathfrak{g}_0$, $H_0 \subset G_0$ a closed subgroup and σ_u a Cartan involution on G_0 with Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$; the pair (G_0, H_0) is a *σ_u-stable pair* if two conditions are satisfied:

1. Letting $K_0 \subset G_0$ denote the connected component of a maximal compact subgroup with respect to σ_u, then $H_0 = (H_0 \cap K_0)\exp(\mathfrak{h}_0 \cap \mathfrak{p}_0)$.
2. The connected Lie subgroup $H \subset G$ corresponding to the complexified Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is closed in the group $\mathrm{Aut}(\mathfrak{g})$ (this group is also the projective or derived group).

The first condition generalizes the notion of symmetric pair; this condition also implies that $\mathfrak{h}_0$ is c_u-stable, and if H_0 is connected, 1. is equivalent to $\mathfrak{h}_0$ being c_u-stable. The important thing about the second condition is that the complex Lie group H is a closed subgroup of the derived group (centerless, connected) of the complexification $G = \mathrm{Aut}(\mathfrak{g})$ of G_0, and it implies a close relationship between the two spaces G_0/H_0 and G/H. In particular, the inclusions $H_0 \subset G_0$ and $H \subset G$ are compatible in the sense that

$$\begin{array}{ccc} G_0 & \xrightarrow{i_G} & G \\ \bigcup & & \bigcup \\ H_0 & \xrightarrow{i_H} & H \end{array} \tag{1.5}$$

commutes. Hence $i_H(H_0)$ intersects the connected component H^0 of H (which need not be connected) and $H = i_H(H_0) \cdot H^0$ (sorry about the lousy notation at this point). From this diagram we get

1. a complex homogeneous space G/H;
2. an inclusion $i_G(G_0)/(H \cap i_G(G_0)) \hookrightarrow G/H$;
3. a covering $G_0/H_0 \longrightarrow i_G(G_0)/(H \cap i_G(G_0))$;
4. a compact subgroup $H_u \subset G_u$ of the compact form of G_0 and a compact homogeneous space G_u/H_u;
5. an inclusion $G_u/H_u \hookrightarrow G/H$.

The compact subgroup H_u is defined as $H \cap G_u$, the compact form being defined by the compact involution σ_u. Given a σ_u-stable pair (G_0, H_0), the space G/H is called the *associated complex homogeneous space*, the space G_u/H_u is called the *associated compact homogeneous space*. All three homogeneous spaces are reductive and there are corresponding decompositions

$$\mathfrak{g}_u = \mathfrak{h}_u + \mathfrak{q}_u, \quad \mathfrak{g} = \mathfrak{h} + \mathfrak{q}, \quad \mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{q}_0, \tag{1.6}$$

where $\mathfrak{q}_u = (\mathfrak{k}_0 \cap \mathfrak{q}_0) + i(\mathfrak{p}_0 \cap \mathfrak{q}_0)$.

Just as the Lie algebra $\mathfrak{g}$ is the space of invariant vector fields, the exterior powers of its dual are invariant differential forms, and for the homogeneous spaces one has the following descriptions of differential forms. Let A_X^p (resp. Ω_X^p) denote the space of smooth real-valued (resp. complex-valued) differential forms of degree p on a manifold (resp. complex manifold) X and A_X^* (resp. Ω_X^*) denote the algebra of smooth real-valued (resp. complex-valued) differential forms or arbitrary degree. Then

$$(A^*_{G_0/H_0})^{G_0} = \left(\bigwedge^* \mathfrak{q}_0^*\right)^{H_0}, \quad (A^*_{G_u/H_u})^{G_u} = \left(\bigwedge^* \mathfrak{q}_u^*\right)^{H_u}, \quad (\Omega^*_{G/H})^{G} = \left(\bigwedge^* \mathfrak{q}^*\right)^{H}, \tag{1.7}$$

in which the superscripts denote the corresponding invariants. This follows for the algebra (individual forms and products), only the exterior derivative needs to be explained for the right handed spaces. In the usual manner the exterior derivative is defined here for $\alpha \in \left(\bigwedge^r \mathfrak{q}_0^*\right)$ by the relation

