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Domination Games Played on Graphs

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Boštjan Brešar • Michael A. Henning
Sandi Klavžar • Douglas F. Rall

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Boštjan Brešar
Faculty of Natural Sciences & Mathematics
University of Maribor
Maribor, Slovenia

Michael A. Henning
Department of Mathematics
and Applied Mathematics
University of Johannesburg
Auckland Park, South Africa

Sandi Klavžar
Faculty of Mathematics and Physics
University of Ljubljana
Ljubljana, Slovenia

Douglas F. Rall
Department of Mathematics
Furman University
Greenville, SC, USA

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Preface

At its most basic level, this book is about dominating sets in graphs. A dominating set of a graph G is a subset D of vertices having the property that each vertex of G is in D or is adjacent to a vertex of D. Because of the many real-world problems that can be modeled by domination in graphs, research in the area has primarily focused on the problem of finding dominating sets of smallest cardinality. From a computational standpoint, this is known to be an intractable problem, and thus researchers have studied the problem when restricted to special classes of graphs or have produced upper and lower bounds on the minimum size of a dominating set in terms of other more easily computed graphical invariants.

When presented with a given graph and asked to find a small dominating set, it would seem natural to choose vertices one at a time based on some criterion such as large vertex degree. However, including vertices based on a single property (or even several properties) is not guaranteed to solve this optimization problem and in fact can lead to dominating sets that are far from being minimum. The domination game was introduced in 2010 as a way to model this vertex-by-vertex approach to building a dominating set of a graph. The game is played by two competitors who take turns adding a vertex to a set of chosen vertices. Each new vertex selected must enlarge the set of vertices dominated by the chosen set. The game ends when no such vertex exists in the graph, that is, once the chosen set is a dominating set. The two players have complementary goals, one seeks to minimize the size of the chosen set while the other player tries to make it as large as possible. The game is not one that is either won or lost. Instead, if both players employ an optimal strategy that is consistent with their goal, the cardinality of the chosen set is a graphical invariant, called the game domination number of the graph.

There are some surprises along the way. For example, there are connected graphs that have a spanning tree whose game domination number is much smaller than the game domination number of the original graph. This is perhaps counterintuitive since the domination number of a connected graph is always at most that of any of its spanning trees. Sometimes new methods and ideas that are developed within a theory are almost as important as its results. The first such method that we mention is the imagination strategy, which is the primary tool for proving results about domination game invariants. A perfect example of this is the proof of the Continuation Principle, a much used monotonicity property of the game domination number. In addition, we expose the discharging method of Bujtás. The power of this method was shown by improving the known upper bound, in terms of a graph's order, on the (ordinary) domination number of graphs with minimum degree between 5 and 50.

In the decade since it was published, the seminal paper [24] has 66 citations in MathSciNet. As a result, the game is quite mature with most of the notation and terminology commonly accepted. Hence we believe that now is the right time for a book on the game that summarizes the previous efforts, stimulates research on closely related topics, and presents a key reference for future developments. The book is intended primarily for research students in graph theory as well as established graph theorists, but it can be enjoyed by anyone with a modicum of mathematical maturity. Much has been done on game domination in the first 10 years, but many questions remain to be (asked and) answered, and a number of interesting conjectures are open. We hope this book can be a "jumping off" point for those interested in domination theory and in games played on graphs.

Prerequisites

A book of this size cannot be fully self-contained. Therefore, we assume some basic mathematical skills and do not explain fundamental concepts such as induction, sets, sequences, or functions. In addition, we assume a reader has some familiarity with basic concepts in graph theory such as connectivity and independence. Some knowledge of combinatorial algorithms would be useful, and in several situations, we discuss computational complexity, although these can be skipped without compromising the remainder of the topics.

Contents

Many of the proofs of results in the study of the domination games are technical and long. Because of space limitations, we are not able to include most of them. Thus, what we do instead is summarize what is known about various aspects of this research area. However, when possible, we do provide some of the proofs in order to give the reader an idea of the proof strategies involved when dealing with a game played on graphs. In all cases, we provide specific citations for the original papers. Some concepts covered in the book are more developed than others, but problems are suggested throughout.

In Chapter 1, we supply most of the graph theory definitions and notation used in the remainder of the book. The domination game and the total domination game are introduced and we meet the two players, Dominator and Staller. Together these two games define four graphical invariants for each graph depending on which player starts and which game is being played. For a small graph, we construct the entire game domination tree, and thereby illustrate why these numbers depend only on the graph. We also present natural bounds relating the game (total) domination number and the (total) domination number of a graph.

