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A State Space Approach to Canonical Factorization with Applications

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Linear Operators & Linear Systems

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Preface

The present book deals with canonical factorization problems for different classes of matrix and operator functions. Such problems appear in various areas of mathematics and its applications. The functions we consider have in common that they appear in the state space form or can be represented in such a form. The main results are all expressed in terms of the matrices or operators appearing in the state space representation. This includes necessary and sufficient conditions for canonical factorizations to exist and explicit formulas for the corresponding factors. Also, in the applications the entries in the state space representation play a crucial role.

The theory developed in the book is based on a geometric approach which has its origins in different fields. One of the initial steps can be found in mathematical systems theory and electrical network theory, where a cascade decomposition of an input-output system or a network is related to a factorization of the associated transfer function.

Canonical factorization has a long and interesting history which starts in the theory of convolution equations. Solving Wiener-Hopf integral equations is closely related to canonical factorization. The problem of canonical factorization also appears in other branches of applied analysis and in mathematical systems theory, in H_∞-control theory in particular.

The first book devoted to the state space factorization theory was published in 1979 as the monograph “Minimal factorization of matrix and operator functions,” Operator Theory: Advances and Applications **1**, Birkhäuser Verlag, written by the first three authors. Some of the factorization results published in the 1979 book appeared there in print for the first time.

The present book is the second book written by the four of us in which the state space factorization method is systematically used and developed further. In the earlier book [20], published in 2008, the emphasis is on non-canonical factorizations and degree 1 factorizations, in particular. In the present book we concentrate on canonical factorizations. Together both books present a rich and far reaching update of the 1979 monograph [11].

In the present book the emphasis is on canonical factorization and symmetric factorization with applications to different classes of convolution equations. For the latter we have in mind the transport equation, singular integral equations, equations with symbols analytic in a strip, and equations involving factorization of non-proper rational matrix functions. A large part of the book will deal with factorization of matrix functions satisfying various symmetries. A main theme will be the effect of these symmetries on factorization and how the symmetries can be used in effective ways to get state space formulas for the factors. Applications to H_∞-control theory, which have been developed in the 1980s and 1990s, will also be included. The text is largely self-contained, and will be of interest to experts and students in mathematics, sciences and engineering.

The authors gratefully acknowledge a visitor fellowship for the second author from the Netherlands Organization for Scientific Research (NWO), and the financial support from the School of Economics of the Erasmus University at Rotterdam, from the School of Mathematical Sciences of Tel-Aviv University and the Nathan and Lily Silver Family Foundation, and from the Mathematics Department of the Vrije Universiteit at Amsterdam. These funds allowed us to meet and to work together on the book for different extended periods of time in Amsterdam and Tel-Aviv.

The authors *Amsterdam – Rotterdam – Tel-Aviv, Summer 2009*

Postscript

On Monday October 12, 2009, Israel Gohberg, the second author of this book, passed away at the age of 81. At that time the preparation of the book was in a final phase and only some minor work had to be done. Israel Gohberg was one of the initiators using state space methods in solving problems appearing in various branches of mathematical analysis and its applications. His fundamental insights and inspiring leadership have been driving forces in our joint work.

Chapter 0

Introduction

This monograph presents a unified approach for solving canonical factorization problems for different classes of matrix and operator functions. The notion of canonical factorization originates from the theory of convolution equations. For instance, canonical factorization, provided it exists, allows one to invert Wiener-Hopf, Toeplitz and singular integral operators, and when the factors are known one can also build explicitly the inverses of these operators. The problem of canonical factorization also appears in various branches of applied analysis, in linear transport theory, in interpolation theory, in mathematical systems theory, in particular, in H_∞-control theory.