$$\begin{aligned} d\,\alpha(X_1, \dots, X_{r+1}) = \sum_{1\le i<j\le r+1} & (-1)^{i+j}\alpha([X_i, X_j], X_1, \dots, \hat{X}_i, \dots, \\ & \dots, \hat{X}_j, \dots, X_{r+1}), \;\; X_i \in \mathfrak{q}_0, \end{aligned} \tag{1.8}$$

and similarly for $\mathfrak{q}$ and $\mathfrak{q}_u$. For $X \in \mathfrak{q}_0$, decompose it as $X = X_{\mathfrak{k}} + X_{\mathfrak{p}},\ X_{\mathfrak{k}} \in \mathfrak{q}_0 \cap \mathfrak{k}_0$, $X_{\mathfrak{p}} \in \mathfrak{q}_0 \cap \mathfrak{p}_0$; then the map

$$\phi : \mathfrak{q}_0 \xrightarrow{\cong} \mathfrak{q}_u, \quad (X) \mapsto X_{\mathfrak{k}} + i\,X_{\mathfrak{p}}, \tag{1.9}$$

defines a linear isomorphism between $\mathfrak{q}_0$ and the compact $\mathfrak{q}_u$.

Lemma 1.1.5 *Let $\alpha_0,\ \alpha_u$ be differential forms on G_0/H_0 and G_u/H_u (the associated compact space) which satisfy*

$$\alpha_u(\phi(X_1), \dots, \phi(X_a), \phi(Y_1), \dots, \phi(Y_b)) = i^b \alpha_0(X_1, \dots, X_a, Y_1, \dots, Y_b). \tag{1.10}$$

Then if α_u is exact $\alpha_u = d\,\beta_u$, then β_u is G_u-invariant and there is a G_0-invariant form β_0 such that $\alpha_0 = d\,\beta_0$; if α_u is closed, then α_0 is closed.

This result allows the passage from the relation (1.7) to cohomology.

Proof From (1.7) it follows that the complexifications of the real algebras are isomorphic to the complex algebra (since H is the complexification of both H_0 and H_u), that is

$$\left(\wedge^* \mathfrak{q}_0^*\right)^{H_0} \otimes_{\mathbb{R}} \mathbb{C} \cong \left(\wedge^* \mathfrak{q}^*\right)^{H} \cong \left(\wedge^* \mathfrak{q}_u^*\right)^{H_u} \otimes_{\mathbb{R}} \mathbb{C}, \tag{1.11}$$

which in turn implies that the algebras of differential forms are isomorphic:

$$(A^*_{G_0/H_0})^{G_0} \otimes \mathbb{C} \cong (\Omega^*_{G/H})^{G} \cong (A^*_{G_u/H_u})^{G_u} \otimes \mathbb{C}. \tag{1.12}$$

The assumption in the lemma tells us that α_u and α_0 which are in the outside spaces have the same image in $(\Omega^*_{G/H})^G$, i.e., the complexified forms are the same. Now since G_u is compact (so cohomology is finite-dimensional), α_u is G_u-invariant and by assumption exact, $\alpha_u = d\,\beta_u$, it follows that β_u is G_u-invariant. The image of β_u in $(\Omega^*_{G/H})^G$ is in the image of $(A^*_{G_o/H_0})^{G_0}$ under the isomorphism of forms induced by (1.11), defining the form β_0 which is clearly also G_0-invariant and $\alpha_0 = d\,\beta_0$. The same argument shows the closedness. □

