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Tauberian Operators

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Preface

Tauberian operators were introduced to investigate a problem in summability theory from an abstract point of view. Since that introduction, they have made a deep impact on the isomorphic theory of Banach spaces. In fact, these operators have been useful in several contexts of Banach space theory that have no apparent or obvious connections. For instance, they appear in the famous factorization of Davis, Figiel, Johnson and Pełczyński [49] (henceforth the *DFJP factorization*), in the study of exact sequences of Banach spaces [174], in the solution of certain summability problems of tauberian type [63,115], in the problem of the equivalence between the Krein-Milman property and the Radon-Nikodým property [151], in certain sequels of James' characterization of reflexive Banach spaces [135], in the construction of hereditarily indecomposable Banach spaces [13], in the extension of the principle of local reflexivity to operators [27], in the study of certain Calkin algebras associated with the weakly compact operators [16], etc. Since the results concerning tauberian operators appear scattered throughout the literature, in this book we give a unified presentation of their properties and their main applications in functional analysis. We also describe some questions about tauberian operators that remain open.

This book has six chapters and an appendix. In Chapter 1 we show how the concept of tauberian operator was introduced in the study of a classical problem in summability theory–the characterization of conservative matrices that sum no bounded divergent sequences–by means of functional analysis techniques. One of those solutions is due to Crawford [45], who considered the second conjugate of the operator associated with one of those matrices. Crawford's solution led Kalton and Wilansky to introduce tauberian operators in [115] as those operators $T\colon X \longrightarrow Y$ acting between Banach spaces for which $T^{**}(X^{**}\backslash X) \subset Y^{**}\backslash Y$, where T^{**} denotes the second conjugate of T.

Chapter 2 displays the basic structural properties of the class of tauberian operators; in particular, the links between tauberian operators, weakly compact operators and reflexivity. We present some basic examples and describe the most important characterizations of tauberian operators: the sequential characterization of Kalton and Wilansky [115], the geometrical characterizations obtained by Neidinger and Rosenthal [135], the characterization in terms of reflexivity of the kernel of the compact perturbations given in [92], the algebraic characterization obtained in [92], and some characterizations in terms of the action of tauberian operators upon basic sequences proved by Holub [103].

We begin Chapter 3 by introducing the cotauberian operators as those operators T such that T^* is tauberian. Next we give the main properties of these operators. Several results show that cotauberian operators form the right class to be taken as the dual class of the tauberian operators. However, this relationship of duality is not perfect: we give an example, obtained in [8], of a tauberian operator T such that T^* is not cotauberian. We also include a perturbative characterization and an algebraic characterization for the cotauberian operators similar to those obtained for the tauberian operators in the previous chapter.

We describe an improved version of the DFJP factorization, obtained in [68], which allows us to show plenty of examples of tauberian and cotauberian operators: every operator $T\colon X \longrightarrow Y$ can be factorized as $T = jUk$, with j tauberian, k cotauberian and U a bijective isomorphism. Moreover, this version behaves well under duality.

The DFJP factorization has received a lot of attention. In particular, several variations of it have been introduced. We describe an isometric variation and a conditional variation. The first one was introduced by Lima, Nygaard and Oja [119] to study the approximation property of Banach spaces, and the second one was introduced by Argyros and Felouzis [13] to construct examples of hereditarily indecomposable Banach spaces. Moreover, following Beauzamy's exposition [21], we show that the intermediate space in the DFJP factorization can be identified with a real interpolation space for certain values of the interpolation parameters.

We treat other situations in which tauberian operators appear. For example, following [32, 35], we show that the natural embedding of certain Orlicz function spaces $L_\Phi(\mu)$ into $L_1(\mu)$ is a tauberian operator if and only if for every Banach space X the natural embedding of the space of vector-valued functions $L_\Phi(\mu, X)$ into $L_1(\mu, X)$ is a tauberian operator.

The aforementioned characterizations show that the tauberian and the cotauberian operators are closely linked to the operator ideal of the weakly compact operators. Following [89, 90, 92], we consider four other operator ideals that admit sequential characterizations. We show that each one has two classes of associated operators similar to the tauberian and the cotauberian operators. The first of these classes is defined in terms of sequences and the second one is defined by duality. We show that both classes admit a perturbative characterization and an algebraic characterization.

Chapter 4 is devoted to the study of tauberian operators $T\colon L_1(\mu) \longrightarrow Y$, where μ is a finite measure and Y is a Banach space [75]. The characterizations of relatively weakly compact subsets of $L_1(\mu)$ are applied to obtain some useful characterizations of these tauberian operators and show that their properties are better than those of the general tauberian operators. For example, the set of tauberian operators from $L_1(\mu)$ into Y is open in the set of all operators, and one of these operators T is tauberian if and only if so is its second conjugate.

In Chapter 5 we describe the main applications of tauberian operators in Banach space theory. Following Schachermayer in [151] and the exposition in [67], we show that, for a Banach space X for which there exists a tauberian operator $T\colon X \times X \longrightarrow X$, the Radon-Nikodým property and the Krein-Milman property are equivalent. We also show that tauberian operators preserve some isomorphic properties: following Neidinger's thesis [133], we show that, given a tauberian operator $T\colon X \longrightarrow Y$ and a bounded subset C of X, some isomorphic properties of the set $T(C)$ are inherited by C and some isomorphic properties of the space Y are inherited by X.

Using the version of the DFJP factorization presented in Chapter 3, we show that some operator ideals $\mathcal{A}$ possess the factorization property: each operator in $\mathcal{A}$ factors through a Banach space whose identity belongs to $\mathcal{A}$. Here we include some results of Heinrich [100] and some extensions of these results obtained in [68]. We also show that these factorization results can be extended in two directions: one of them by showing that we can obtain a uniform factorization of this kind for the operators of a compact set of operators [73], and the other one (see [71, 72]) by showing that the definition of some operator ideals can be extended to holomorphic mappings $f\colon X \longrightarrow Y$ acting between Banach spaces X and Y, and that in some cases these maps can be written as $f = T \circ g$ or $f = g \circ T$, where g is another holomorphic mapping and T is an operator that belongs to the same operator ideal as f.

