Alina Galvão Spinillo
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Mathematical Reasoning of Children and Adults

Teaching and Learning from an Interdisciplinary Perspective

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ISBN 978-3-030-69656-6 ISBN 978-3-030-69657-3 (eBook)
https://doi.org/10.1007/978-3-030-69657-3

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

We dedicate this book to Analucia Dias Schliemann, David William Carraher, and Terezinha Nunes – those who, as teachers, have turned us into learners forever

Acknowledgments

We are grateful for the financial support received from the following institutions: Pro-Rectory of Research of the Federal University of Pernambuco (PROPESQ), Foundation of Support for Research of the State of Pernambuco (FACEPE), the National Association for Research and Post-graduate Programs in Psychology (ANPEPP), and the Brazilian Society of Mathematical Education (Section Pernambuco). This financial support allowed many of the authors of the chapters of this book to meet in May 2019 during the IV Brazilian Symposium of Psychology of Mathematical Education (IV SBPEM). The discussions carried out at that event served as a basis for the preparation of this book.

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Chapter 1
Mathematical Reasoning: The Learner, the Teacher, and the Teaching and Learning

Alina Galvão Spinillo, Síntria Labres Lautert, and Rute Elizabete de Souza Rosa Borba

Abstract This introductory chapter presents to the readers the foundations of this book and the chapters that constitute it. The chapters are presented in the light of the three instances that permeate the entire book – the learner, the teacher, and the teaching and learning. They are discussed having as background a field of knowledge of an interdisciplinary nature named psychology of mathematics education.

Keywords Mathematical reasoning · Learner · Teacher · Teaching · Learning

1.1 Introduction

An introductory chapter of a book is always a challenge; especially when it intends to go beyond the presentation of the chapters that compose it. Therefore, we proposed to discuss three key aspects that are addressed throughout this book: the learner, the teacher, and the teaching and learning of mathematics. These aspects are relevant topics in the field of psychology of mathematics education (see Resnick and Ford 1981; Gutiérrez and Boero 2006), which is the background that permeates this book dealing with different facets of mathematical reasoning of children and adults. Some of the chapters analyze the didactic challenges that teachers face in the classroom, such as how to interpret students' reasoning, the use of digital technologies, and their knowledge about mathematics. Other chapters examine students' opinions about mathematics, and others analyze the ways in which students solve situations that involve basic and complex mathematical concepts.

The approaches adopted in the description and interpretation of the data obtained in the studies documented in this book have in common the fact that they point out

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A. G. Spinillo et al. (eds.), *Mathematical Reasoning of Children and Adults*,
https://doi.org/10.1007/978-3-030-69657-3_1

the limits, the development, and the possibilities of students' thinking. Another characteristic that deserves to be highlighted is that this book brings didactic and cognitive perspectives to the learning scenarios in the school setting. Topics in each chapter are current and have been discussed at international scientific events in the field of mathematics education. These themes, and the way they are investigated, reflect the interests that researchers from different countries have in mathematical knowledge, its development, and issues related to teaching and learning.

1.2 The Learner

What about the one who learns mathematics in the school context? Who is this individual who, while being the target of the teaching process, is also the one who has an impact on this process, generating challenges for the one who teaches? Regardless of the answers given to these questions, which vary according to the approach adopted, one thing is certain: the learner is the focus of teaching and interferes with this process, requiring the teachers to systematically review their ways of teaching for learning to occur. Two points may be considered about the one who learns mathematics in the school context: the perception that the learner has about mathematics and the teaching of mathematics, and the fact that mathematical knowledge changes over time and under different circumstances.

1.2.1 The Learner's Perception of Mathematics and the Teaching of Mathematics Matters

The conception of the nature of mathematics and on the teaching of mathematics influence the learning of mathematics. This statement is supported by research carried out with students from different countries through different methodological resources.

Borthwick (2011), for example, carried out research on how Elementary School students perceived mathematics classes. The results showed that learners think that students should be encouraged to work more collaboratively, and to explain to others and listening to the explanations of colleagues about the strategies they adopt when solving mathematics activities. Taylor et al. (2005) investigated students' opinions about the teacher's role in learning mathematics. They reported that students perceive the teacher as a knowledge transmitter and, due to this, they assume a passive attitude toward their own process of learning. These studies revealed that the role attributed to teachers influences students' perception of how mathematics is learned and how they are positioned in their learning.