An important special case of σ_u-stable pair occurs when a real Lie group G (playing the role of G_0 in the previous discussion, for ease of notation) has an *involutory automorphism* $\sigma \neq 1_G$, which is a Lie group automorphism with $\sigma^2 = 1$, H is the set of points of G which are invariant under σ; this is a closed subgroup, hence a Lie subgroup and (G, H, σ) is a symmetric pair. The tangent map of σ at e, $T_e\sigma : T_e(G) \cong \mathfrak{g} \longrightarrow \mathfrak{g}$ is a Lie algebra homomorphism and because $\exp(tT_e\sigma(X)) = \sigma(\exp(tX))$ for $X \in \mathfrak{g}$ and $t \in \mathbb{R}$, the condition that $\exp(tT_e\sigma(X)) = \exp(tX)$ (which is the invariance under $T_e\sigma$) is equivalent to the invariance of $\exp(tX)$ under σ, the fixed point set in $\mathfrak{g}$ of $T_e\sigma$ is the Lie algebra $\mathfrak{h}$ of H and $(\mathfrak{g}, \mathfrak{h}, T_e\sigma)$ is a symmetric Lie algebra. Furthermore, from $\sigma(ghg^{-1}) = g\sigma(h)g^{-1}$, one obtains for the tangent map the relation $T_e\sigma \circ \mathrm{ad}(g) = \mathrm{ad}(g) \circ T_e\sigma$. Since $T_e\sigma$ also has square 1, the eigenvalues are ± 1; since $\mathfrak{h}$ is the eigenspace of $= 1$, it follows that the eigenspace of -1 is a complementary subspace $\mathfrak{m} \subset \mathfrak{g}$, with $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Since $\mathfrak{h}$ is a subalgebra, it is closed under the bracket, hence if $\mathfrak{g}$ is reductive, then the relations

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h} \tag{1.13}$$

are satisfied. The second relation holds since $\mathfrak{g}$ is reductive, the third follows from: $X, Y \in \mathfrak{m} \Rightarrow \sigma([X, Y]) = [\sigma(X), \sigma(Y)] = [-X, -Y] = [X, Y]$. Let H^0 denote the connected component of H, let H_1 be a subgroup with $H^0 \subset H_1 \subset H$, and consider the homogeneous space $M = G/H_1$, viewed as the base of a principal H_1-bundle and view G as the total space of this bundle over M; by Proposition 1.1.1, there is a G-invariant principal connection on $G \longrightarrow M$, which has connection form $\boldsymbol{\omega}$ whose value at $e \in G$, $\boldsymbol{\omega}_e$, is the natural projection of $\mathfrak{g}$ onto $\mathfrak{h}$ with kernel $\mathfrak{m}$.

Lemma 1.1.6 *The G-invariant connection whose connection form $\boldsymbol{\omega}_e$ at $e \in G$ has the kernel $\mathfrak{m}$ is the only G-invariant connection which is also invariant under σ.*

Proof By Proposition 1.1.1 any G-invariant connection determines a subspace $\mathfrak{m}' \subset \mathfrak{g}$ as the kernel of the projection operator of $\mathfrak{g}$ to $\mathfrak{h}$ (defining the value of the connection form at $e \in G$); invariance of the connection under σ amounts to invariance of the connection one-form under $T_e\sigma$, or $T_e\sigma(\mathfrak{m}') = \mathfrak{m}'$, which implies $\mathfrak{m}' = \mathfrak{m}$ ($\mathfrak{m}$ defined here as the -1-eigenspace of $T_e\sigma$).

The triple (G, H, σ) above is a *symmetric pair* and the connection on the principal bundle $G \longrightarrow G/H$ just described is called the *canonical connection* on that principal bundle. Since $\mathfrak{m}$ is the -1-eigenspace of $T_e\sigma$, $T_e\sigma(X) = -X$ for $X \in \mathfrak{m}$ and $T_e\sigma([X, Y]) = [T_e\sigma(X), T_e\sigma(Y)] = [X, Y]$ for $X, Y \in \mathfrak{m}$, resulting in $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$. For the curvature form of X, Y, which is just the projection of $[X, Y]$ to $\mathfrak{h}$ by (1.3), one has $\boldsymbol{\Omega}_e(X, Y) = -[X, Y] \subset \mathfrak{h}$ for $X, Y \in \mathfrak{m}$. It is customary to assume that G is *connected* when a symmetric pair (G, H, σ) is given; in what follows if nothing to the contrary is stated this assumption will be made. The discussion of connected components is given in Sect. 1.2.2.