We also give some applications of the isometric variation of the DFJP factorization to study the approximation property of Banach spaces, due to Lima, Nygaard and Oja [119], and following Astala and Tylli [16], we characterize the weakly compact approximation property of Banach spaces in terms of the weak Calkin algebra.

In Chapter 6 we consider some classes of operators that have a similar behavior to that of tauberian operators. Some of these classes were named *semigroups* in [89,90,92], following Lebow and Schechter [118] who did it for the semi-Fredholm operators. Finally, the notion of an *operator semigroup* was axiomatized in [1] as a counterpart to Pietsch's concept of an operator ideal [139].

Every operator ideal $\mathcal{A}$ has two semigroups $\mathcal{A}_+$ and $\mathcal{A}_-$ associated in a similar way as the weakly compact operators have the tauberian and the cotauberian operators. We summarize the main properties of these two operator semigroups and show other general constructions that provide semigroups.

We describe a strongly tauberian operator and its dual class, introduced by Rosenthal [147]. Moreover, we show how tauberian operators have been useful in distinguishing between the different concepts of local representability of operators that have appeared in the literature.

We study in some detail the ultrapower-stable operator semigroups. For that purpose, we consider two different types of finite representability for operators: local representability and local supportability. As an application, we investigate the class of supertauberian operators, which is the largest ultrapower-stable semigroup contained in the class of tauberian operators, and their dual class: the cosupertauberian operators.

Each chapter ends with a section of Notes and Remarks where we include some comments, complementary results and bibliographical references.

This book is addressed to graduate students and researchers interested in functional analysis and operator theory. The prerequisites for reading this book are a basic knowledge of functional analysis, including the consequences of the Hahn-Banach theorem and the open mapping theorem. Familiarity with the rudiments of Fredholm theory for operators and some parts of Banach space theory, like criteria for the existence of basic subsequences from a given sequence, Rosenthal's ℓ_1-theorem, ultraproducts and the principle of local reflexivity would be helpful. For the convenience of the reader, a brief exposition of these prerequisites has been included in Appendix A.

Our intention has been to present a self-contained exposition of the fundamental results of the subject. When describing the applications, sometimes we give a reference instead of a complete proof.

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Manuel González and Antonio Martínez-Abejón
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Notation

Henceforth, capital letters X, Y and Z denote Banach spaces. Most of the time we work with real scalars but, in a few places, we need complex scalars. Moreover, B_X and S_X are the closed unit ball and the unit sphere of X, X^* is the first dual of X, X^{**} the second dual (or bidual), and $X^{*(n)}$ the n-th dual.

Given a Banach space X, its elements will be denoted by small letters x, y, z; the elements of its dual X^* by x^*, y^*, and the elements of X^{**} by x^{**}, y^{**}, etc. Given $x \in X$ and $x^* \in X^*$, $\langle x^*, x\rangle$ denotes the value attained by x^* at x. We denote by $J_X \colon X \longrightarrow X^{**}$ the canonical embedding of X into X^{**}. In most cases we identify X with $J_X(X)$.

The symbol w will stand for the weak topology $\sigma(X, X^*)$ on X. Thus, in X^* w is $\sigma(X^*, X^{**})$ and w^* is $\sigma(X^*, X)$ when this notation is sufficiently clear. For instance, if we say that x^{**} is a w^*-cluster point of a subset A of X, w^* stands for the topology $\sigma(X^{**}, X^*)$ of X^{**}.

The norm closure of a subset A of X is denoted by $\overline{A}$; its closure with respect to w is represented by $\overline{A}^{\sigma(X,X^*)}$ or $\overline{A}^{w}$; the annihilator of A in X^* is

$$A^{\perp} := \{x^* \in X^* \colon \langle x^*, x\rangle = 0,\ \forall x \in A\}.$$

Analogously, given a subset B of X^*, its closure with respect to the weak* topology of X^* is denoted by $\overline{B}^{\sigma(X^*,X)}$ or $\overline{B}^{w^*}$. Moreover, $B_{\perp}$ denotes the annihilator $\{x \in X \colon \langle x^*, x\rangle = 0,\ \forall x^* \in B\}$.

The subspaces of a Banach space X that we consider are not necessarily closed; given a nonempty subset A of X, $\operatorname{span}\{A\}$ represents the subspace generated by A and $\overline{\operatorname{span}}\{A\}$ is the norm-closure of $\operatorname{span}\{A\}$.

Given a pair of Banach spaces X and Y, $\mathcal{L}(X, Y)$ denotes the set of all bounded linear maps –henceforth operators– acting between X and Y.

An isomorphism is an injective operator $T \in \mathcal{L}(X, Y)$ with closed range (not necessarily bijective). Note that for every isomorphism $T \colon X \longrightarrow Y$, there exists a constant $d > 0$ such that $d^{-1} \le \|T(x)\| \le d$ for all $x \in S_X$. So we shall say that T is a *d-injection*, or a *metric injection* if $d = 1$.

We will say that *we identify two Banach spaces* X and Y when there is a bijective isomorphism $A \colon X \longrightarrow Y$. Similarly, we will say that *we identify two* *operators* $S \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(V,X)$ when there are two bijective isomorphisms $A\colon Y \longrightarrow V$ and $B\colon X \longrightarrow Z$ so that $S = BTA$.

The null operator and the identity on X are denoted by 0_X and I_X. Given $T \in \mathcal{L}(X,Y)$, its kernel and range are $N(T)$ and $R(T)$, its co-kernel is $Y/\overline{R(T)}$, its *conjugate operator* is $T^*\colon Y^* \longrightarrow X^*$, its second conjugate is T^{**} and $T^{*(n)}$ represents the n-th conjugate operator of T.

The class of all operators is denoted by $\mathcal{L}$. Given a class of operators $\mathcal{A}$, its component of operators acting between X and Y is

$$\mathcal{A}(X,Y) := \mathcal{A} \cap \mathcal{L}(X,Y).$$

In the case $X = Y$ we usually write $\mathcal{A}(X)$ instead of $\mathcal{A}(X,X)$.