The various matrix and operator functions that are considered in this book have in common that they appear in a natural way as functions of the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B \tag{1}$$

or (after a suitable transformation) can be represented in this form. In the above formula λ is a complex variable, and A, B, C, and D are matrices or linear operators acting between appropriate Banach or Hilbert spaces, which in this book often will be finite dimensional. When the underlying spaces are all finite dimensional, A, B, C, and D can be viewed as matrices and the function W is a rational matrix function which is analytic at infinity. From mathematical systems theory it is known that, conversely, any rational matrix function which is analytic at infinity admits a representation of the above form. In systems theory the right hand side of (1) is called a *state space realization* of the function W, and one refers to the space in which A is acting as the *state space.*

The method of factorization employed in this book uses realizations as in (1), and for this reason it is referred to as the *state space method.* It allows one to deal with factorization from a geometric point of view. This state space factorization approach has its origins in different fields, for instance, in the theory of non-selfadjoint operators [27], [141], in mathematical systems theory and electrical network theory [23], [95], [94], and in the factorization theory of matrix polynomials [67], [131]. In all three areas a state space representation of the function to be factored is used, and the factors are also expressed in state space form.

The first book to deal with factorization problems in a systematic way using the state space approach is the monograph [11] of the first three authors. This monograph appeared in 1979, very soon after the first main results were obtained. In fact, some of the factorization results were published in [11] for the first time.

The present book is the second book written by the four of us in which the state space factorization method is systematically used and developed further. In our first book [20], published in 2008, the emphasis is on non-canonical factorizations and degree 1 factorizations, in particular. In the present book we concentrate on canonical factorizations. As a result the overlap between the main parts of the two books is minor. Together both books present a rich and far reaching update of the 1979 monograph [11].

In the present book special attention is paid to various factorizations with additional symmetries such as spectral factorization, inner-outer factorization, and J-spectral factorization. The latter require elements of the theory of spaces with an indefinite metric. Factorizations with symmetries appear in a natural way in H_∞-control problems and the related Nehari approximation problem. In fact, the latter problems are the main topic of the final part of the book. We also deal with applications to problems in the theory of algebraic Riccati equations, to inversion problems for Wiener-Hopf, Toeplitz and singular integral operators, and to Riemann-Hilbert problems. The linear transport equation from mathematical physics is another important area of application in this book. It requires infinite dimensional realizations of a special type.

We have made an effort to make the text reasonably self-contained. For that reason we included some known material about realizations, minimal factorizations of rational matrix functions, angular operators, and the theory of matrices in indefinite inner product spaces. In the final part we also briefly review elements of control theory of linear systems.

Not counting the present introduction, the book consists of 20 chapters grouped into 7 parts. We shall now give a short description of the contents of the book.

Part I. The first part has a preparatory character. In the first chapter we review the role of canonical factorization in inverting Wiener-Hopf integral operators and block Toeplitz operators. Also the role of this factorization in solving singular integral equations is described. The second chapter presents in detail the elements of the state space method that are used in this book.

Part II. This part starts with the canonical factorization theorem for rational matrix functions in state space form. This theorem is then used to invert explicitly Wiener-Hopf, Toeplitz and singular integral operators with a rational matrix symbol, with the inverses being presented explicitly in state space formulas. For rational matrix symbols the solution to the homogeneous Riemann-Hilbert boundary value problem is also given in state space form. In the first chapter of this part we consider proper rational matrix functions, that is, rational matrix functions that are analytic at infinity. The case of non-proper rational symbols is treated in the second chapter of this part. In this case the realization (1) is replaced by

$$W(\lambda) = I + C(\lambda G - A)^{-1}B, \tag{2}$$

where I is an identity matrix, G and A are square matrices, and B and C are matrices of appropriate sizes. A square rational matrix function, proper or not, always admits such a realization. We develop this realization result, and prove a canonical factorization theorem for the realization (2). As an application we solve the homogeneous Riemann-Hilbert boundary value problem for an arbitrary rational matrix symbol.

Part III. In this part we carry out a program analogous to that of the second part, but now for certain classes of non-rational matrix and operator functions. For instance, for matrix functions analytic on a strip but not at infinity we develop a realization theory, prove a canonical factorization theorem in state space form, and develop its applications to Wiener-Hopf integral equations. A new feature is that the problems involved require us to employ realizations with an unbounded main operator A and deal with curves cutting through the spectrum of this main operator. In this part it is also shown that, after an appropriate modification, the state space method can be used to solve the integro-differential equation appearing in linear transport theory, which forces us to use realizations of operator-valued functions. In the final chapter of this part we make an excursion into non-canonical Wiener-Hopf factorization for analytic operator-valued functions on a curve, and identify the so-called factorization indices in state space terms.