As important as knowing how learners perceive the teaching of mathematics is to know their opinion about the nature of mathematics. Hawera and Taylor (2011) asked students *What do you think math is all about?* They found that, in general, children perceived mathematics in a dichotomous way: as something that needs to be learned or as something that needs to have meaning and could be related to situations outside the school context. In other studies, children were asked a more specific question: *What is mathematics for?* or *What is the use of learning math?* (Spinillo 2018). Taken together, these investigations revealed that the uses attributed to mathematics can be characterized as school or academic, relating to everyday situations, professional activities, and referring to intellectual gains.

Following this same line of interest, Laurendon, Silva, and Spinillo (Chap. 13) investigated low-income children and adults from the same socioeconomic background and with the same level of schooling. The children were Elementary School students, and the adults were students of a teaching program aimed at people with late schooling, who for social and economic reasons did not have access to education or who were unable to complete schooling at an appropriate age. Despite belonging to the same social class and being at the same level of schooling, adults and children differed in the uses they assign to mathematics: children attribute school uses to mathematics while adults attribute uses related to daily and professional activities. The authors discuss the role played by school and out-of-school experiences in the participants' conceptions about the uses of mathematics.

The dichotomous perception about mathematics and the teaching of mathematics that seems to emerge from the results of these studies make it difficult for the learner to recognize the relations between mathematics learned at school and mathematics presented in practical everyday situations. This dichotomy may compromise their ability to capitalize on the potential that learning situations can provide.

1.2.2 The Learners' Mathematical Knowledge Changes

The learners' mathematical knowledge changes according to the development resulting from advancing age and school years, and according to the teaching experiences provided in the classroom context. Examples of these changes, especially in relation to basic mathematical skills relevant to the acquisition of later complex mathematical knowledge, are addressed in two chapters of this book dealing with number sense.

Both Serrazina and Rodrigues (Chap. 2), and Spinillo, Correa, and Cruz (Chap. 3) emphasize that number sense is a holistic and polysemic term. Nevertheless, there is a convergence in defining number sense as a good intuition about numbers, their relationships, and properties, which allow the individual to deal efficiently and flexibly with situations that involve mathematics. Due to its nature, many studies aimed to identify the components of number sense (Berch 2005). These components have been explored from a development perspective through longitudinal and cross-sectional studies in Kindergarten and Elementary School children (Yang et al. 2008). The main objective of these investigations was to describe the rate of progress presented by students and to show that number sense was a strong predictor of later performance in mathematics. However, as Spinillo, Correa, and Cruz (Chap. 3) highlight, it is also relevant to know, in a developmental perspective, which components of number sense are understood before others. The results revealed that some notions related to number sense are already mastered by children since the beginning of Elementary School, while others are acquired with the advance of schooling, and others remained a challenge for them, even with the advance of schooling.

Changes in the learners' mathematical knowledge also occur as a function of teaching experiences promoted in the classroom. Even though number sense cannot be a product of direct instruction, it can be developed from the way in which mathematical concepts are taught (Greeno 1991). Based on this assumption, intervention studies encouraged learners to use flexible ways, such as mental calculation to solve problems. Mental calculation is considered a transversal ability that should be integrated into the teaching-learning process of mathematics throughout schooling instead of procedural algorithms (Carvalho and Ponte 2019).

According to Serrazina and Rodrigues (Chap. 2), the efficient use of calculation strategies needs number sense and using such strategies leads to its development. The authors discuss data from previous studies describing strategies used by the learners and the actions conducted by the teacher to illustrate how the interplay between students' reasoning and teachers' actions may contribute for the development of mathematical thinking. This development is exemplified by passages extracted from dialogues that show the changes observed in the learners' procedures when solving arithmetical problems. These changes revealed progress derived from an environment based on discussion fostered by the teacher who systematically invited students to explain their solution procedures and to justify their ways of reasoning.

These two chapters illustrate the changes occurring in the mathematical reasoning of the learner according to schooling and didactic situations proposed in the classroom. In turn, Gómez and Morales (Chap. 4) analyze these changes from a broad review of the literature on research involving learners from early childhood to adulthood. The central topic of the chapter is numerical magnitude, a subject considered an essential property of numbers (Fazio et al. 2014), and its understanding presents learners with challenges of increasing complexity across the schooling process. An interdisciplinary approach is adopted to discuss empirical data that contribute to the understanding of how people's minds and brains process numerical magnitude (a topic associated with number sense) and to the understanding of children's learning and development. Initially, empirical arguments are presented in defense of the hypothesis that the educated human mind processes numbers in a manner similar to a mental number line. Then, Gómez and Morales focus on studies about the processing of numerical magnitude across different number systems. Throughout the chapter, the authors keep an interdisciplinary dialogue between neuroscience and mathematics education.