Given a closed subspace E of a Banach space X, $J_E\colon E \longrightarrow X$ denotes the natural embedding of E into X, and $Q_E\colon X \longrightarrow X/E$ represents the quotient operator; we recall that ${Q_E}^*$ maps $(X/E)^*$ onto $E^\perp$ isometrically; moreover, since $N({J_E}^*) = E^\perp$, the operator ${J_E}^*$ induces an isometry from $X^*/E^\perp$ onto E^* that maps $x^* + E^\perp$ to $x^* \circ J_E$; thus, we identify $(X/E)^*$ with $E^\perp$, $X^*/E^\perp$ with E^*, $(X/E)^{**}$ with $X^{**}/E^{\perp\perp}$ and E^{**} with $E^{\perp\perp}$.

Given a set I of indices, $\ell_p(I)$ denotes the Banach space of all families of real numbers $(x_i)_{i\in I}$ endowed with the norm $\|(x_i)_{i\in I}\|_p := \left(\sum_{i\in I} |x_i|^p\right)^{1/p}$ if $1 \leq p < \infty$, and $\|(x_i)_{i\in I}\|_\infty := \sup_{i\in I} |x_i|$. Given a family of Banach spaces $\{X_i : i \in I\}$, we denote by $\ell_p(I,X)$ the Banach space of all families $(x_i)_{i\in I}$ with $x_i \in X_i$ endowed with the norm $\|(x_i)_{i\in I}\|_p := \|(\|x_i\|)_{i\in I}\|_p$. However, in the case $I = \mathbb{N}$ and $X_i = X$ for all i, we just write $\ell_p(X)$, and given a couple X_1 and X_2 of Banach spaces, $\ell_p(\{1,2\}, X_i)$ is denoted by $X_1 \oplus_p X_2$.

Chapter 1

The origins of tauberian operators

In 1976, Kalton and Wilansky [115] coined the term *tauberian* to designate those bounded operators $T\colon X \longrightarrow Y$ acting between Banach spaces that satisfy

$$T^{**}(X^{**} \setminus X) \subset Y^{**} \setminus Y\,. \tag{1.1}$$

In this chapter we intend to answer the two following questions:

Question 1. Why are they called *tauberian*?

Question 2. When and why did those operators first appear?

1.1 Tauberian conditions in summability theory

In order to answer Question 1, we need to go back in time to 1897, when Tauber proved that if

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = \lambda \tag{1.2}$$

and

$$\lim_n a_n/n = 0, \tag{1.3}$$

then

$$\sum_{n=0}^{\infty} a_n = \lambda. \tag{1.4}$$

This is a conditioned converse of Abel's theorem which states that (1.2) is a consequence of (1.4) without the mediation of any hypothesis such as (1.3). Since then, it has been customary to classify certain types of theorems into abelian (or direct) or tauberian according to the following abstract and rather vague scheme: consider a category $\mathcal{A}$ and let p_1 and p_2 denote a pair of properties. Suppose that the following statement holds:

(1.5) *let f be a fixed morphism in the category $\mathcal{A}$; if x verifies p_1, then $f(x)$ verifies p_2.*

Let us also assume that its converse fails but it becomes true when an additional condition (t), like (1.3) in Tauber's theorem, is satisfied. In that case, condition (t) is a *tauberian condition*, the statement

(1.6) *if the condition (t) holds and $f(x)$ verifies p_2, then x verifies p_1;*

is a *tauberian theorem*, and statement (1.5) is an *abelian theorem.* Indeed, Hardy [97] described the above classification with the following words:

> "A tauberian theorem may be defined as the corrected form of the false converse of an abelian theorem. An abelian theorem asserts that, if a sequence or function behaves regularly, then some average of it behaves regularly."

It is not simple at all to provide a more precise definition of a tauberian theorem in regard to the variety of fields where tauberian theorems occur: [37], [55], [167], [168], and so on.

Let us now fix an operator $T\colon X \longrightarrow Y$ (henceforth, when we say *operator* we mean bounded linear operator) and consider the following statement:

(1.7) *(x_n) contains a weakly convergent subsequence if (Tx_n) is convergent and the tauberian condition of boundedness of (x_n) holds.*

The main result in [115] establishes that statement (1.1) is satisfied by T if and only if (1.7) is so. The formal similitude between statements (1.6) and (1.7) demonstrate the tauberian character of those operators satisfying (1.1), which answers Question 1.

1.2 Tauberian matrices

With regard to Question 2, we shall see that the concept of tauberian operator deepens its roots in summability theory, a branch of mathematics whose original purpose was assigning limits to sequences that are not convergent in the usual sense. One of the typical techniques in summability theory is the matrix method: consider an infinite matrix $A = (a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty}$. A sequence of complex numbers $x = (x_i)_i$ is said to be A-summable (or A-limitable) if the sequence $Ax := (\sum_{j=1}^{\infty} a_{ij}x_j)_i$ is well defined and convergent. In that case, $\lim_i Ax$ is denoted $\lim_A x_i$ and assigned to the sequence x. Thus, denoting by c the set of all convergent sequences of real numbers, answers to the following questions are required:

- What is the set ω_A formed by all the sequences x for which Ax exists?
- What is the set c_A formed by all the A-summable sequences?
- Does c_A contain c?
- If $c \subset c_A$, does A preserve limits?

Remark 1.2.1. When $c \subset c_A$, matrix A is called *conservative*. Moreover, if $\lim_i x_i = \lim_A x_i$ for all $(x_i) \in c$, then A is called *regular*.

A genuine example of the interest in regular matrices that sum bounded divergent sequences is provided by Féjer's theorem, which uses the Cesàro matrix to recover any function $f \in L_p(0, 2\pi)$ from its Fourier series.

Intensive research on matrix methods was only possible after the discovery in 1911 of the classical Toeplitz-Silverman conditions which assert that a matrix $A = (a_{ij})_{i=1}^{\infty}{}_{j=1}^{\infty}$ is conservative if and only if

(i) $\|A\| := \sup_i \sum_j |a_{ij}| < \infty$;

(ii) *there exists* $s := \lim_i s_i$, *where* $s_i := \sum_j a_{ij}$;

(iii) *there exists* $a_j := \lim_i a_{ij}$ *for each* j.

Indeed, the Toeplitz-Silverman conditions allow us to identify every conservative matrix A with an operator $S_A\colon c \longrightarrow c$ and also with an operator $T_A\colon \ell_\infty \longrightarrow \ell_\infty$, both of them defined by the expression Ax when x belongs respectively to the domains c or ℓ_∞, so $\|S_A\| = \|T_A\| = \|A\|$.