Part IV. The fourth part deals with factorization of rational matrix functions that have Hermitian values on the imaginary axis, the real line or the unit circle. In the analysis of such functions, minimal realizations play an important role. These are realizations of which the order of the state matrix in (1) is a small possible. Also the so-called state space similarity theorem, which tells us that a minimal realization is unique up to a basis transformation in the state space, enters into the analysis. These facts are reviewed in the first chapter of this part. In this first chapter, using the notion of local minimality, also the concept of a pseudo-canonical factorization relative to a curve is introduced and studied for rational matrix functions with singularities on the given curve. The effect on minimal realizations of the function having Hermitian values on the imaginary axis, the real line or the unit circle is described in the second chapter of this part. This then leads to the construction of special canonical and pseudo-canonical factorizations with additional relations between the factors. Included are spectral factorization for positive definite rational matrix functions and pseudo-spectral factorization for nonnegative rational matrix functions. In the final chapter we present (without proofs) some background material on matrices in indefinite inner product spaces, and review the main results from this area that are used in this book.

Part V. In this part the canonical factorization theorem is presented in a different way using the notion of an angular subspace and Riccati equations. In this case one has to look for angular subspaces that are also spectral subspaces, and the solutions of the Riccati equation must have additional spectral properties. These results, which have a preliminary character, are presented in the first chapter of this part. In the second chapter we introduce the symmetric algebraic Riccati equation, and describe spectral factorization as well as pseudo-spectral factorization in terms of Hermitian solutions of such a Riccati equation. In the final chapter of this part we continue the study of rational matrix functions that take Hermitian values on certain curves. The emphasis will be on rational matrix functions that have Hermitian values for which the inertia is independent of the point on the curve. Such functions may still admit a symmetric canonical factorization, provided we allow for a constant Hermitian invertible matrix in the middle. Such a factorization is commonly known as a J-spectral factorization. Necessary and sufficient conditions for its existence are given, first in terms of invariant subspaces and then in terms of solutions of a corresponding symmetric algebraic Riccati equation. We also study the question when a function which admits a left J-spectral factorization admits a right J-spectral factorization too.

Part VI. In this part we study rational matrix functions that are unitary or of the form identity matrix plus contractions, and rational matrix functions that have a positive real part. Because of the state space similarity theorem, these additional symmetries can be restated in terms of special properties of the minimal realizations of the rational matrix functions considered. These reformulations involve an algebraic Riccati equation. The results are known in systems theory as the bounded real lemma and the positive real lemma, respectively. They allow us to solve related canonical and pseudo-canonical factorization problems in state space form. In the final chapter of this part realizations are used to analyze rational matrix functions of which the values on the imaginary axis are J-unitary matrices. Solutions to various factorization problems are given. Special attention is paid to factorization of J-unitary rational matrix functions into J-unitary factors. In this chapter we also discuss problems of embedding a contractive rational matrix function into a unitary rational matrix function of larger size.

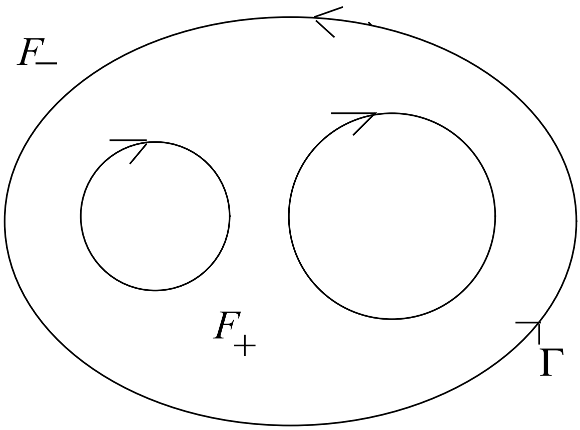
Part VII. In this part the state space theory of J-spectral factorization, developed in the final chapter of the fifth part, is used to solve H_∞ problems. The first chapter of this part contains the solution of the Nehari interpolation problem for rational matrix interpolants. The second chapter presents a short review of elements of control theory that play an important role in the third (and final) chapter of this part. This final chapter is about H_∞-control. Here we use the J-spectral factorization theory to obtain the solutions of some of the main problems in this area, namely the standard problem, the one-sided problem, and the full model matching problem.