Chapter 2 is devoted to the standard domination game. Proofs in the realm of the domination game are often quite different from proofs of results in domination theory. Here we present the imagination strategy and illustrate its use in proving the Continuation Principle. Two of the outstanding conjectures, the $\frac{3}{5}$-conjectures, concerning upper bounds on the game domination number are presented, and we prove a weaker upper bound while illustrating the discharging method of Bujtás. Although their nontrivial proofs are not included, the formulas for the game domination numbers of paths and cycles are given. In this chapter, we also present what is known about the domination game played on subgraphs and trees as well as graphs whose game domination numbers attain the theoretical minimum or maximum values as a function of their domination number. Chapter 2 is concluded with a presentation of the computational complexity of the natural decision problems associated with the game domination numbers.

Chapter 3 is devoted to the total domination game. At this time, not as much is known about the total domination version of the game, but this chapter covers topics in a nearly parallel fashion to Chapter 2. We present the Total Continuation Principle and the best known upper bound for the game total domination number of a graph (all of whose components have at least three vertices) in terms of its order. The main conjecture in this area is the $\frac{3}{4}$-conjecture. The truth of this conjecture in some large classes of graphs has been verified, and the strongest of these is presented, without proof. Versions of critical and perfect graphs with respect to the game total domination number are introduced and known results are summarized. A complete characterization is given of trees whose game total domination number is equal to the total domination number. The chapter ends with a presentation of the computational complexity of the game total domination number.

In Chapter 4, we consider several new invariants that have arisen from a modification of the domination games. Given a linear order of the vertices of a graph G, an online (total) domination algorithm would select a vertex v precisely when at least one vertex in the (open) closed neighborhood of v is not (totally) dominated by the vertices preceding v in the order. The largest number of vertices selected by such an algorithm over all possible orderings of the vertex set is called the Grundy (total) domination number of G. Equivalently, this is the number of vertices chosen in the (total) domination game if both players adopt Staller's goal. Some upper and lower bounds for these numbers are given and the effects of vertex or edge deletion on these invariants are also studied. There are graphs for which every permutation of the vertices results in a (total) dominating set of the same cardinality. A characterization of some classes of these graphs is presented. Several other versions of these dominating sequences are introduced, one of which is related to the concept of zero-forcing in graphs, which in turn has connections to linear algebra.

Chapter 5 introduces and summarizes what is known about more than a dozen other games played on graphs or hypergraphs. Most of these are either modifications of the domination games presented in the first four chapters or are related games played on hypergraphs, which can in some instances lend insight into the domination games on graphs. An example of the former type is the connected domination game in which the set of vertices chosen by the two players at any point in the game must not only satisfy the rules of the domination game but must also induce a connected subgraph. The transversal game on hypergraphs is an example of the latter. In all of these games, there are problems waiting to be solved, so the area is rich for further research.

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Maribor, Slovenia — Boštjan Brešar
Johannesburg, South Africa — Michael A. Henning
Ljubljana, Slovenia — Sandi Klavžar
Greenville, SC, USA — Douglas F. Rall

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Chapter 1
Introduction

In this chapter, we first list the concepts that are essential for the book. These include domination and total domination, further graph invariants, graph products, and hypergraphs. After that we introduce the key players of the book, the domination game and the total domination game. The two games are not games in the sense of combinatorial game theory where one has winners and losers, instead they lead to the game domination number and the game total domination number. Using the game tree of a domination game we demonstrate that these numbers are well-defined graphical invariants. We conclude the introduction by presenting some basic bounds on both game invariants in terms of the (total) domination number.

1.1 Some Useful Notions and Notation

Let $G = (V(G), E(G))$ be a finite graph. The *order* and the *size* of G are denoted by $n(G)$ and $m(G)$, respectively. A graph G with $m(G) = 0$ is a *totally disconnected graph*, and a graph with $n(G) = 1$ is *trivial*. If $v \in V(G)$, then the (*open*) *neighborhood*, $N_G(v)$, of v in G is the set of vertices adjacent to v. The *degree*, $\deg(v)$, of v is $|N_G(v)|$. The minimum and the maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A vertex of degree 0 is an *isolated vertex*, while a vertex of degree 1 is a *leaf* or a *pendant vertex*. A graph is *isolate-free* if it has no isolated vertices. The unique neighbor of a leaf is called the *support vertex* of the leaf. A support vertex is a *strong support vertex* if at least two of its neighbors have degree 1. The *closed neighborhood* of v in G is defined by $N_G[v] = N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$, the (*open*) *neighborhood* of S in G is defined as $N_G(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of S in G is defined as $N_G[S] = N_G(S) \cup S$. When there is no possible confusion we may omit the subscript G in the above notation.