1.1.2 Compact Homogeneous Spaces

In this section the following notations are used: G denotes a compact Lie group, $H \subset G$ a closed subgroup with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$; $T \subset G$ and $S = H \cap T$ are maximal tori in G and H, respectively with Lie algebras $\mathfrak{t}, \mathfrak{s}$. The root space decomposition of the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ are $\mathfrak{g}_{\mathbb{C}} = \mathfrak{c} + \oplus_{\alpha \in \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c})} (\mathfrak{g}_{\mathbb{C}})_\alpha$ and $\mathfrak{h}_{\mathbb{C}} = \mathfrak{d} + \oplus_{\alpha \in \Phi(\mathfrak{h}_{\mathbb{C}}, \mathfrak{d})} (\mathfrak{h}_{\mathbb{C}})_\alpha$ in which $\mathfrak{c} \subset \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{d} \subset \mathfrak{h}_{\mathbb{C}}$ are Cartan subalgebras. The root system of the complex Lie algebra may be identified with the root system of the compact group: $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{c}) = \Phi(G, T)$, $\Phi(\mathfrak{h}_{\mathbb{C}}, \mathfrak{d}) = \Phi(H, S)$; we will use this notation in this section. There is a natural inclusion of the root systems $\Phi(H, S) \subset \Phi(G, T)$, and the *complementary roots* are the roots in $\Phi(G, T)$ which are not roots of $\Phi(H, S)$. For each root let $\mathfrak{g}_\alpha \subset \mathfrak{g}$, $\alpha \in \Phi(G, T)$ denote the corresponding root space in the compact algebra $\mathfrak{g}$ and similarly for $\alpha \in \Phi(H, S)$. Consider the homogeneous space G/H and the corresponding decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of the Lie algebra, where $\mathfrak{m} \cong T_e(G/H)$. For the complementary roots, the root subspaces $\mathfrak{g}_\alpha$ are contained in $\mathfrak{m}$; this is the $\mathbb{R}$-span of the elements $\mathbf{y}_\alpha, \mathbf{z}_\alpha$ of a Weyl basis (6.35).

Suppose G/H has an almost complex structure invariant under G, given by an endomorphism J of the tangent bundle, and its restriction to the reference point defines a complex structure on $\mathfrak{m}$; since $\mathfrak{m} = \sum \mathfrak{g}_\alpha$ for the complementary roots α and J_e commutes with the isotropy group, J_e induces on each $\mathfrak{g}_\alpha$ a complex structure, and these determine and are determined by J_e. In each $\mathfrak{g}_\alpha$, either the complex structure defines the same orientation as the adjoint representation or the opposite, and correspondingly for each pair $\pm\alpha$ of complementary roots the sign ε_α is defined ($+1$ is the orientations coincide and -1 if they are opposite), and the $\varepsilon_\alpha \alpha$ are the *roots of the almost complex structure*. This endomorphism of $\mathfrak{m}$ is extended to one of $\mathfrak{g}$ by setting it 0 on $\mathfrak{h}$; it then has a natural extension $J_{\mathbb{C}}$ to the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ which acts as follows (β_i the complementary roots):

$$(J_{\mathbb{C}})_{|(\mathfrak{g}_{\mathbb{C}})_{\varepsilon_i \beta_i}} = \text{ multiplication by } i, \quad (J_{\mathbb{C}})_{|(\mathfrak{g}_{\mathbb{C}})_{-\varepsilon_i \beta_i}} = \text{ multiplication by } -i. \tag{1.14}$$

One has, as the natural analogue of the splitting principle, the corresponding result for Chern classes (here G is a compact Lie group, H is a closed Lie subgroup): let G/H be complex homogeneous and $\eta' = \eta_{\mathbb{C}}$ be the complexification of the bundle along the fibers $\eta = (T(P/H) \longrightarrow B, G/H, G)$ of a principal G-bundle $\xi = (P \longrightarrow B, G, G)$ defined by the complex structure J on $T_e(G/H)$, which induces a complex representation of H, hence on η', called the *complex isotropy representation* of H in $\mathbb{C}^m$, where $2m = \dim G/H$ (the real part of which is the usual isotropy representation). Then

$$\boldsymbol{\rho}^*(c(\eta')) = \prod(1 + \varepsilon_i\beta_i), \tag{1.15}$$

where $\pm\beta_i$ are the complementary roots, $\varepsilon_i\beta_i$ are the weights of the complex isotropy representation defining the complex structure on η' (i.e., the roots of the complex structure) and $\boldsymbol{\rho} : G/T \longrightarrow G/H$ is the natural projection, under which the lift of η to G/T splits as a sum of line bundles, the first Chern classes of which are in $H^2(G/T, \mathbb{Z})$ which is identified with $H^1(T, \mathbb{Z})$ by transgression in the fiber.

