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Walter Dittrich

The Development of the Action Principle

A Didactic History from Euler-Lagrange to Schwinger

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This volume is dedicated to my dear wife
Ginny Dittrich.

Preface

This work is devoted to the development of the history of the principle of stationary action. It will be divided into two parts: in the first, we will pay tribute to the founders of the action principle, while the second part provides many different worked-out, selected examples which cover in great detail the achievements between the seventeenth and the beginning of the twentieth century. It is exciting to see how the greatest scientists of their time struggled with a final, mathematically satisfying formulation. Foremost Daniel Bernoulli, Euler, d'Alembert and Laplace shaped the history of the early development of the action principle. Later on, Lagrange, Hamilton and last, not least, Schwinger finalized and polished the early attempts to transform almost all of nature's formidable problems into one dynamical principle.

How this principle is put to work and how much we have learned since Euler's fundamental contributions to the lemniscate problem (which spurred Gauß' motivation to develop the elliptic functions) and his introduction of the first classical field theory, namely, of the mathematically founded theory of hydrodynamics, is the leading theme in the present work.

In the nineteenth and twentieth centuries, we see the action principle celebrating triumphs in all branches of classical and quantum theory. The discovery of Einstein's field equations in gravity by Hilbert via the action principle and all the other applications of the action principle in classical and quantum theories of modern times can be traced back to the magnificent achievements by Euler and his contemporaries, and are valid still today.

Tübingen, Germany
April 2021

Walter Dittrich

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My sincerest thanks go to Ms. Ute Heuser, who has guided almost all of my book publications with great commitment through Springer's sometimes challenging publication procedures. The huge world-wide proliferation of our—my and my co-authors'—books can be credited in no small part to her. Needless to say, it is always a pleasure to speak with her on the phone or to read her "good news" announcing the acceptance of a new manuscript for publication. *Ad multos annos* is my greatest wish.

Short Historical Introduction

The first chapter of this history is primarily characterized by the study of the *curva elastica.* It was the curva elastica that was the starting point of one of the most exciting developments in the history of mathematics. The functional representation of this curve remained obscure for a long period, since it could not be expressed with functions known at the time.

In particular, the *curva elastica* had aroused strong emotions and led to a rift between the Bernoulli brothers due to hefty priority conflicts. Johann went so far as to publically announce that he would not return to Basel until his brother, Jakob, was dead—the same older brother who had introduced him to the secrets of mathematics. Jakob died in 1705, but his family did not allow Johann access to his brother's inheritance. Before his death, Jakob had, however, made great progress regarding the *curva elastica.* He succeeded in reducing the problem in the rectification of the *curva elastica* to the so-called *lemniscate.* This Bernoullian *lemniscate* is a special case of the Cassini curves and looks like the mathematical symbol for infinity. The name comes from the Greek λεμνισκοζ, meaning a ribbon or band. Count Fagnano achieved an equally important breakthrough in the same direction. By using the doubling equation for the *lemniscate* arc, he made a contribution of lasting value. Self-confidently, he had the title page of this work decorated with a *lemniscate* arc. Fagnano had sent his research that had been published in 1750 to the Berlin Academy, of which he had previously become a member. These works landed in the hands of Euler on December 23, 1750. The details were passed on by Jacobi, who, in preparing an edition of Euler's complete works, wrote: *On this occasion I also discovered an unusually important day in the history of mathematics, [the day] on which our academy requested Euler to examine the work that had been submitted by Fagnano before acceptance for publication. As a result of Euler's investigation, the theory of elliptic functions emerged.* Of all Euler's discoveries, the addition theorem for *lemniscate* integrals and their generalization had the greatest impact on the further development of analysis in the nineteenth century.

Nevertheless, it should be said that in spite of the brilliant results of Euler's work on the *lemniscate* produced, he was still dealing with elliptic or, better, *lemniscate* or Fagnano integrals, and not elliptic functions, which are the inverse of elliptic integrals. The latter were first introduced in the history of mathematics by Gauß.

Just as the history of elliptic integrals began with the lemniscate integral, so did the history of the doubly periodic *lemniscate* or of the elliptic functions start with their inversion and continuation into the complex plane. Their beginnings and the developments that followed explosively already emerged during Gauß' lifetime in the work done by Abel, Jacobi, Eisenstein, Weierstraß, etc., and were in fact due to Gauß' further strokes of genius. Unfortunately, he never published anything on this success in the realm of elliptic functions.