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If X and Y are subsets of vertices in a graph G, then the set X *dominates* the set Y in G if $Y \subseteq N[X]$. When X consists of a single vertex x we say that x *dominates* each vertex from Y, where clearly $Y \subseteq N[x]$. In particular, if a set X dominates $V(G)$, then X is a *dominating set* of G; that is, every vertex in $V(G) - X$ is adjacent to a vertex in X. In other words, X is a dominating set of G if its closed neighborhood is $V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. The maximum cardinality of a minimal dominating set of G is the *upper domination number*, $\Gamma(G)$, of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$*-set*. A vertex in G is *dominating* if it is adjacent to all other vertices of G. For more information on domination in graphs see the book [65].

Replacing closed neighborhoods with open neighborhoods leads one to the concept of total domination. In particular, if X and Y are subsets of vertices in a graph G, then the set X *totally dominates* the set Y in G if $Y \subseteq N(X)$. In particular, a vertex u *totally dominates* a vertex v if $v \in N(u)$. In the case that a set X totally dominates $V(G)$, then X is a *total dominating set* of G; that is, every vertex of G is adjacent to a vertex in X. Note that G has a total dominating set only if it has no isolated vertices. The *total domination number*, $\gamma_t(G)$, of a graph G with minimum degree at least 1 is the minimum cardinality of a total dominating set of G. A total dominating set of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$*-set*. For more information on total domination in graphs see the book [77].

A set of vertices A in a graph G is *independent* if any two distinct vertices in A are not adjacent, and the maximum size of an independent set in G is the *independence number*, $\alpha(G)$, of G. If A is an independent set in G, then the set B, where $B = V(G) - A$, is a *vertex cover* of G. Note that every edge in G is incident with at least one vertex in B. The minimum size of a vertex cover in G is the *vertex cover number*, $\beta(G)$, of G. A *matching* in G is a set of edges of G, no two of which share an endvertex. The maximum cardinality of a matching in G is the *matching number*, $\alpha'(G)$, of G.

A graph G is *connected* if there is a path joining every two vertices of G; otherwise, G is *disconnected*. A maximal connected subgraph of a graph is a *(connected) component*. The *distance* $d_G(u, v)$ between two vertices u and v in a connected graph G is the minimum length of a u, v-path in G. The *diameter* $\mathrm{diam}(G)$ of G is the maximum distance among all pairs of vertices of G. The *distance* between two sets S and T of vertices of G is the minimum distance between a vertex of S and a vertex of T; that is, $d_G(S, T) = \min\{d_G(u, v) : u \in S, v \in T\}$. A set $P \subseteq V(G)$ is a 2*-packing* in G if every two distinct vertices in P are at distance at least 3.

Let G and H be two graphs. In the four standard graph products $G * H$, where $* \in \{\circ, \boxtimes, \times, \square\}$, the vertex set of the product of graphs G and H is equal to $V(G) \times V(H)$. Two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $V(G * H)$ if

- either $g_1g_2 \in E(G)$ or ($g_1 = g_2$ and $h_1h_2 \in E(H)$) holds, when $* = \circ$, and $G \circ H$ is the *lexicographic product* of G and H;

- ($g_1g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1h_2 \in E(H)$) or ($g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$) holds, when $* = \boxtimes$, and $G \boxtimes H$ is the *strong product* of G and H;
- $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$ holds, when $* = \times$, and $G \times H$ is the *direct product* of G and H;
- ($g_1g_2 \in E(G)$ and $h_1 = h_2$) or ($g_1 = g_2$ and $h_1h_2 \in E(H)$) holds, when $* = \square$, and $G \square H$ is the *Cartesian product* of G and H.

For more information on graph products see the book [63].

The *disjoint union* of two graphs G and H will be denoted with $G \sqcup H$. The disjoint union is an associative operation, and we will denote by nG the disjoint union of n copies of a graph G. The *join* $G + H$ of G and H is obtained from $G \sqcup H$ by adding all possible edges joining $V(G)$ and $V(H)$. The join is also an associative operation on graphs, and the join of two or more totally disconnected graphs is called a *complete multipartite graph*. The *corona* $G \odot H$ of graphs G and H is the graph obtained from G by adding a disjoint copy H_v of H for each vertex v of G and joining v to each vertex of H_v. In particular, $G \odot K_1$ is called the *corona* of G, also denoted $\mathrm{cor}(G)$ in the literature.

A subgraph Q of a graph G is a *k-clique* or simply a *clique* if Q is isomorphic to the complete graph K_k. A vertex $u \in V(G)$ is *simplicial* if $N_G(u)$ induces a clique. A pair of vertices u and v are *closed twins*, or simply called *twins*, in a graph G if $N_G[u] = N_G[v]$ and *open twins* if $N_G(u) = N_G(v)$. A vertex u is an *(open) twin vertex* if u and v are (open) twins for some vertex v. A graph is *twin-free* if it contains no twins, and *open twin-free* if it contains no open twins.