Searching for criteria to decide whether or not a conservative matrix sums a bounded divergent sequence became an engaging activity during the 1950s: [125], [161], [171], [172], etc. The next decade brought new characterizations with an undoubtedly algebraic character. Thus, Copping [44] obtained the following result:

(1.8) *Let A be a conservative matrix such that T_A is injective. Then A sums no bounded divergent sequence if and only if there is a conservative matrix B which is a left inverse of A.*

In 1964, Wilansky [168] improved Copping's result by replacing the injectivity of T_A with the weaker condition of injectivity of S_A. For the same matrices that same year, Berg [28] obtained the following characterization:

(1.9) *Let A be a conservative matrix such that S_A is injective. Then A sums no bounded divergent sequence if and only if A is not a left-topological divisor of zero, that is, there exists $\varepsilon > 0$ such that for every norm one element $x \in c$, $\|Ax\| \geq \varepsilon$.*

Obviously, if S_A is injective, then A is a left-topological divisor of zero if and only if the range of S_A is not closed. A definitive improvement dropped the hypothesis of injectivity of S_A in (1.9):

(1.10) *A conservative matrix A sums no bounded divergent sequence if and only if the operator $S_A\colon c \longrightarrow c$ has closed range and finite-dimensional null-space.*

Wilansky called *tauberian* the conservative matrices that sum no bounded divergent sequence [170].

Statement (1.10) was obtained with different proofs by Crawford in 1966 [45], Whitley in 1967 [166], and Garling and Wilansky in 1972 [63]. Each of the above mentioned papers meant a new stage in the increasing presence of functional analysis in summability theory, which paved the way for the first appearance of tauberian operators. Crawford's main contribution to the attainment of (1.10) is the introduction of duality techniques by means of the following result:

(1.11) *Given a conservative matrix A, we have ${T_A}^{-1}(c) \subset c$ if and only if ${S_A^{**}}^{-1}(c) \subset c$.*

Note that, in general, the operators T_A and S_A^{**} are not equal. Indeed, T_A is represented by the matrix A, but since the canonical embedding of c into its bidual space, ℓ_∞, maps every sequence (x_i) to $(\lim_i x_i, x_1, x_2, \ldots)$, the operator S_A^{**} is represented by the matrix

$$P = \begin{pmatrix} s & a_1 & a_2 & \cdots \\ s_1 - s & a_{11} - a_1 & a_{12} - a_2 & \cdots \\ s_2 - s & a_{21} - a_1 & a_{22} - a_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Crawford overcomes that difficulty by substitution of c for an isomorphic space, c_0, and taking advantage of the fact that for every operator $L\colon c_0 \longrightarrow c_0$, both L and L^{**} are representable by the same matrix. Thus, he considers the surjective isomorphism $U\colon c_0 \longrightarrow c$ that maps e_1 to the constant sequence $(1, 1, \ldots)$ and e_i to e_{i-1} for $i > 1$, and takes the operator $R := U^{-1} S_A U$ which is matrix representable by P. Next, by means of classical techniques of matrix summability, Crawford obtains the following result:

(1.12) ${T_A}^{-1}(c) \subset c$ *if and only if* $(R^{**})^{-1}(c_0) \subset c_0$;

and since R is an isomorphism, statement (1.12) yields (1.11).

1.3 Tauberian operators

Garling and Wilansky's innovation with respect to Crawford's proof is that they study a general operator $T\colon X \longrightarrow Y$ satisfying ${T^{**}}^{-1}(Y) \subset X$ prior to consideration of the particular case $X = Y = c$. Thus they deduce the following results:

(1.13) *Let X and Y be a pair of Banach spaces, and $T\colon X \longrightarrow Y$ an operator. Consider the conditions*

(i) $T^{**-1}(Y) \subset X$,

(ii) $N(T^{**}) \subset X$,

(iii) $N(T)$ *is reflexive.*

Then (i)⇒(ii)⇒(iii) *and neither implication can be reversed.*

(1.14) *Moreover, for T with closed range the three conditions are equivalent.*

Garling and Wilansky obtained (1.10) with the following argument: if A is a conservative matrix that sums no bounded divergent sequence, then Crawford's result (1.11) yields $S_A^{**-1}(c) \subset c$, and by condition (i) in (1.13) it follows that $N(S_A^{**})$ is reflexive, and therefore finite-dimensional because c contains no infinite-dimensional reflexive subspace. They offer no new proof of the fact that $R(S_A)$ is closed. Conversely, if $R(S_A)$ is closed and $N(S_A)$ is finite-dimensional, then $N(S_A)$ is trivially reflexive, so (1.14) shows that $S_A^{**-1}(c) \subset c$, hence (1.11) yields A sums no bounded divergent sequence.

As far as we know, Crawford's statement (1.11) contains the first application of tauberian operators, but condition (i) in (1.13) is the first appearance of tauberian operators with the same level of generality given in (1.1). Garling and Wilansky stimulated interest in tauberian operators posing the following questions:

Question 1.3.1. For which pairs of non-reflexive Banach spaces X and Y can the assumption "closed range" be dropped in (1.14)?

Question 1.3.2. For which non-reflexive Banach spaces X and Y does condition (i) in (1.13) imply $R(T)$ closed?

Sufficient and necessary conditions for the equivalence between the three clauses of (1.13) were found by Kalton and Wilansky in [115], published in 1976. Their paper, which only uses functional analysis and Banach space theory, popularized the term *tauberian* for the operators defined in (1.1).

Full answers to Questions 1.3.1 and 1.3.2 are still unknown. However, the following sufficient condition was shown in [115]:

(1.15) *If X contains no reflexive infinite-dimensional subspace and $T\colon X \longrightarrow Y$ is tauberian, then T is upper semi-Fredholm.*

Let us recall that an operator $T\colon X \longrightarrow Y$ is said to be *upper semi-Fredholm* if it has closed range and finite dimensional kernel.