As the description of the contents given above shows, the emphasis in the book is mainly on rational matrix functions and finite dimensional realizations. An exception is Part III. The latter part deals with non-rational matrix functions and operator-valued functions, and it uses realizations that have an infinite dimensional state space. Other exceptions are Chapter 2 in Part I and Chapter 12 in Part V. For the material in the other chapters of the book, in particular, in Parts IV–VII, often extensions to an infinite dimensional setting exist; they require appropriate modifications. See, e.g., the books [5], [35], [42], [73], and the references therein.

A few remarks about terminology and notation

At the end of this book, after the bibliography, the reader will find a List of Symbols and an Index. The latter contains in alphabetical order the various terms that are used in this book with references to the pages where they are introduced. In addition, we would like to mention the following.

In the sequel, whenever convenient, a $p \times q$ matrix with complex entries will be identified with the (linear) operator from $\mathbb{C}^q$ into $\mathbb{C}^p$ defined by the canonical action of the matrix on the standard orthogonal basis of $\mathbb{C}^q$. Conversely, a linear operator from $\mathbb{C}^q$ into $\mathbb{C}^p$ is identified with its $p \times q$ matrix representation with respect to the standard orthogonal bases of $\mathbb{C}^q$ and $\mathbb{C}^p$.



Throughout the word "operator" refers to a bounded linear transformation acting between Banach or Hilbert spaces (finite or infinite dimensional). We assume the reader to be familiar with Sections I.1 and I.2 in [51] which contain the standard spectral theory of operators, including the notion of a Riesz projection and the corresponding functional calculus (see, also Chapter V in [144]). In particular, we shall often use the notions of a Cauchy domain and Cauchy contour which are defined as follows. A *Cauchy domain* is an open set in the complex plane $\mathbb{C}$ consisting of a finite number of components such that its boundary is composed of a finite number of simple closed non-intersecting rectifiable curves. A *Cauchy contour* Γ is the positively oriented boundary of a bounded Cauchy domain. We write F_+ for the interior domain of Γ, and F_- for the exterior domain, i.e., the complement of the closure $\overline{F_+}$ of F_+ in the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. The picture on the previous page illustrates this notion. We shall also work with the extended real line and the extended imaginary axis as contours on the Riemann sphere $\mathbb{C}_\infty$. For the real line the orientation will be from left to right and for the imaginary axis from bottom to top. Thus for the extended real line the interior domain is the open upper half plane, which will be denoted by $\mathbb{C}_+$; for the extended imaginary axis it is the open left half plane, which is denoted by $\mathbb{C}_{\mathrm{left}}$.

We shall also freely use the Lesbesgue integral and related L_p spaces (see, e.g., Appendix 2 in [53]). Functions which are equal almost everywhere (shorthand: a.e.) are often identified, sometimes without explicitly mentioning this.

Finally, when dealing with inner-outer factorization, we shall always assume that the outer factor is invertible outer (see Section 17.6). In the outer-co-inner factorizations considered in this book, the outer factor will be assumed to be invertible outer as well.

Part I
Convolution equations, canonical factorization and the state space method

This part has a preparatory character. It consists of two chapters. In the first chapter we review the role of canonical factorization in inverting Wiener-Hopf integral operators and block Toeplitz operators. The role of this factorization in solving singular integral equations is described as well. The second chapter presents in detail the basic elements of the state space method that are used throughout this book. The central notion is that of a realization of a matrix or operator function. Three important operations on realizations are studied.

Chapter 1

The role of canonical factorization in solving convolution equations

This chapter has a preparatory character. We review (without giving proofs) the role of canonical factorization in inverting systems of convolution equations. The chapter consists of three sections. Section 1.1 deals with Wiener-Hopf integral equations, Section 1.2 with block Toeplitz equations, and Section 1.3 with singular integral equations.

1.1 Wiener-Hopf integral equations and factorization

In this section we outline the factorization method of [61] to solve systems of Wiener-Hopf integral equations. Such a system may be written as a single vector-valued *Wiener-Hopf equation*

$$\phi(t) - \int_0^\infty k(t-s)\phi(s)\,ds = f(t), \qquad t \geq 0. \tag{1.1}$$

Here ϕ and f are m-dimensional vector functions and $k \in L_1^{m\times m}(-\infty,\infty)$, that is, the kernel function k is an $m \times m$ matrix function whose entries are in $L_1(-\infty,\infty)$. We assume that the given vector function f has its component functions in the Lebesgue space $L_p[0,\infty)$, and we express this property by writing $f \in L_p^m[0,\infty)$. Throughout this section p will be fixed and $1 \leq p < \infty$. The problem we shall consider is to find a solution ϕ of equation (1.1) that also belongs to the space $L_p^m[0,\infty)$.