Let G/H again denote a compact homogeneous space provided with an almost complex structure, i.e., a complex structure J_e on the tangent space $T(G/H)_e$ at the base point and let $\pm\beta_i$, $i = 1, \ldots, k$ denote the complementary roots and $\varepsilon_i\beta_i$ the roots of the almost complex structure. This structure is *integrable*, if the torsion tensor[1] $S(X, Y)$ vanishes for any two vector fields on G/H; since this space is homogeneous the vector fields are determined by the corresponding Lie algebra, and this reduces to evaluating the expression for S in the Lie algebra. Let $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ denote the complexified Lie algebras of G and H and $\mathfrak{t}_{\mathbb{C}}$ the complexified Lie algebra of the maximal torus of G, respectively; the decomposition into root spaces and an Abelian subalgebra is compatible with complexification, and one has $(\mathfrak{g}_{\mathbb{C}})_\alpha \oplus (\mathfrak{g}_{\mathbb{C}})_{-\alpha} = \mathfrak{g}_\alpha \otimes_{\mathbb{R}} \mathbb{C}$. By (1.14) this amounts to the statement that $\mathfrak{h}_{\mathbb{C}} + (\mathfrak{g}_{\mathbb{C}})_{\varepsilon_1\beta_1} + \cdots + (\mathfrak{g}_{\mathbb{C}})_{\varepsilon_k\beta_k}$ is a Lie algebra, and the conclusion is the first statement of

Theorem 1.1.7 *Given a compact homogeneous manifold G/H with almost complex structure given by the set of roots $\Psi = \{\varepsilon_i\beta_i\}$ and the set of roots $\Phi(H, S)$ of the subgroup H, the almost complex structure is integrable and G/H is homogeneous complex if and only if $\Psi \cup \Phi(H, S)$ is a closed set of roots, and in this case Ψ is closed and contained in the system of positive roots for some basis of $\Phi(G, T)$.*

The second statement follows from the properties of closed sets of roots: for some ordering of the roots $\Phi(H, S) = \Phi^+(H, S) \cup -\Phi^+(H, S)$, $\Psi \cup \Phi^+(H, S)$ contains the set of positive roots for that order, while both $\Psi \cup \Phi(H, S)$ and $\Psi \cup \Phi^+(H, S)$ are closed, so Ψ is also ($\Psi \cap -\Psi = \emptyset$).

As a corollary of this one obtains a theorem first proved by Wang (see [528, 529]).

Corollary 1.1.8 *Let G be compact, semisimple and $H \subset G$ closed and connected, with* $\mathrm{rank}(G) = \mathrm{rank}(H)$. *Then the homogeneous space G/H is complex homogeneous (has an invariant complex structure) if and only if H is the centralizer of a torus.*

Without the assumption on the rank, the statement remains true when the formulation is that the semisimple part of H is the semisimple part of the centralizer of a torus. Assuming the ranks of G and H coincide, T will denote a common maximal torus; the torus *centralized* by H will be denoted S.

[1] $S(X, Y) := [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]$ for an almost complex structure J.

Proof First observe that as a corollary of the statement that under the assumption that $\mathrm{rank}(G) = \mathrm{rank}(H)$, if $H^2(G/H, \mathbb{R}) = 0$ (which holds if H is semisimple), every 2-cohomology class vanishes, in particular this is the case for the first Chern class c_1 of the complex tangent bundle of G/H; on the other hand, by the splitting principle, lifting to G/T (let $\boldsymbol{\rho} : G/T \longrightarrow G/H$ and τ be the transgression in the bundle $(G \longrightarrow G/T, T, G)$) $\boldsymbol{\rho}^*(c_1) = -\tau(\sum \varepsilon_i \beta_i)$ in the notation used above. Since this vanishes, one has the relation $\sum \varepsilon_i \beta_i = 0$; this implies that the set of roots of the complex structure is not closed, hence by the theorem, that G/H does not have a complex structure.