The reader will realize that (1.15), combined with Crawford's result (1.11), yields an immediate proof of (1.10). This observation was made by Wilansky in [170, Section 17.6]. But the most important fact concerning [115] is that it led to further research focused on tauberian operators. In fact, Kalton and Wilansky suggested that Statement 1.15 could be extended to more Banach spaces X other than those with no reflexive infinite-dimensional subspaces. In particular, as c_0 is isomorphic to a Banach space of continuous functions, they posed the following question:

Question 1.3.3. Given a pair of spaces of continuous functions, $C(K)$ and $C(L)$, is a tauberian map $T\colon C(K) \longrightarrow C(L)$ an isomorphism in some sense?

Kalton and Wilansky also asked in [115] about duality of tauberian operators:

Question 1.3.4. When is it true that an operator $T\colon X \longrightarrow Y$ is tauberian if and only if T^{**} is so?

Question 1.3.4 was suggested by the fact that its answer is positive when T has a closed range.

Besides, an operator $T\colon X \longrightarrow Y$ is tauberian if and only if the operator $T^{co}\colon X^{**}/X \longrightarrow Y^{**}/Y$, given by $T^{co}(x^{**} + X) := T^{**}(x^{**}) + Y$, is injective. So Kalton and Wilansky asked:

Question 1.3.5. Given an operator $T \in \mathcal{L}(X, Y)$, when is T^{co} an isomorphism?

Answers to these questions and subsequent results have been collected and organized in the chapters indicated in the next section.

1.4 Notes and Remarks

As we have already said, the first work entirely devoted to tauberian operators is [115], which came to light in 1976 from the hands of Kalton and Wilansky. But there are two other papers concerning tauberian operators, [49] and [174], published respectively in 1974 and in 1976. The authors of [115] and [174], prior to submission, were acquainted with the contents of the three mentioned papers, but a closer look at them reveals that actually [49], [115] and [174] are mathematically independent and pursue different ends. Thus, in [174], Yang extends the theory of Fredholm operators to the case of tauberian operators with closed range. His results lead to a presentation of reflexivity in Banach spaces from a homological point of view. In [49], Davis, Figiel, Johnson and Pełczyński obtain their famous factorization for weakly compact operators, which is the main source of examples of tauberian operators. It shall be the subject of further study in Chapter 3. Finally, as has been thoroughly explained in Chapter 1, paper [115] can be regarded as the continuation of the work of Garling and Wilansky [63] published in 1972, putting an endpoint to a longstanding problem in summability theory: the characterization of tauberian matrices. These arguments have led us to consider [63] and [115] as the seminal papers in the study of tauberian operators. Let us notice that the role played by tauberian operators in the solution of the aforementioned problem of tauberian matrices has been recognized by some summability theorists [116, p. 262].

Since this book is not primarily concerned with summability theory, the reader interested in that subject should consult [37], [116] or [170]. The first two references are very exhaustive monographs, while the third one is concise but contains most of the material about summability dealt with in Chapter 1, including the results in Crawford's Ph.D. thesis. The historical exposition about tauberian operators described in this chapter has been borrowed from [86].

Proofs for statements (1.13) and (1.15), as well as sufficient and necessary conditions for the equivalence between the statements (1.13), can be found in Chapter 2.

Question 1.3.3 was partially solved by Lotz, Peck and Porta [124], who proved that a compact space K is scattered if and only if every injective tauberian operator from $C(K)$ into a Banach space Y is an isomorphism.

Regarding Question 1.3.4, it is immediate, after Proposition 2.1.3, that T is tauberian provided that T^{**} is so as well. However, we shall see in Chapter 3 that the converse fails.

A partial answer to Question 1.3.1 is given in Proposition 2.1.12, which states that if X is a weakly sequentially complete Banach space, then every operator $T\colon X \longrightarrow Y$ with property (N) is tauberian. Moreover, if X is contained in a space L-embedded in its bidual, then T^{co} is an isomorphism. This fact, proved by Bermúdez and Kalton [29] and included in Chapter 6, means a partial answer to Question 1.3.5.

The operators T for which T^{co} is an isomorphism have been studied by Yang [175] and by Rosenthal, who called them *strongly tauberian* [147]. The most important structural properties and applications of strongly tauberian operators are dealt with in Chapter 6.

Chapter 2

Tauberian operators. Basic properties

This chapter is devoted to the general properties and characterizations of tauberian operators, with special emphasis on their relationship to reflexivity.

Tauberian operators and their most elementary properties are formally introduced in Section 2.1. One of them is the following: an operator $T \in \mathcal{L}(X,Y)$ is tauberian if and only if $T(B_X)$ is closed and $N(T^{**}) = N(T)$, which implies that $N(T)$ is reflexive.

Section 2.2 exhibits the main characterizations of tauberian operators which will be used throughout this book, sometimes without explicit mention. In particular, it contains Kalton and Wilansky sequential characterizations for tauberian operators (Theorem 2.2.4) and for operators T with $N(T^{**}) = N(T)$ (Theorem 2.2.2), which are derived from the Eberlein-Smulian theorem. A sequel is given in Theorem 2.2.7, which proves that an operator $T \in \mathcal{L}(X,Y)$ is tauberian if and only if, for every compact operator $K \in \mathcal{L}(X,Y)$, the kernel $N(T+K)$ is reflexive.

Section 2.3 pays particular attention to the research of Neidinger and Rosenthal on the action of tauberian operators over closed convex sets, which has a significant impact on the study of the Radon-Nikodým and the Krein-Milman properties, as we shall see in Chapter 5. Its main result states that $T \in \mathcal{L}(X,Y)$ is a tauberian operator if and only if $T(B_E)$ is closed for every closed subspace E of X. This characterization is a consequence of a fundamental theorem of James, which asserts that a Banach space X is reflexive if and only if every functional $x^* \in X^*$ attains its norm on B_X.

Finally, Section 2.4 describes some results, due to Holub, on the action of tauberian operators over shrinking basic sequences and boundedly complete basic sequences. Note that the closed linear span of a basic sequence is a reflexive subspace if and only if that sequence is both shrinking and boundedly complete.

2.1 Basic facts about tauberian operators

Let us start by recalling the definition of a tauberian operator formally introduced by Kalton and Wilansky in [115].

Definition 2.1.1. An operator $T \in \mathcal{L}(X, Y)$ is said to be *tauberian* whenever $T^{**-1}(Y) \subset X$.