The usual method (see [61]) for solving equation (1.1) is as follows. First assume that (1.1) has a solution ϕ in $L_p^m[0,\infty)$. Extend ϕ and f to the full real line by putting

$$\phi(t) = 0, \qquad f(t) = -\int_0^\infty k(t-s)\phi(s)\,ds, \quad t < 0.$$

Then $\phi, f \in L_p^m(-\infty,\infty)$ and the full line convolution equation

$$\phi(t) - \int_{-\infty}^\infty k(t-s)\phi(s)\,ds = f(t), \qquad -\infty < t < \infty$$

is satisfied. By applying the Fourier transformation and leaving the part of f that is given in the right-hand side, one gets

$$W(\lambda)\Phi_+(\lambda) - F_-(\lambda) = F_+(\lambda), \qquad \lambda \in \mathbb{R}, \tag{1.2}$$

where

$$W(\lambda) = I_m - \int_{-\infty}^\infty e^{i\lambda t}k(t)\,dt, \qquad F_+(\lambda) = \int_0^\infty e^{i\lambda t} f(t)\,dt, \tag{1.3}$$

$$\Phi_+(\lambda) = \int_0^\infty e^{i\lambda t}\phi(t)\,dt, \qquad F_-(\lambda) = \int_{-\infty}^0 e^{i\lambda t} f(t)\,dt. \tag{1.4}$$

Here I_m is the $m\times m$ identity matrix. Note that the functions K and F_+ are given, but the functions Φ_+ and F_- have to be found. In fact in this way the problem to solve (1.1) is reduced to that of finding two functions Φ_+ and F_- such that (1.2) holds, while furthermore Φ_+ and F_- must be as in (1.4) with $\phi \in L_p^m[0,\infty)$ and $f \in L_p^m(-\infty, 0]$.

To find Φ_+ and F_- of the desired form such that (1.2) holds, one factorizes the $m\times m$ matrix function W appearing in (1.2). This function is called the *symbol* of the integral equation (1.1). Note that W is continuous on the real line, and by the Riemann-Lebesgue lemma $\lim_{\lambda\in\mathbb{R},\,\lambda\to\infty} W(\lambda)$ exists and is equal to I_m.

Assume that the symbol admits a factorization of the following form:

$$W(\lambda) = \big(I_m + G_-(\lambda)\big)\big(I_m + G_+(\lambda)\big), \qquad \lambda \in \mathbb{R}, \tag{1.5}$$

where

$$G_+(\lambda) = \int_0^\infty e^{i\lambda t} g_+(t)\,dt, \qquad G_-(\lambda) = \int_{-\infty}^0 e^{i\lambda t} g_-(t)\,dt,$$

with $g_+ \in L_1^{m\times m}[0,\infty)$ and $g_- \in L_1^{m\times m}(-\infty,0]$ while, in addition, the determinants

$$\det\big(I_m + G_+(\lambda)\big), \qquad \det\big(I_m + G_-(\lambda)\big)$$

do not vanish in the closed upper and lower half plane, respectively. We shall refer to the factorization (1.5) as a *right canonical factorization* of W *with respect to* *the real line*. Under the conditions stated above the functions $\big(I_m + G_+(\lambda)\big)^{-1}$ and $\big(I_m + G_-(\lambda)\big)^{-1}$ admit representations as Fourier transforms:

$$\big(I_m + G_+(\lambda)\big)^{-1} = I_m + \int_0^\infty e^{i\lambda t}\gamma_+(t)\,dt, \tag{1.6}$$

$$\big(I_m + G_-(\lambda)\big)^{-1} = I_m + \int_{-\infty}^0 e^{i\lambda t}\gamma_-(t)\,dt, \tag{1.7}$$