Changes also occur in the mathematical reasoning of children with sensory limitations, as shown by investigations on numerical cognition in deaf individuals (Nunes 2004). Most research compares deaf and hearing children; however, there are studies that specifically target deaf sign language users (Madalena et al. 2020).

This is the case with the research carried out by Bedoya-Ríos and Dorneles (Chap. 5). In this study, a set of strategies used by Brazilian and Colombian deaf children when performing a numerical estimation task are described. For the authors, the task of estimating is a direct way to access the understanding children have about the representation of the numerical system which is the basis of later mathematical development. The changes identified were associated with the strategies used by participants that expressed different types of knowledge that emerged when they solved a numerical estimation task. The main conclusion was that deaf children are not familiar with the number line and have a limited knowledge about numerical estimation.

Considering these two aspects, it is possible to return to the question posed in the beginning of this section of this chapter: What about the one who learns mathematics in the school context? What can be said is that: (i) the learner has opinions about the nature and the teaching of mathematics, and that these perceptions need to be known by teachers in order to review and adapt their ways of teaching to promote the learning of their students; and (ii) the learner's mathematical knowledge changes due to several factors, and that this also needs to be known by teachers. These aspects are a fundamental knowledge to be mastered by those who want to teach mathematics, as discussed below.

1.3 The Teacher

Research on the mathematics teacher has, to a large extent, been performed considering Lee Shulman's and Deborah Ball and collaborators' theoretical presumptions. These authors have presented what they consider to be essential knowledge that teachers must have, in order to efficiently exercise teaching in their classrooms. Shulman (1986) deals with teaching knowledge more generally – common to all areas of knowledge – and Ball and Bass (2003), and Ball et al. (2008) discuss, more specifically, the teaching knowledge necessary for teaching mathematics. We will present the central assumptions defended by these authors and relate them to research dealt with in this book.

Shulman (1986) questioned if *subject matter content knowledge* would be sufficient for classroom teaching and learning. One could ask, for example: Is it enough for teachers to have a good mathematical mastery in order to competently teach algebra (as addressed on Chap. 11) or multiplication and division (as dealt with on Chap. 12)?

Ribeiro, Aguiar, Trevisan, and Elias (Chap. 11) argue on the importance of whole-class discussions in the teaching of algebra and they view these moments not only beneficial to students' learning, but also as means of (re)construction of teachers' mathematical knowledge to teach. In this sense, the knowledge of students' mathematical reasoning is important in building teachers' knowledge in a broad way and their domain of algebra per se is not enough.

In a similar direction, Santos, Araújo Gomes, and Gomes (Chap. 12) stress the importance of teachers' knowledge on problem posing, in particular in the formulation of multiplication and division problems. This ability goes beyond content knowledge concerning these arithmetic operations, but also involves knowing different problem types and the adequation of these types to different school levels (see Spinillo et al. 2017).

The examples here discussed indicate that, with regard to the mastery of the contents to be taught, it is not enough that the teacher is able to present, in a clear way, to the students the concepts that one wishes to develop. The teacher also needs to explain the principles and rules associated with the concepts. Additionally, it is necessary to know how to justify why the knowledge of a certain concept is necessary, both theoretically and in practice. The teacher also needs to know how a given concept is related to other concepts – within the same area of knowledge and outside it (Shulman 1986). Knowing how to motivate the learning of mathematics is also expected from the teacher. In this way, content knowledge is broader than it may appear to be. Teacher's content knowledge has aspects for him/her needed, even if it is not necessary for other professionals who use the concepts he/she teaches. The teacher must know that a concept applies in a given situation – as other professionals using the concept do – but also why it applies and how its use is justified and can be valued.

In addition, it is desirable for the teacher to understand why a specific content is central (or peripheral) to the area he/she teaches. This knowledge allows him/her to understand how the content is placed in the curriculum – the order of presentation and its deepening throughout schooling. In this direction, Shulman (1986) highlights another domain of teaching knowledge: *curricular knowledge*. In this domain, he includes: content teaching programs at different levels of teaching and instructional materials that meet the programs. Efficient teaching of content also permeates this teaching knowledge and, in the case of the topics in this book, it is necessary for the teacher to know how number sense, counting, multiplication and division, combinatorics, algebra and functions are placed in the curriculum so that he/she stimulates the development of students in these areas.