A graph G is *a chordal graph* if all induced cycles of G are triangles (that is, they induce K_3). A graph G is a *split graph* if $V(G)$ can be partitioned into a complete subgraph and an independent set. Note that trees and split graphs are chordal graphs. A *cograph* is a graph in which there is no induced path P_4. If H is a graph, and G contains no induced subgraph isomorphic to H, then G is an *H-free* graph. The graph $K_{n\times 2}$ obtained from the even-ordered complete graph K_{2n} by removing all edges of a perfect matching is the *cocktail-party graph*. Given a graph G, the *line graph* $L(G)$ of G is the graph with $V(L(G)) = E(G)$ and $ef \in E(L(G))$ if edges e and f are adjacent in G.

A *hypergraph* $\mathcal{H} = (V, F)$ consists of the non-empty set of vertices V and a family F of non-empty subsets of V called *edges*. The concept of hypergraph generalizes that of graph in which every edge has cardinality 2. A hypergraph in which all edges have the same cardinality k is a *k-uniform hypergraph*. (A graph is thus a 2-uniform hypergraph.) The *complete k-uniform hypergraph* on n vertices has all k-subsets of V as its edges. For a graph G, the *open (closed) neighborhood hypergraph*, abbreviated ONH (resp. CNH), of G is the hypergraph H with vertex set $V(H) = V(G)$ and with edge set $E(H) = \{N_G(x) : x \in V(G)\}$ (resp. $E(H) = \{N_G[x] : x \in V(G)\}$) consisting of the open (closed) neighborhoods of vertices in G.

If $n \in \mathbb{N}$, then we will use the notation $[n] = \{1, \dots, n\}$, and let $[n]_0 = \{0, 1, \dots, n\}$. For notation and graph theory terminology not defined herein, we in general follow [109], and for domination-related concepts we follow [65].

1.2 Domination and Total Domination Game

The domination game is played on a graph G by two players, *Dominator* and *Staller*, who take turns choosing a vertex from G. Whenever a vertex is chosen it must dominate at least one vertex not dominated by the vertices previously chosen. We call such a vertex a *playable* vertex. When either player chooses a vertex we call this a *move* in the game. For emphasis we will sometimes refer to a move as a *legal move*. When convenient we might also refer to the chosen vertex as a move. The game ends when no move is possible. Note that at this stage the set of selected vertices is a dominating set of G. The players have opposite goals. Dominator wishes that the game is ended in as few moves as possible, and Staller wishes to maximize the total number of moves.

If the first move is made by Dominator, then the game will be called a *D-game* and the sequence of moves will be denoted by $d_1, s_1, d_2, s_2, \dots$ Otherwise (if Staller starts the game), the game will be called an *S-game*; in this case the sequence of moves will be denoted $s_1', d_1', s_2', d_2', \dots$

The *game domination number*, $\gamma_{\rm g}(G)$, of G is the number of moves in a D-game when both players follow a strategy to achieve their goals. The *Staller-start game domination number*, $\gamma_{\rm g}'(G)$, of G is the number of moves in an S-game when both players follow a strategy to achieve their goals.

The *total domination game* is defined just as the domination game, except that whenever a player selects a new vertex in the course of the game, the selected vertex must **totally** dominate at least one vertex that was not **totally** dominated by vertices previously selected by the players. The *game total domination number* and the *Staller-start game total domination number* are denoted by $\gamma_{\rm tg}(G)$ and $\gamma_{\rm tg}'(G)$, respectively.

We first argue that the invariants $\gamma_{\rm g}$, $\gamma_{\rm g}'$, $\gamma_{\rm tg}$, and $\gamma_{\rm tg}'$ are well-defined. For this purpose consider the *game tree* of a domination game played on a graph G defined as follows. Its vertices are all the possible positions in the game, where a *position* is a sequence of legal moves played up to some point in a game played on G. The tree is rooted in the initial empty sequence λ. A position $(x_1, \dots, x_{i+1})$, $i \geq 0$, is a child of a position $(x_1, \dots, x_i)$. Note that $N[x_{i+1}] - N[\{x_1, \dots, x_i\}] \neq \emptyset$. If $N[\{x_1, \dots, x_k\}] = V(G)$, then the position $(x_1, \dots, x_k)$ is *terminal* and represents a leaf of the game tree. In Figure 1.1 an example of a game tree is shown, where positions $(x_1, \dots, x_i)$ are briefly written $x_1 \dots x_i$. In the figure some branches of the tree are omitted due to symmetry. For instance, playing x' as the first move is equivalent/symmetric to playing x, hence the option x' is not drawn. Similarly, the first move y' is not shown. We do likewise in the rest of the game tree. Say, if z is