For the implication "complex structure" ⇒ "centralizes a torus", write H (locally) as a product $H^{ss} \cdot S$ where H^{ss} is semisimple and S is a torus; by the preceding remark, the torus part S has positive dimension (otherwise H would be semisimple and $H^2(G/H, \mathbb{R}) = 0$). If $Z(S) \subset G$ denotes the centralizer of S (which we want to show is just H), then $Z(S) = Z(S)^{ss} \cdot S_1$, where S_1 is a torus with $S \subset S_1$, $Z(S)^{ss}$ contains H^{ss}, and $Z(S)^{ss} \cap S_1$ is finite; a dimension argument shows that $S = S_1$ ($\mathrm{rank}(G) = \mathrm{rank}(H^{ss}) + \dim(S) = \mathrm{rank}(Z(S)^{ss}) + \dim(S_1)$), and $\mathrm{rank}(Z(S)^{ss}) = \mathrm{rank}(H^{ss})$ and consequently also $\mathrm{rank}(Z(S)) = \mathrm{rank}(H)$, and applying the above consideration to $Z(S)/H = Z(S)^{ss}/H^{ss}$, the homogeneous space $Z(S)/H$ does not have a complex structure. On the other hand, considering, in addition to the G roots complementary to H (say $\varepsilon_i \beta_i$, $i \in K$), also the $Z(S)$ roots complementary to those of H (say $\varepsilon_i \beta_i$, $i \in J$ for a subset $J \subset K$), the latter define a complex structure on the tangent space at $e \in Z(S)/H$, and since this is a subsystem of the G-roots complementary to H, which is closed, the latter set is closed, hence defines a complex structure on $Z(S)/H$, provided $Z(S) \neq H$, a contradiction. Hence $Z(S) = H$, and H is centralizer of a torus.

Conversely, suppose H centralizes the torus $S \subset T$, and let β_j be one of the complementary roots. Choose the sign ε_i as follows: for a regular element $s \in S$, H is the centralizer of s; set $\varepsilon_i := \mathrm{sign}(\beta_i(s))$, so that $\varepsilon_i \beta_i > 0$ and the set of $\varepsilon_i \beta_i$ is the set of positive roots for the order determined by S, hence closed; the almost complex structure on G/H defined by $\varepsilon_i \beta_i$ thus fulfills the criteria of the Theorem 1.1.7, and is therefore integrable. □

This applies in particular to the *flag spaces* G/T where T is a maximal torus of G. Since in this case it is clear that the subgroup centralizes any subtorus, there are various (a priori non-equivalent) complex structures. Let $S \subset H$ be a maximal torus.

Lemma 1.1.9 ([95], 13.7) *Let $\Phi^+(H, S)$ be a system of positive roots for H. The set of roots of an invariant complex structure on G/H form a closed system Ψ such that $\Psi \cup \Phi^+(H, S)$ is a positive system of roots for G and conversely, a closed set Ψ of complementary roots such that $\Psi \cup \Phi^+(H, S)$ is the set of positive roots of G for some ordering (set of simple roots) is the system of roots for an invariant complex structure.*

This gives the tool to determine the number of invariant complex structures (note that these are not complex analytically equivalent, but they may or may not be equivalent under a diffeomorphism of G/H, see the discussion below).