Shulman (1986) adds a third knowledge indispensable to the practice of teaching: the *pedagogical content knowledge,* that is, how to make a certain content teachable. For this achievement, the teacher needs to know various ways of representing the concepts involved in the contents, the most powerful analogies, illustrations, examples, explanations, and demonstrations. This knowledge allows the content to be understandable to the group of students with whom he/she works. This pedagogical knowledge also involves the understanding of what makes content easier or more difficult, the knowledge of possible previous understandings of students and the misconceptions they have, as well as the mastery of diverse strategies to assist students in overcoming their difficulties.

This pedagogical content knowledge is necessary for teachers to be able to formulate problems, as addressed by Santos, Araújo Gomes and Gomes (Chap. 12). This dimension goes beyond the knowledge of the concepts involved, but also includes distinct problem types, possible solution representations, and resolution procedures. As stated by Chapman (2012), problem posing depends on mathematical content knowledge, but goes beyond. Previous experience with problem solving and creativity are also required, as well as the knowledge of different meanings for each arithmetic operation.

In summary, Shulman (1986) argues that teaching involves the conceptual and operational mastery of the content to be taught, as well as an understanding of what needs to be learned and how it can be taught. Teaching needs to be flexible and adaptable to the characteristics of the group of learners it is intended for, the complexities of the content and the conditions of the school environment, always keeping in mind students' meaningful learning. These elements are indispensable to the basic knowledge for teaching and the interaction of *content*, *curriculum*, and *pedagogical knowledge* is necessary for teaching and learning to occur in the best possible way.

The knowledge base for teaching, proposed by Shulman (1986) is not particular to a field of knowledge, but it applies to the various areas of teaching. Deborah Ball in numerous studies, based on Shulman's assumptions, discussed specific teaching knowledge for the mathematics teacher's practice.

Ball and Bass (2003) state that the quality of mathematics teaching and learning depends on what teachers develop with their students and what teachers can do depends on their knowledge of mathematics, on how its learning happens, and how to teach it. In this sense, it is necessary to help teachers to develop broad knowledge that supports their teaching practice – that makes them able to understand the ideas exposed by their students, that enables them to elaborate strategic questions, and, also, that allows them to analyze the potential of mathematical tasks, such as those exposed in textbooks.

In the direction of indicating Mathematical Knowledge for Teaching (MKT), Ball et al. (2008) presented subdomains of the knowledge proposed by Shulman (1986). There are six subdomains proposed by them, the first three closest to the knowledge of specific content and the last three associated with pedagogical content knowledge: (1) common content knowledge, (2) specialized content knowledge, (3) horizon content knowledge, (4) knowledge of content and students, (5) knowledge of content and teaching, and (6) knowledge of content and curriculum.

Common content knowledge can be used in a variety of situations and is not knowledge exclusively used by teachers. Other professionals have this knowledge domain and use it to solve different problems. For example, this knowledge is necessary for the correct application of algebraic thinking and, in another example, is needed when multiplying or dividing to find the answer of problems that involve these operations.

In turn, the *specialized content knowledge* is geared toward teaching, and is therefore a specific domain for teachers. One can take as an example (as discussed in Chap. 11) the knowledge that teachers have regarding algebraic generalizations (recognition of patterns and regularities) and appropriate justifications of generalizations.

The *horizon content knowledge* allows the teacher to recognize relationships between varied contents of Mathematics, as well as to know ways of deepening the mathematical contents. Teachers need this knowledge to decide which multiplicative problems (as those presented in Chap. 12) can be taught in each school level.

From the pedagogical knowledge block, the first highlight of Ball et al. (2008) is *knowledge of content and students*. Based on this knowledge, teachers can predict their students' possibilities and difficulties in learning the mathematical content presented in the classroom, as well as knowing what can make a content interesting and motivating. With the use of this knowledge, teachers have the ability to anticipate students' correct reasonings and possible errors. As mentioned by Ribeiro, Aguiar, Trevisan, and Elias (Chap. 11), Steinbring (1998) points out the importance of teachers' awareness of students' mathematical knowledge and recognizes that teachers are not able to directly guide pupils' learning, but they are able to provide learning environments that may enrich students' learning processes.