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About the Author

Author (r) with J. Schwinger, 1981. Photo: Tübingen University, Germany

Prof. Dr. Walter Dittrich was head of the quantum electrodynamics group at the University of Tübingen until his retirement in 2001 and is still actively publishing papers and books in classical and quantum physics. He received his doctorate under Prof. Heinz Mitter at Heisenberg's institute in Munich and continued pre- and postdoc work at Brown University, Harvard and MIT. He profited immensely from lectures by and discussions with Profs. Herb Fried, Ken Johnson, Steve Weinberg, Julian Schwinger and, later on, at the Institute for Advanced Study (IAS) in Princeton, Steve Adler and David Gross at Princeton University. He started his work on gauge theories and QED in collaboration with Schwinger in the late 1960s. He was visiting professor at UCLA, Berkeley, Stanford and the IAS. He has over 30 years of teaching experience and is one of the key scientists in developing the theoretical framework of quantum electrodynamics.

Chapter 1
Curva Elastica

In the following, we will consider the problem of determining the forms which an infinitesimally thin rod can take when held by constant forces at the end only. A measure for the rod's bending is denoted by the flexural rigidity B. (If the rod had a finite cross section, we would find for B the product EI, where E is Young's modulus for the material of the wire, and I is its moment of inertia.) We are dealing instead with a line - the neutral line -, which is neither stretched nor compressed; so the rod is in equilibrium under the action of the force couple $\boldsymbol{F}$ and $-\boldsymbol{F}$, i.e., the force acting as a couple on the curve line is zero (See Fig. 1.1.).

The bending moment (torque) at the point $P(x, y)$ of the curve is a flexural couple $\boldsymbol{D} = D\hat{z}$ in the z direction at (x, y) with magnitude

$$D = D_z = F \cdot y\,, \tag{1.1}$$

where $\boldsymbol{F}$ is constant all along the arc, and is, in fact, the magnitude of the single force in the string which connects the two endpoints on the x line. y is the distance from the x axis.

If ρ is the radius of curvature at the point (x, y) in a Cartesian coordinate system where the x axis is horizontal, the y axis is vertical, and radius ρ of curvature positive when the curve is concave upward (∪), it follows from calculus (Newton) that ($y' = dy/dx$):

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{\frac{3}{2}}}\,. \tag{1.2}$$

However, since we want the bending moment positive when the curve is concave downward (∩), we have to replace ρ by $-\rho$:

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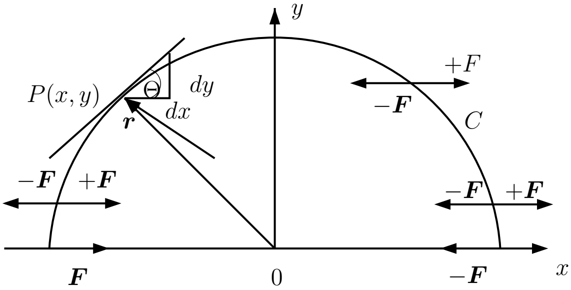

Fig. 1.1 Curva elastica

$$\frac{1}{\rho} = \frac{-y''}{(1+y'^2)^{\frac{3}{2}}} \,. \tag{1.3}$$

Writing

$$D = F \cdot y = B\frac{1}{\rho} = B\kappa \,, \qquad y = \frac{B}{F}\frac{1}{\rho} \,, \tag{1.4}$$

we see that the curvature κ is proportional to the coordinate y at the point (x, y): $\kappa \sim y$. At this stage it is convenient to introduce the substitution $a^2 = \frac{4B}{F}$ so that

$$y = \frac{a^2}{4}\frac{1}{\rho} = \frac{a^2}{4}\kappa \,. \tag{1.5}$$

Then Eq. (1.3) together with (1.2) turns into

$$y = \frac{-a^2}{4}\frac{y''}{(1+y'^2)^{\frac{3}{2}}} \,. \tag{1.6}$$

Note: It took several years from the beginning of the 18th century for giants like Leibniz, Jacob and Johann Bernoulli, as well as Euler, to discover and then understand the full meaning of Eq. (1.6). Also Daniel Bernoulli and Fagnano delivered important contributions. The highpoint in solving the equation for the *elastica* was reached when Gauß (in January of 1797, when he was just 19 years old) turned his attention to the *lemniscate* problem. This was one of the brilliant moments in the history of mathematics and theoretical physics. Gauß noticed that the *curva elastica* (*lemniscata*) opened a window on a brand new field of mathematics when he introduced the a.g.M. (arithmetic-geometric-mean or elliptical functions). He also extended the new functions into the complex plane! The followers, Legendre, Weierstrass, Jacobi and many others, extended and completed the whole problem which began with such a simple physical object as the *curva elastica*. We will cast a brief glance at the achievements that were made over the centuries, but which are still of great importance up to the present.