Proof of 1.1.9 First observe that if $\widetilde{\sigma} \in \mathrm{Aut}(G)$ leaves a maximal torus T invariant, then the tangent map $T\widetilde{\sigma}$ induces a map of the root system of G; if in addition $\widetilde{\sigma}$ leaves H invariant, then it induces a map $\sigma : G/H \longrightarrow G/H$. If Ψ is the set of roots defining a complex structure on G/H and $\Psi' = T\widetilde{\sigma}(\Psi)$ is the image under the induced map, then Ψ' is again the set of roots of a complex structure and σ maps the complex structure defined by Ψ to that defined by Ψ'. If $\widetilde{\sigma}$ is conjugation by an element $g \in N(T) \cap H$, then σ is multiplication by g on G/H, hence preserves the given complex structure and consequently $T\widetilde{\sigma}$, which is an element of the Weyl group, preserves the set Ψ of roots defining the complex structure. Now let Ψ be a given subset of roots defining a complex structure on G/H; by Theorem 1.1.7 Ψ is closed and contained in the system of positive roots of G for some ordering of the roots; let Φ^+ denote this set of positive roots of G, hence $\Phi^+ = \Psi \cup \Phi^+(H, S)'$ for a system of positive roots $\Phi^+(H, S)'$ of H. It follows that there is an element w of the Weyl group of H such that $w(\Phi^+(H, S)') = \Phi^+(H, S)$ is the given positive set of roots of H, w is induced by an inner automorphism of H, which hence leaves Ψ invariant as just explained.

Conversely, assume Ψ is a closed system of complementary roots such that $\Phi^+(H, S) \cup \Psi$ is the set of positive roots with respect to an ordering of the roots, call this o; it must be shown it defines an invariant complex structure. If a root $\alpha \in \Phi^+(H, S)$ is the sum of two positive roots for the ordering o, then both the positive roots also belong to $\Phi^+(H, S)$, which shows that the simple roots of $\Phi^+(H, S)$ are also simple roots for the order o, hence we can write the set of simple roots for the ordering o as $\alpha_1, \dots, \alpha_n$, where $\alpha_i \in \Phi^+(H, S)$ for $i = 1, \dots, n-k$, and the elements of $\Phi^+(H, S)$ are the linear combinations with positive coefficients of the $\alpha_1, \dots, \alpha_{n-k}$, those of Ψ can be similarly written as positive linear combinations of $\alpha_1, \dots, \alpha_n$ for which at least one of the coefficients of α_i is positive for some $i > n-k$. It follows from this that there is an element s in the center of H for which $0 < \beta(s) < \frac{1}{2}$ for all $\beta \in \Psi$, and so as in (1.14) using the $\beta \in \Psi$ one defines $J_{\mathbb{C}}$ on the complexification $\sum_{\beta\in\Psi}(\mathfrak{g}_{\mathbb{C}})_\beta$ as $J_{\mathbb{C}} = \mathrm{Ad}(s)$, a complex structure. When restricted to $\mathfrak{g}/\mathfrak{h}$, $\mathrm{Ad}(s)$ has only imaginary eigenvalues; since Ψ is in the complement of the root system of H (indeed $\Phi^+(H, S) \cup -\Phi^+(H, S) \cup \Psi$ is closed), $\mathrm{Ad}(s)$ commutes with H and the complex structure $J_{\mathbb{C}}$ when restricted to $\mathfrak{g}/\mathfrak{h} \cong T_e(G/H)$ defines a complex structure on that space whose roots are Ψ. □

Proposition 1.1.10 *Let G/H be homogeneous and let q denote the dimension of the center of H, and n the rank of G. If $q = 1$ (resp. $q = n$, i.e., $H = T$) then the number of invariant complex structures is equal to 2 (resp. the order of the Weyl group $W(G)$). For any two such there is a homeomorphism of G/H induced from an automorphism (resp. an inner automorphism) of G which fixes H and maps the one complex structure to the other.*