The subdomain of *knowledge of content and teaching*, in turn, allows the teacher to evaluate advantages and disadvantages of certain activities in the classroom – including the instructions given and the representations used to teach a specific content.

Finally, *knowledge of content and curriculum* also supports the teacher inside and outside the classroom. Knowing what is prescribed in official documents and what is presented in didactic materials allows the teacher to better plan his/her classes, as well as allows his/her teaching practice to be supported by diverse curricular information.

Hill et al. (2005) measured Elementary School teachers' mathematical knowledge for teaching and examined the relations between this knowledge and students' mathematical development. Subsequently, Hill et al. (2007) presented qualitative data that indicates that the quality of mathematical instruction is related to teachers' mathematical knowledge for teaching. Aiming to further understand the role of teacher knowledge, Baumert and Kunter (2013) investigated the impact of *content knowledge* (CK) and *pedagogical content knowledge* (PCK) on instruction. This longitudinal study involved teachers and students in tenth grade classes. The main theoretical assumption, confirmed in the data collected, is that PCK is not possible without a significant level of CK, but CK, on its own, does not provide basis for teachers to deliver instruction that is cognitively active nor to provide support for students' learning. In this sense, relations between CK and cognitive activation, and individual learning support were expected to be statistically lower than the relations with PCK. Results pointed out that PCK determined extensively the cognitive structure of mathematical learning opportunities. This applied to a very limited extent to CK, thus CK had a lower predictive power for students' progress, with a direct impact only on the alignment of tasks to the curriculum of grade 10.

Certainly, there is a positive association between the knowledge that teachers have and the quality of teaching that occurs with students. Thus, there is a strong relation between what the teacher knows and what his/her knowledge enables him/her to accomplish in his/her classrooms. A good mastery of teacher knowledge – acquired through teacher training and experience in the classroom – will, certainly, provide strong links between the student, the mathematics taught, what the students effectively learn, and the development of their reasoning.

1.4 The Teaching and Learning

Weaving reflections on teaching requires considering that teaching and learning are two inseparable processes and that it cannot be said that there was teaching if there was effectively no learning by the learner. This leads us to defend the need to provide learners with teaching situations that promote a conceptual understanding of mathematics. Baumert et al. (2013), for example, emphasize structural features of teaching that should be considered in insightful learning processes, among which (i) the construction of challenging tasks; (ii) monitoring the learning processes; and (iii) providing appropriate feedback and support when difficulties arise. In addition to these characteristics, it is also important to highlight the role of representations on these situations, as discussed in some of the chapters presented in this book that focus on teaching and learning mathematics.

This book discusses contexts of interaction between the adult and the learner during situations proposed in the school context or in investigative individual situations. These contexts are taken with a view of illustrating teaching possibilities on specific mathematical topics (counting, combinatorics, and functional and algebraic reasoning). They contribute to a deeper understanding of teaching and learning, clarifying aspects that need to be considered when seeking to develop mathematical reasoning. The studies reported in this book are based on the idea that teaching should start with motivating and simple situations that serve as a basis for, gradually, solving more complex tasks, as the adult interacts with the child/adolescent. These interactions occur through the resolution of a variety of activities, using different resources (pencil and paper, illustrative cards, and different digital technologies).

Oliveira, Yokoyama, and Koslinski (Chap. 6), based on the theory of counting (Sarama and Clements 2009) and on the five principles of counting documented in the literature (Gelman and Gallistel 1978), evaluate the initial conceptions of very young children (Nursey and Kindergarten). In this research, the factors related to the participants (age/level of education) and the factors related to the tasks presented (number of objects and how the objects are organized for counting) were explored. The study revealed that the number and the arrangements of the objects influenced the way children counted, and that this was observed either when correct or incorrect strategies were adopted.

What does this empirical evidence bring to the debate about teaching and learning? First, very young children have schemes of action that guide them in solving a set of mathematical situations. These schemes are crucial for the construction of new knowledge. However, for children to advance in their schemes, it is necessary to provide them with a variety of situations involving counting, above all, situations in which they can organize the objects in different ways. The activities proposed in the test developed by Oliveira, Yokoyama, and Koslinski (Chap. 6) can both help the teacher to identify the counting skills of their students and can serve as a source to create activities to be explored with students in the classroom. The second aspect highlighted in this chapter refers to the impact that representations, whether symbolic or not, have on ways of reasoning.