The notion of weakly compact operator is inseparable from that of tauberian operator. As working definition, we adopt the following:

Definition 2.1.2. An operator $T \in \mathcal{L}(X, Y)$ is said to be *weakly compact* whenever $T^{**}(X^{**}) \subset Y$.

The action of a tauberian operator in its domain is, to some degree, opposite to the action of a weakly compact operator. Indeed, let us agree to call non-trivial any element $x \in X^{**} \setminus X$. Thus, an operator T is tauberian if no non-trivial element is mapped by T^{**} to a trivial element, while T is weakly compact if T^{**} maps each non-trivial element to a trivial one.

Henceforth, the class of all tauberian operators and that of all weakly compact operators will be respectively denoted by $\mathcal{T}$ and $\mathcal{W}$. According to our notation, their respective components of operators acting between the spaces X and Y will be represented by $\mathcal{T}(X, Y)$ and $\mathcal{W}(X, Y)$.

The most basic properties regarding the interaction between the classes $\mathcal{T}$ and $\mathcal{W}$ are included in the following result. Its proof is straightforward.

Proposition 2.1.3. *Let T and S be a pair of operators in $\mathcal{L}(X, Y)$, and U an operator in $\mathcal{L}(Y, Z)$. Then the following statements hold:*

(i) *if both T and U are tauberian, then UT is tauberian;*

(ii) *if UT is tauberian, then T is tauberian;*

(iii) *T is tauberian and weakly compact if and only if X is reflexive;*

(iv) *if T is tauberian and S is weakly compact, then $T + S$ is tauberian.*

Note that, unlike $\mathcal{W}$, the class $\mathcal{T}$ is far from being an operator ideal. In particular, for each Banach space X, the identity operator $I_X : X \longrightarrow X$ is tauberian, while the null operator $0_X : X \longrightarrow X$ is weakly compact.

Proposition 2.1.4. *Let Z be a closed subspace of X. Then the following statements hold:*

(i) *the natural embedding $J_Z : Z \longrightarrow X$ is tauberian;*

(ii) *the quotient operator $Q_Z : X \longrightarrow X/Z$ is tauberian if and only if Z is reflexive.*

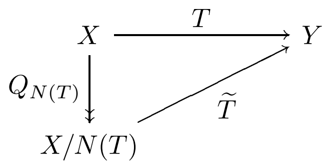
Proof. (i) Since $Z^{\perp\perp} \cap X = Z$ and $Z^{\perp\perp}$ is identified with Z^{**}, the proof of the statement is easy.

(ii) It is enough to observe that $(Q_Z)^{**}$ can be identified with the quotient operator $Q_{Z^{\perp\perp}}$, and that Z is reflexive if and only if $Z = Z^{\perp\perp}$. $\square$

Every operator $T \in \mathcal{L}(X, Y)$ can be factorized as

$$T = \widetilde{T} \circ Q_{N(T)} \tag{2.1}$$

where $\widetilde{T}\colon X/N(T) \longrightarrow Y$ is given by $\widetilde{T}(x + N(T)) := Tx$ for every $x \in X$. That yields the following commutative diagram:



Theorem 2.1.5. *For every $T \in \mathcal{L}(X, Y)$, the following statements hold:*

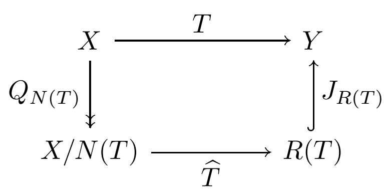
(i) *the operator T is tauberian if and only if $\widetilde{T}$ is tauberian and $N(T)$ is reflexive;*

(ii) *assume that $R(T)$ is closed; then T is tauberian if and only if $N(T)$ is reflexive.*

Proof. (i) Let us assume that T is tauberian. Thus, as $T = \widetilde{T} \circ Q_{N(T)}$, Proposition 2.1.3 shows that $Q_{N(T)}$ is tauberian, hence $N(T)$ is reflexive by Proposition 2.1.4. In order to prove that $\widetilde{T}$ is tauberian, note that $N(T) = N(T)^{\perp\perp}$, so we identify $(X/N(T))^{**}$ with $X^{**}/N(T)$, and consequently, $(\widetilde{T})^{**}$ can be regarded as a map between $X^{**}/N(T)$ and Y^{**}. Thus, given $x^{**} + N(T) \in X^{**}/N(T)$ such that $(\widetilde{T})^{**}(x^{**} + N(T)) = T^{**}x^{**} \in Y$, we have $x^{**} \in X$, so $x^{**} + N(T) \in X/N(T)$, concluding that $\widetilde{T}$ is tauberian.

For the converse, if $\widetilde{T}$ is tauberian and $N(T)$ is reflexive, then $Q_{N(T)}$ is tauberian by Proposition 2.1.4, and by Proposition 2.1.3 we see that $T = \widetilde{T} \circ Q_{N(T)}$ is tauberian.

(ii) The 'only if' implication is a consequence of (i). For the 'if' part, since $R(T)$ is closed, T factorizes as



where $\widehat{T}$ maps every $x + N(T)$ to Tx. Note that $\widehat{T}$ is tauberian because it is a surjective isomorphism. Moreover, since $R(T)$ is closed and $N(T)$ is reflexive, Proposition 2.1.4 yields that $J_{R(T)}$ and $Q_{N(T)}$ are both tauberian. Therefore, $T = J_{R(T)} \circ \widehat{T} \circ Q_{N(T)}$ is tauberian. $\square$

The argument of the following lemma will be applied on many occasions.

Lemma 2.1.6. *For every $T \in \mathcal{L}(X,Y)$ and every bounded subset A of X, we have:*

(i) $T^{**}(\overline{A}^{w^*}) = \overline{T(A)}^{w^*}$;

(ii) *if A is convex, then* $T^{**}(\overline{A}^{w^*}) \cap Y = \overline{T(A)}$.

In particular, $T^{**}(B_{X^{**}}) = \overline{T(B_X)}^{w^*}$ *and* $T^{**}(B_{X^{**}}) \cap Y = \overline{T(B_X)}$.