with $\gamma_+ \in L_1^{m\times m}[0,\infty)$ and $\gamma_- \in L_1^{m\times m}(-\infty, 0]$. Using the factorization (1.5) and omitting the variable λ, equation (1.2) can be rewritten as

$$(I_m + G_+)\Phi_+ - (I_m + G_-)^{-1}F_- = (I_m + G_-)^{-1}F_+. \tag{1.8}$$

Let $\mathcal{P}$ be the projection acting on the Fourier transforms of $L_p^m(-\infty,\infty)$-functions according to the following rule:

$$\mathcal{P}\left(\int_{-\infty}^\infty e^{i\lambda t}h(t)\,dt\right) = \int_0^\infty e^{i\lambda t}h(t)\,dt.$$

Applying $\mathcal{P}$ to (1.8) one gets

$$(I_m + G_+)\Phi_+ = \mathcal{P}\big((I_m + G_-)^{-1}F_+\big),$$

and hence

$$\Phi_+ = (I_m + G_+)^{-1}\mathcal{P}\big((I_m + G_-)^{-1}F_+\big), \tag{1.9}$$

which is the formula for the solution of equation (1.2). To obtain the solution ϕ of the original equation (1.1), i.e., to obtain the inverse Fourier transform of Φ_+, one can employ the formulas (1.6) and (1.7). In fact

$$\phi(t) = f(t) + \int_0^\infty \gamma(t,s)f(s)\,ds, \qquad t \geq 0,$$

where the $m \times m$ matrix function $\gamma(t,s)$ is given by

$$\gamma(t,s) = \gamma_+(t-s) + \gamma_-(t-s) + \int_0^{\min(t,s)} \gamma_+(t-r)\gamma_-(r-s)\,dr.$$

We conclude the description of this factorization method by mentioning that the equation (1.1) has a unique solution in $L_p^m[0,\infty)$ for each f in $L_p^m[0,\infty)$ if and only if its symbol admits a factorization as in (1.5). For details, see [50], [61].

Let T be the *Wiener-Hopf integral operator* on $L_p^m[0,\infty)$ associated with equation (1.1), that is, T is the operator on $L_p^m[0,\infty)$ given by

$$(T\phi)(t) = \phi(t) - \int_0^\infty k(t-s)\phi(s)\,ds, \qquad t \geq 0.$$

The function W in the left-hand side of (1.3) is also referred to as the *symbol* of T. Obviously the operator T is invertible if and only if the equation (1.1) has a unique solution in $L_p^m[0,\infty)$ for each f in $L_p^m[0,\infty)$. Thus the results reviewed above can be summarized as follows.

Theorem 1.1. *Let T be the Wiener-Hopf integral operator on $L_p^m[0,\infty)$ with symbol W. Then T is invertible if and only if W admits a right canonical factorization with respect to the real line. Furthermore, if (1.5) is such a factorization of W, then the inverse of T is the integral operator given by*

$$(T^{-1}f)(t) \;=\; f(t) + \int_0^\infty \gamma(t,s)f(s)\,ds, \qquad t \geq 0,$$

where the kernel function γ is defined by

$$\gamma(t,s) = \begin{cases} \gamma_+(t-s) + \displaystyle\int_0^s \gamma_+(t-r)\gamma_-(r-s)\,dr, & 0 \leq s < t, \\ \gamma_-(t-s) + \displaystyle\int_0^t \gamma_+(t-r)\gamma_-(r-s)\,dr, & 0 \leq t < s \end{cases} \tag{1.10}$$

with γ_- and γ_+ as in (1.6) and (1.7), respectively.

To illustrate the method, let us consider a special choice for the right-hand side f (cf., [61]). Take

$$f(t) = e^{-iqt}x_0, \tag{1.11}$$

where x_0 is a fixed vector in $\mathbb{C}^m$ and q is a complex number with $\Im q < 0$. Then

$$F_+(\lambda) \;=\; \int_0^\infty e^{i(\lambda-q)t}x_0\,dt \;=\; \frac{i}{\lambda-q}x_0, \qquad \Im\lambda \geq 0.$$