Since the most remote times, individuals have been faced with social practices and means of production that allow them to create and produce new forms of interaction between themselves and the objects of the world, enabling them to acquire the basic skills necessary to successfully deal with the demands they face in everyday life. These basic skills, according to Nunes (1997), are driven by the use of *cultural amplifiers*, which enable individuals to act, perceive, and reason beyond the limits of their capabilities. Representation systems are a type of cultural amplifier that have evolved, becoming more and more abstract due to the uses and meanings that each social context requires.

Therefore, the different forms of representations are a key aspect for learning mathematics, which should be involved in teaching situations. First, because representations are one of the aspects that constitute concepts (together with operative invariants and contexts) as proposed by Vergnaud (1983). Second, because representations are crucial for the establishment of relations between the individual and the symbolic collection of their culture (Da Rocha Falcão 1997).

The impact of representations on ways of thinking is evidenced in solving combinatorial problems, when Borba, Lautert, and Silva (Chap. 7) point out that illustrative cards helped Kindergarten children to reflect on the relationships present in combinatorial reasoning more than drawings did, and in discussions about the use of tables as tools to think about functions, as indicated by Brizuela, Blanton, and Kim (Chap. 8). Representations are also highlighted in generalizations of information through algebraic notations after teaching situations proposed by Teixeira, Magina, and Merlini (Chap. 9) and on how technological tools can stimulate algebraic understanding, as advocated by Castro Filho, Castro, and Freire (Chap. 10).

In the studies discussed by Castro Filho, Castro, and Freire (Chap.10), the software acted as a symbolic context that led the user to reason about several abstract ideas in mathematics. The manipulations generated reflections that allowed to raise and test hypotheses about the mathematical properties present in the proposed activities involving algebraic relations. Although the domain of representations does not guarantee conceptual understanding per se, the use of diverse and increasingly elaborate representations are powerful tools, as it depends both on the logic of the individual and on their ability to use the representation systems. This is because representations have a dual function when triggered via teaching situations: expressing and expanding/limiting the knowledge about a certain mathematical concept by the learners.

Both the research presented by Borba, Lautert, and Silva (Chap. 7) and the study conducted by Brizulela, Blanton, and Kim (Chap. 8) point out that Kindergarten children can find simple solutions to complex concepts that were not taught in school using appropriate representations. For example, the use of the table served as an important representational tool through which the child reasoned and justified his/her ideas about algebraic relations. When generalizing about notation using tables, the child was reasoning about functional relations, making inferences, and drawing conclusions. Although these studies reveal a variety of possible forms of representation for the development of mathematical reasoning, this alone is not sufficient for the conceptual construction, as it depends largely on the reflection on the ways of thinking and the role of verbal explanation in the acquisition of knowledge.

Another aspect presented in this book, refers to the relevance of intervention studies that promote situations in which the learner is requested to *think about thinking*, to think about their way of solving mathematical tasks such as those involved in solving a problem posed to them. The ability to make their own thought as an object of analysis and reflection involves, among other aspects, awareness of the acts and processes of knowing and reasoning in a given situation. This became known in the literature as *metacognition* (Flavell 1976, 1979). The term *meta* designates the knowledge and intentional control that individuals have over their own cognitive activity, which can, in turn, be triggered and self-regulated by individuals. This allows to understand the active participation of the learner in the learning situation, because it makes him/her a spectator of his/her thought processes. In other words, the learner becomes, simultaneously, agent and object of reflection of his/her own learning (de Jou and Sperb 2006). Some of the studies presented in this book are examples of how metacognitive skills can be triggered via teaching situations.

For these skills to be developed, it is essential that the teacher be aware that metacognition is a powerful didactic resource. Unlike that teacher who performs metacognitive activities in a spontaneous way, the teacher who knows this resource makes better use of opportunities in the school context, because he/she systematically asks his/her students to demonstrate their knowledge, communicate their ways of thinking, discuss the implemented forms of solution, whether these are appropriate or not. Such didactic actions allow the student to review their own perspective and change it when necessary.