Proof. (i) Since T^{**} is weak* continuous and $\overline{A}^{w^*}$ is weak* compact, we have

$$T^{**}(\overline{A}^{w^*}) = \overline{T(A)}^{w^*}.$$

(ii) The weak closure of $T(A)$ equals $\overline{T(A)}^{w^*} \cap Y$, so statement (i) yields $\overline{T(A)}^{w} = T^{**}(\overline{A}^{w^*}) \cap Y$, and since the weak closure of any convex set equals its norm closure, we get $\overline{T(A)} = T^{**}(\overline{A}^{w^*}) \cap Y$.

The remaining results are a consequence of Goldstine's theorem, which states that $B_{X^{**}} = \overline{B_X}^{w^*}$. □

The following characterizations are fundamental in the study of tauberian operators.

Theorem 2.1.7. *For every operator $T \in \mathcal{L}(X,Y)$, the following statements are equivalent:*

(a) *T is tauberian;*

(b) $N(T^{**}) = N(T)$ *and* $T(B_X)$ *is closed;*

(c) $N(T^{**}) = N(T)$ *and* $\overline{T(B_X)}$ *is contained in* $R(T)$.

Proof. (a)⇒(b) The equality $N(T^{**}) = N(T)$ is immediate. In order to prove that $T(B_X)$ is closed, take $y \in \overline{T(B_X)}$. By Lemma 2.1.6, there exists $x^{**} \in B_{X^{**}}$ so that $y = T^{**}x^{**}$. But T is tauberian, so $x^{**} \in B_X$, hence $y \in T(B_X)$.

(b)⇒(c) Trivial.

(c)⇒(a) Let x^{**} be a norm-one element in X^{**} such that $y := T^{**}x^{**} \in Y$. By Lemma 2.1.6, $y \in \overline{T(B_X)}$, and by hypothesis, $\overline{T(B_X)}$ is contained in $R(T)$, so $y = Tz$ for some $z \in X$. Thus $x^{**} - z \in N(T^{**})$, and as $N(T^{**}) = N(T)$ by assumption, it follows that $x^{**} \in X$, which proves that T is tauberian. □

It is convenient to name those operators T for which $N(T)$ equals $N(T^{**})$. We adopt the following notation introduced by Kalton and Wilansky in [115].

Definition 2.1.8. An operator $T \in \mathcal{L}(X,Y)$ is said to have *property* (N) whenever $N(T^{**}) = N(T)$.

Proposition 2.1.9. *An operator $T \in \mathcal{L}(X,Y)$ has property (N) if and only if $N(T)$ is reflexive and* $\overline{R(T^*)}^{w^*} = \overline{R(T^*)}$.

Proof. For every operator T, $N(T)^\perp = \overline{R(T^*)}^{w^*}$ (see Theorem 4.14 in [148]). Moreover, $N(T)$ is reflexive if and only if $N(T)$ equals $N(T)^{\perp\perp}$. Thus the result is a consequence of:

$$N(T) \subset N(T)^{\perp\perp} = \left(\overline{R(T^*)}^{w^*}\right)^\perp \subset \overline{R(T^*)}^\perp = N(T^{**}). \qquad \square$$

Theorem 2.1.7 and Proposition 2.1.9 show that the following implications hold for every operator T:

'T tauberian $\Rightarrow$ T has property (N)'
'T has property (N) $\Rightarrow$ $N(T)$ is reflexive'.

Theorem 2.1.5 and the examples below show that the converse implications are valid when T has closed range, but fail in general.

Example 2.1.10. Let $C \in \mathcal{L}(c_0, c_0)$ be the Cesàro operator, defined by

$$C(x_n)_n := \left(\frac{1}{n}\sum_{k=1}^n x_k\right)_n.$$

The operator C has property (N) but is not tauberian.

Proof. Indeed, C^{**} is injective and $C^{**}((1,-1,1,-1,\ldots)) \in c_0$. $\square$

Example 2.1.11. The operator $T\colon c_0 \longrightarrow \ell_2$ defined by $T(x_n) := (x_n/n)$ has property (N) but it is not tauberian. Moreover, T is weakly compact.

Proof. In fact, T is weakly compact because ℓ_2 is reflexive. Moreover, since c_0 is not reflexive, T cannot be tauberian. However, T^{**} maps every $(x_n) \in \ell_\infty$ to (x_n/n). So T^{**} is injective, which implies that T has property (N). $\square$

The context of Example 2.1.11 describes very well the opposite character of tauberian operators and weakly compact operators. Indeed, $\mathcal{L}(c_0, \ell_2) = \mathcal{W}(c_0, \ell_2)$ and $\mathcal{T}(c_0, \ell_2) = \emptyset$. Therefore, having property (N) is much weaker than being tauberian. However, every operator $T\colon X \longrightarrow Y$ with property (N) is tauberian if X is weakly sequentially complete.

Proposition 2.1.12. *Let X be a weakly sequentially complete Banach space, and let $T\colon X \longrightarrow Y$ be an operator. If T has property (N), then T is tauberian.*

Proof. According to Theorem 2.1.7, we only need to prove that the identity $N(T) = N(T^{**})$ implies that $T(B_X)$ is norm closed. To do so, take an element $y \in \overline{T(B_X)}$ and choose a sequence (x_n) in B_X so that $T(x_n) \underset{n}{\longrightarrow} y$. By Rosenthal's ℓ_1-theorem (Theorem A.3.10), (x_n) contains a weakly Cauchy subsequence or a subsequence (u_n) equivalent to the unit vector basis of ℓ_1.

In the latter case there would exist $u^{**} \in \overline{\{u_{2n} - u_{2n+1}\}}^{w^*} \setminus X$, and therefore $T^{**}(u^{**}) = 0$, in contradiction with $N(T^{**}) = N(T)$. Hence, the only possibility is that (x_n) contains a weakly Cauchy subsequence (v_n). But X is sequentially weakly complete, so there exists $z \in B_X$ such that $v_n \xrightarrow[n]{w} z$. Thus, we have a convex block sequence (z_n) of (v_n) such that $z_n \xrightarrow[n]{} z$. But $\|z_n\| \leq 1$ for all n, so $y = T(z) \in T(B_X)$. □

The following examples show that property (N) is stronger than having reflexive kernel.

Example 2.1.13. The operator $T \in \mathcal{L}(c_0, c_0)$ that maps every element $(x_n)_n$ to $(x_n - x_{n+1})_n$ has reflexive kernel but fails property (N). Moreover, $T(B_{c_0})$ is not closed.