Now observe that

$$\frac{i}{\lambda-q}\Big(\big(I_m + G_-(\lambda)\big)^{-1} - \big(I_m + G_-(q)\big)^{-1}\Big)x_0$$

is the Fourier transform of an $L_p^m(-\infty,0]$-function and hence it vanishes when the projection $\mathcal{P}$ is applied. It follows that in the present case the formula for Φ_+ may be written as

$$\Phi_+(\lambda) \;=\; \frac{i}{\lambda-q}\big(I_m + G_+(\lambda)\big)^{-1}\big(I_m + G_-(q)\big)^{-1}x_0.$$

Recall that the solution ϕ is the inverse Fourier transform of Φ_+. So we have

$$\phi(t) \;=\; e^{-iqt}\left(I_m + \int_0^t e^{iqs}\gamma_+(s)\,ds\right)\big(I_m + G_-(q)\big)^{-1}x_0. \tag{1.12}$$

1.2 Block Toeplitz equations and factorization

In this section we consider the discrete analogue of a Wiener-Hopf integral equation, that is, a *block Toeplitz equation* . So we consider an equation of the type

$$\sum_{k=0}^{\infty} a_{j-k}\xi_k \;=\; \eta_j, \qquad j=0,1,2,\ldots\,. \tag{1.13}$$

Throughout we assume that the coefficients a_j are given complex $m\times m$ matrices satisfying

$$\sum_{j=-\infty}^{\infty} \|a_j\| \;<\; \infty, \tag{1.14}$$

and $\eta = (\eta_j)_{j=0}^{\infty}$ is a given vector from $\ell_p^m = \ell_p(\mathbb{C}^m)$. The problem is to find $\xi = (\xi_k)_{k=0}^{\infty} \in \ell_p^m$ such that (1.13) is satisfied. We shall restrict ourselves to the case $1 \le p \le 2$; the final results however are valid for $2 < p \le \infty$ as well.

Assume $\xi \in \ell_p^m$ is a solution of (1.13). Then one can write (1.13) in the form

$$\sum_{k=-\infty}^{\infty} a_{j-k}\xi_k \;=\; \eta_j, \qquad j=0,\pm1,\pm2,\ldots\,, \tag{1.15}$$

where $\xi_k = 0$ for $k<0$ and η_j is defined by (1.15) for $j<0$. Multiplying both sides of (1.15) by λ^j with $|\lambda|=1$ and summing over j, one gets

$$a(\lambda)\xi_+(\lambda) - \eta_-(\lambda) \;=\; \eta_+(\lambda), \qquad |\lambda|=1, \tag{1.16}$$

where

$$a(\lambda) = \sum_{j=-\infty}^{\infty} \lambda^j a_j, \qquad \eta_+(\lambda) = \sum_{j=0}^{\infty}\lambda^j\eta_j, \tag{1.17}$$

$$\xi_+(\lambda) = \sum_{j=0}^{\infty}\lambda^j\xi_j, \qquad \eta_-(\lambda) = \sum_{j=-\infty}^{-1}\lambda^j\eta_j.$$

In this way the problem to solve (1.13) is reduced to that of finding two sequences ξ_+ and η_- such that (1.16) holds, while moreover, ξ_+ and η_- must be as in (1.2) with $(\xi_j)_{j=0}^{\infty}$ and $(\eta_{-j-1})_{j=0}^{\infty}$ from ℓ_p^m.

The usual way (cf., [61] or the book [40]) of solving (1.16) is again by factorizing the *symbol* $a(\lambda)$ of the given block Toeplitz equation. Assume that $a(\lambda)$ admits a *right canonical factorization with respect to the unit circle* . By definition this means that $a(\lambda)$ can be written as

$$\begin{aligned} a(\lambda) &= h_-(\lambda)h_+(\lambda), \qquad |\lambda|=1, \\ h_+(\lambda) &= \sum_{j=0}^{\infty}\lambda^j h_j^+, \qquad h_-(\lambda) = \sum_{j=-\infty}^{0}\lambda^j h_j^-, \end{aligned} \tag{1.18}$$

where $(h_j^+)_{j=0}^\infty$ and $(h_{-j}^-)_{j=0}^\infty$ belong to the space $\ell_1^{m\times m}$ of all absolutely convergent sequences of complex $m\times m$ matrices, $\det h_+(\lambda)\neq 0$ for $|\lambda|\leq 1$ and $\det h_-(\lambda)\neq 0$ for $|\lambda|\geq 1$ (including $\lambda=\infty$). Then h_+^{-1} and h_-^{-1} also admit a representation of the form