Thinking about thinking is not something the individuals spontaneously do. There is always a demand that provokes them to communicate what they know or to explicitly show what they are doing. Examples of such situations are evident in the interventions carried out by the adult in Brizuela, Blanton, and Kim's study (Chap. 8) when the children were asked: "Do you see any relationships here?" "Anything this is saying that's going on, any patterns in these numbers? Tell me about them" or "What does that mean?". In doing so, the examiner highlighted the child's thinking, redirected the resolution process (when necessary), encouraged the use of mathematical notations (using tables), and discussed possible contradictions in the solutions found by the child. Tables were used in this scenario both to communicate and to allow an understanding of the algebraic relationships present in the proposed situation.

In addition, the interventions addressed to the child aimed to transform implicit knowledge into explicit knowledge. To transform implicit into explicit knowledge, it is essential to understand, from the point of view of teaching, the role played by verbal explanation by the part of students. This is because language has the dual function of both expressing thought and constituting it.

In fact, teaching situations that aim to promote conceptual understanding should be provocative and stimulate verbal explanations by the part of students. Verbal explanation (whether in collective situations as in the classroom context or individually as in controlled situations such as in research contexts) encourages learners to reflect on their own ways of reasoning. In Borba, Lautert, and Silva's study (Chap. 7), for example, children were encouraged to reflect and discuss with the examiner about the relations of choice and ordering of elements and the exhaustion of possibilities using different combinatorial situations (*arrangements, combinations, permutations, and products of measures*). In a systematic way, the examiner asked the children to talk about the strategies they adopted to solve the problems, in order to draw attention to the invariant principles present in the situations, which needed to be considered in the resolution of what was proposed.

Teaching activities should be proposed in such a way as to also guarantee a diversity of interactions between students, making it possible to compare their ways of thinking with that of others, as well as requesting justifications for how they performed a particular activity. By asking for explanations and justifications about their ways of reasoning, the teacher could know what their students know and, above all, what errors they present, since their errors express the difficulties they encounter when learning (Spinillo and Lautert 2012). The relevance of encouraging communication and discussion in the classroom is not new aspect mentioned in the literature about the teaching of mathematics. In fact, several studies call attention to this issue, stressing the positive impact these didactical opportunities have in learning (e.g., Moschkovich 2010; Sfard 2012).

In whole classroom discussion or in small groups, teachers may raise questions that draw students' attention to the fact that there are other ways of thinking when dealing with a given situation (Does everyone think so? Who thought otherwise?), independently of whether they were correct or incorrect. The relevant aspect to be stressed here is that students may either express their ways of reasoning and listen to others' ways of reasoning. One example of this is presented by Teixeira, Magina, and Merlini (Chap. 9) when students needed to reach a consensus when solving tasks in pairs. Such a challenge allowed students to interact with each other, compare different procedures and opinions, and, then, decide how they could solve the problems presented to them.

One may conclude that, as stressed by Baumert et al. (2013), one of the major challenges faced by the teachers is to create a learning environment capable of stimulating students to think and to be cognitively engaged when solving classroom tasks. The teaching process involves a cognitive activity that requires conceptual knowledge and forms of organization that enable teachers to develop an effective professional practice.

1.5 Final Remarks

Our final considerations concern how *the learner*, *the teacher*, and *the teaching and learning* are intrinsically related. If one wants to understand how students can better learn mathematics, it is necessary to consider what teachers know and how teaching and learning mathematics occur.

It is important to highlight that learners' perceptions about mathematics and on how it should be taught varies in different groups of individuals. Some perceptions are associated more to school uses and others to day-to-day activities, when what should be expected is that students recognize the importance of both school and out-of-school mathematics in learning situations.

Another relevant aspect is that learners' mathematical knowledge changes over time and is related to the circumstances in which teaching, and learning occur. In this sense, it is important – as reported in chapters of this book – to investigate what mathematics is mastered by different groups of students, their learning difficulties, and how conceptual development may be influenced by schooling.

From the teachers' perspective, they need to be aware of how students perceive mathematics and what may influence their mathematical development. In this direction, a large range of knowledge is required by teachers: conceptual and operational mastery of the school contents, understanding of how learning occurs, and how mathematics can be taught.

The research presented in the chapters of this book points out as good classroom teaching-learning practices: stimulating students to explain procedures, to interact in whole class discussions, and to justify their thinking in verbal explanations. In turn, teachers, need to understand students' ideas, to be able to elaborate strategic questions, and to be capable of analyzing potentialities of mathematical activities. In this direction, teachers may provide rich learning environments – in particular with diversified mathematical situations and varied symbolic representations, aiming to provide active participation of the learners in teaching situations.