Proof. Indeed, T is injective, but $N(T^{**})$ is the space of all constant sequences, so T fails property (N). In order to show that $\overline{T(B_{c_0})} \not\subset R(T)$, let us take $(z_i) \in c_0$ so that $|\sum_{i=1}^n z_i| \leq 1/2$ for all n and $\sum_{i=1}^\infty z_i$ does not converge. Thus $(z_i) \notin R(T)$. In fact,

$$R(T) = \{(y_i) \in c_0 \colon \sum_{i=1}^\infty y_i \text{ converges}\}.$$

Moreover, given $(y_i) \in R(T)$,

$$T^{-1}\big((y_i)\big) = \Big(\sum_{j=1}^\infty y_j, \sum_{j=2}^\infty y_j, \sum_{j=3}^\infty y_j, \dots\Big).$$

For every $n \in \mathbb{N}$, let $P_n \colon c_0 \longrightarrow c_0$ be the projection with $R(P_n) = \text{span}\{e_i\}_{i=1}^n$ and $N(P_n) = \overline{\text{span}}\{e_i\}_{i=n+1}^\infty$, where $\{e_i\}_{i=1}^\infty$ is the unit vector basis of c_0. Thus $P_n\big((z_i)\big) \in T(B_{c_0})$ for all n, so $(z_i) \in \overline{T(B_{c_0})}$. □

The following example exhibits an operator $T \colon X \longrightarrow Y$ which has reflexive kernel but is not tauberian, despite $T(B_X)$ being closed.

Example 2.1.14. The operator $T \in \mathcal{L}(\ell_1, \ell_2)$, defined by $T(x_n) := (x_n)$, maps B_{ℓ_1} onto a closed set and has reflexive kernel, but fails property (N).

Proof. Let $L \colon \ell_2 \longrightarrow c_0$ be the operator given by $L(x_n) := (x_n)$. Thus T is the conjugate of L, so $T(B_{\ell_1})$ is weak* compact, hence norm closed.

Clearly T is injective, so $N(T)$ is trivially reflexive. However, $R(T^*)$ is a separable subspace of ℓ_∞, hence $N(T^{**}) \neq \{0\}$. □

The following example shows that, given an operator $T \colon X \longrightarrow Y$, the conditions $T(B_X)$ closed and $\overline{R(T^*)} = \overline{R(T^*)}^{w^*}$ are not enough to assure that T is tauberian.

Example 2.1.15. The null operator $0_{\ell_1} \colon \ell_1 \longrightarrow \ell_1$ maps B_{ℓ_1} onto a closed set and satisfies the identity $\overline{R({0_{\ell_1}}^*)} = \overline{R({0_{\ell_1}}^*)}^{w^*}$, but its kernel is not reflexive, so 0_{ℓ_1} is not tauberian.

By virtue of Theorem 2.1.5, the first examples of non-trivial tauberian operators are the operators with closed range and finite dimensional kernel, usually called *upper semi-Fredholm operators* (see Section A.1). Tauberian operators with closed range were studied by Yang [174], who called them generalized semi-Fredholm transformations.

Since the class $\mathcal{T}$ contains Φ_+, the following question arises naturally and establishes a general pattern followed by many researchers:

Question 2.1.16. Which properties of the operators in Φ_+ can be transferred to the operators in $\mathcal{T}$, and vice versa?

For instance, all the statements in Proposition 2.1.3 are valid if the words *'tauberian'* and *'weakly compact'* are respectively replaced by *'upper semi-Fredholm'* and *'compact'*.

Of course, there are reasonable properties which cannot be transferred from Φ_+ to $\mathcal{T}$. The topological structure of $\mathcal{T}$ offers an example in that direction. In fact, it is well known that the components of the class Φ_+ are always open. That assertion follows from the fact that if $T \in \Phi_+(X,Y)$, then X can be decomposed as $X = N(T) \oplus X_1$ where $T|_{X_1}$ is an isomorphism; thus, denoting $\beta := \inf\{\|Tx\| : x \in S_{X_1}\}$, given any operator $S \in \mathcal{L}(X,Y)$ such that $\|T - S\| < \beta$, it follows that $N(S) \subset N(T)$ and that $S|_{X_1}$ is an isomorphism, so $S \in \Phi_+$. Nevertheless, the following example shows that the components of $\mathcal{T}$ are not always open.

Example 2.1.17. Given a non-reflexive space X, the operator $T\colon \ell_2(X) \longrightarrow \ell_2(X)$ defined by

$$T\big((x_n)\big) := (x_n/n), \quad (x_n) \in \ell_2(X)$$

is tauberian and belongs to the topological boundary of $\mathcal{T}\big(\ell_2(X)\big)$.

Proof. We can identify the bidual of $\ell_2(X)$ with $\ell_2(X^{**})$ and T^{**} maps every (x_n^{**}) to (x_n^{**}/n). So it is clear that T is tauberian.

In order to prove that T belongs to the boundary of $\mathcal{T}\big(\ell_2(X)\big)$, it is enough to realize that for every positive integer k, the operator $T_k\colon \ell_2(X) \longrightarrow \ell_2(X)$ defined by

$$T_k(x_n) := \Big(x_1, \frac{x_2}{2}, \dots, \frac{x_k}{k}, 0, 0, \dots\dots\Big)$$

satisfies $\|T - T_k\| = 1/(k+1)$ and it is not tauberian because its kernel is not reflexive. □

Nevertheless, Example 2.1.17 can still be used to trace an analogy between Φ_+ and $\mathcal{T}$. Indeed, the set of all upper semi-Fredholm operators acting between X and Y with complemented range in Y equals the set

$$\mathcal{K}_l(X,Y) := \{T \in \mathcal{L}(X,Y) \colon I_X - LT \in \mathcal{K} \text{ for some } L \in \mathcal{L}(Y,X)\}$$

where $\mathcal{K}$ denotes the class of all compact operators (see [160, IV.13 Problems]). Note that the inclusion of $\mathcal{K}_l(X,Y)$ in $\Phi_+(X,Y)$ is strict in general because every Banach space non-isomorphic to a Hilbert space contains non-complemented closed subspaces.