Proof For $q = n$, Lemma 1.1.9 implies that the set of systems of roots of complex structures on G/T corresponds to the set of positive roots, i.e., to basis of the root system, on which the Weyl group acts simply transitively. By the definition of the Weyl group as $N(T)/T$, it is clear that these arise from inner automorphisms and hence extend to homeomorphisms of G/T. For $q = 1$, by assumption there is only one simple root of a basis of the root system $\Phi(G, T)$ which is complementary; we may suppose this to be the last one, so $\Phi(H, S)$ is generated by a set of positive roots $\alpha_1, \ldots, \alpha_{n-1}$; let α_n be the remaining basis element of $\Phi(G, T)$. Two complex structures are given by sets of complementary roots Ψ, Ψ' such that (by Lemma 1.1.9) $\Psi \cup \Phi^+(H, S)$ and $\Psi' \cup \Phi^+(H, S)$ are both systems of positive roots (for some bases) for G, and we may assume that $\alpha_n \in \Psi$. If $\alpha_n \in \Psi'$ also then $\Psi = \Psi'$, while if $-\alpha_n \in \Psi'$ then the complex structure is the complex conjugate one: since $-\alpha_n \in \Psi'$ it can be taken as a basis element, and the positive roots in the ordering defined by Ψ' are positive linear combinations of $\alpha_1, \cdots, \alpha_{n-1}, -\alpha_n$, and hence for any complementary root (which is not in the span of $\alpha_1, \ldots, \alpha_{n-1}$) the coefficient of $-\alpha_n$ is positive, hence $\Psi' = -\Psi$.

In general an automorphism of $\mathfrak{t}$ which permutes the roots extends to an automorphism of $\mathfrak{g}$, in particular this is the case for a reflection ($\alpha \mapsto -\alpha$ for some root α), hence also for the map $\alpha_n \mapsto -\alpha_n$ above. This defines an automorphism of G (we may assume G simply connected here) leaving T invariant, mapping $\Psi \mapsto -\Psi$ and leaving the roots of H invariant (spanned by $\alpha_1, \ldots, \alpha_{n-1}$ above), giving an automorphism of G/H mapping the complex structure to the complex conjugate one. The example of projective space shows that this automorphism is in general not inner. □

More generally, if Ψ, Ψ' are two root systems of complex structures such that there is an automorphism of the ambient space $(H^1(T, \mathbb{Z})^*)$ mapping Ψ to Ψ' and leaving the roots of H invariant, then the two complex structures are equivalent under an automorphism of G/H which is induced by an automorphism of G leaving H invariant, i.e., there is a G-diffeomorphism mapping one complex structure to the other while fixing H; an example of this is given by the statement concerning the two complex structures for $q = 1$ in Proposition 1.1.10. There are, however, examples of Ψ, Ψ' which do not fulfill this condition and hence the corresponding complex structures are not (necessarily) equivalent under a diffeomorphism of the homogeneous space, an example of which is provided by the following. Take $G = U(4)$ and $H = U(2) \times T^2$, maximal torus $T \subset U(4)$ with coordinates $x_1, \ldots, x_4 \in H^1(T, \mathbb{Z})$, subgroup $SU(4) \subset U(4)$ with corresponding maximal torus $ST \subset SU(4)$. The inclusion $ST \subset T$ identifies $H^1(ST, \mathbb{Z})$ with the quotient $H^1(T, \mathbb{Z})/\mathbb{Z}(x_1 + x_2 + x_3 + x_4)$. To get two sets of roots for complex structures Ψ and Ψ', apply Lemma 1.1.9, defining Ψ (resp. Ψ') as the set of positive simple roots for some ordering of the roots (choice of Weyl chamber). The roots of $U(4)$ are $\pm(x_i - x_j)$, $1 \leq i < j \leq 4$, the roots of H are $\pm(x_1 - x_2)$, and given the usual order $x_1 > x_2 > x_3 > x_4$ take the roots $\beta_1 = x_1 - x_3$, $\beta_2 = x_1 - x_4$, $\beta_3 = x_2 - x_3$, $\beta_4 = x_2 - x_4$, $\beta_5 = x_3 - x_4$ as the set of roots for a complex structure (defining Ψ), and take $\beta_1, -\beta_2, \beta_3, -\beta_4, -\beta_5$ as the set of roots of a second complex structure for the order $x_4 > x_1 > x_2 > x_3$ (defining Ψ') (here the lemma is applied implying both Ψ and Ψ' are the set of roots of a complex structure). The flag spaces of $U(4)$ and $SU(4)$ are the same, $U(4)/T \cong SU(4)/ST$, and since $SU(4)$ is simply connected, the transgression is an isomorphism $H^1(ST, \mathbb{Z}) \cong H^2(SU(4)/ST, \mathbb{Z})$. Hence the Chern classes of the complex structures, given by (1.15) are to be viewed modulo