Acknowledgments Thanks to the Coordenação de Aperfeiçoamento de Pessoal de Ensino Superior (CAPES) of the Brazilian Ministry of Education for the support given to the Post-graduate Programme on Cognitive Psychology and to the Post-graduate Programme on Mathematical and Technological Education, both at the Federal University of Pernambuco.

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Part I
Mathematical Reasoning: Developmental Approach

Chapter 2
Number Sense and Flexibility of Calculation: A Common Focus on Number Relations

Lurdes Serrazina and Margarida Rodrigues

Abstract There is a consensus that twenty-first century citizens need competencies such as flexibility, creativity, critical thinking, taking different perspectives, and considering multiple solutions for which number sense and mental calculation flexibility play a crucial role. Number sense implies to know numbers and its relations with two central characteristics: (i) progressive development throughout schooling; (ii) global character, implying knowledge about numbers and operations and their flexible use in making mathematical judgments and solving problems. Mental calculation flexibility involves to know how to use dynamically number sense, number patterns, and being attentive to problem characteristics to achieve a solution, which, in turn, depends on students' knowledge of numbers and operations and their relations and how they are able to create a network of number relations. This chapter discusses those inter-relations based on relevant literature and illustrated by data from Portuguese elementary school. Through data analysis of excerpts of students' resolutions of open tasks and the respective discussion, the data illustrate the relevance of constructing number relations and how the teacher's role, particularly the teacher's actions in whole class discussion, can create a classroom environment conducive to the development of both number sense and flexibility in calculation.

Keywords Number sense · Flexibility of calculation · Number relations · Early years

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A. G. Spinillo et al. (eds.), *Mathematical Reasoning of Children and Adults*,
https://doi.org/10.1007/978-3-030-69657-3_2

2.1 Introduction

Renato, a first grader, when the challenge of representing the day number, on 19 March, wrote 100:100 + 18 = 19. In its turn, Dario represented 19 as 38:2, justifying that because 19 + 19 is 38. Another student represents 19 as 3 × 5 + 4 (Serrazina and Rodrigues 2017).

These students were able to create multiple representations of 19, as they established multiple relations between numbers and operations, that is, they used their number sense. What does it mean to have number sense? For Mcintosh et al. (1992, p. 4), number sense reflects a "propensity for and an ability to use numbers and quantitative methods as a means of communicating, processing and interpreting information." But the expression *number sense* is not consensual in literature (Andrews and Sayers 2015; Berch 2005). In a review of literature, Whitacre et al. (2020) consider number sense as polysemy, identifying three constructs: (i) *approximate number sense* (ANS) – "a basic neurological ability, related to visual and auditory perception" (p. 10); (ii) *early number sense* (ENS) – "learned skills that involve explicit number knowledge, such as counting items using number words and comparing numbers represented symbolically as numerals" (p. 10); (iii) *mature number sense* (MNS) – a construct "that features prominently in mathematics education literature" (p. 11), including multidigit and rational number sense, and like ENS, is learned. Flexibility is one of the characteristics of this construct. For the authors, ENS is more related to skills that preschool and primary graders should learn while MNS is more related with conceptual structures "rather than directly observable skills" (p. 12). In this chapter, the expression "number sense" is used encompassing ENS and MNS constructs as referred by Mcintosh et al. (1992, p. 3), being its "acquisition a gradual, evolutionary process, beginning long before formal schooling begins." This idea of number sense "reflects a set of understandings and skills" (Andrews and Sayers 2015, p. 258). For Reys (1994, p. 115), number sense is not identified as something that a person has or not, but a person with number sense is able to:

> look at a problem holistically before confronting details, look for relationships among numbers and operations and will consider the context in which a question is posed; choose or invent a method that takes advantage of his or her own understanding of the relationships between numbers or between numbers and operations and will seek the most efficient representation for the given task; use benchmarks to judge number magnitude; and recognize unreasonable results for calculations in the normal process of reflecting on answers.

This author also considers that number sense is a process that develops and matures with experience and knowledge. For us, the idea of number sense has essentially two characteristics. One is its progressive development, that is, it should be developed throughout schooling, not learnt at once (Abrantes et al. 1999; NCTM 2000; Serrazina and Rodrigues 2017). The other includes a global intuition about numbers and operations. This general understanding means that number sense comprises knowledge about numbers and operations and their flexible use in making mathematical judgments and solving problems (Mcintosh et al. 1992). So, the dif-