$$h_+^{-1}(\lambda)=\sum_{j=0}^{\infty}\lambda^j\gamma_j^+,\qquad h_-^{-1}(\lambda)=\sum_{j=-\infty}^{0}\lambda^j\gamma_j^-, \tag{1.19}$$

with $(\gamma_j^+)_{j=0}^\infty$ and $(\gamma_{-j}^-)_{j=0}^\infty$ from $\ell_1^{m\times m}$. Defining the projection $\mathcal{P}$ by

$$\mathcal{P}\Big(\sum_{j=-\infty}^{\infty}\lambda^j b_j\Big)=\sum_{j=0}^{\infty}\lambda^j b_j,$$

one gets from (1.16) and (1.18)

$$\xi_+ = h_+^{-1}\mathcal{P}\big(h_-^{-1}\eta_+\big). \tag{1.20}$$

Here, for convenience, the variable λ is omitted. The solution of the original equation (1.13) can now be written as

$$\xi_k=\sum_{s=0}^{\infty}\gamma_{ks}\eta_s,\qquad k=0,1,\dots\,, \tag{1.21}$$

where

$$\gamma_{ks}=\begin{cases}\displaystyle\sum_{r=0}^{s}\gamma_{k-r}^+\gamma_{r-s}^-, & s\leq k,\\ \displaystyle\sum_{r=0}^{k}\gamma_{k-r}^+\gamma_{r-s}^-, & s\geq k.\end{cases}$$

Note that for $s=k$ both sums in the above formula define the same matrix.

The assumption that $a(\lambda)$ admits a right canonical factorization as in (1.18) is equivalent to the requirement that for each $\eta=(\eta_j)_{j=0}^\infty$ in ℓ_p^m the equation (1.13) has a unique solution $\xi=(\xi_k)_{k=0}^\infty$ in ℓ_p^m. For details we refer to [61], [40].

Let T be the block Toeplitz operator on ℓ_p^m associated with the Toeplitz equation (1.13), that is, T is the operator on ℓ_p^m given by

$$T\xi=\eta \quad\Longleftrightarrow\quad \sum_{k=0}^{\infty}a_{j-k}\xi_k=\eta_j,\quad j=0,1,2,\dots\,.$$

The function a appearing in the left-hand side of (1.17) is also referred to as the *symbol* of T. Obviously T is invertible if and only if for each $\eta=(\eta_j)_{j=0}^\infty$ in ℓ_p^m the equation (1.13) has a unique solution $\xi=(\xi_k)_{k=0}^\infty$ in ℓ_p^m. This allows us to summarize the results reviewed above as follows.

Theorem 1.2. *Let T be the block Toeplitz operator on ℓ_p^m with symbol $a(\lambda)$ satisfying* (1.14). *Then T is invertible if and only $a(\lambda)$ admits a right canonical factorization with respect to the unit circle. Furthermore, if* (1.18) *is such a factorization of the function $a(\lambda)$, then the inverse of T is given by*

$$T^{-1} = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

where the matrices γ_{ks} are defined by

$$\gamma_{ks} = \begin{cases} \sum_{r=0}^{s} \gamma_{k-r}^{+} \gamma_{r-s}^{-}, & s \le k, \\ \sum_{r=0}^{k} \gamma_{k-r}^{+} \gamma_{r-s}^{-}, & s \ge k, \end{cases} \tag{1.22}$$

with γ_j^+ and γ_j^- being determined by (1.19).

By way of illustration, we consider the special case when

$$\eta_j = q^j \eta_0, \qquad j = 0, 1, \ldots . \tag{1.23}$$

Here η_0 is a fixed vector in $\mathbb{C}^m$ and q is a complex number with $|q| < 1$. Then clearly

$$\eta_+(\lambda) = \frac{1}{1 - \lambda q} \eta_0, \qquad |\lambda| \le 1,$$

and one checks without difficulty that formula (1.21) becomes

$$\xi_k = q^k \sum_{s=0}^{k} q^{-s} \gamma_s^+ h_-^{-1}(q^{-1}) \eta_0, \qquad k = 0, 1, \ldots . \tag{1.24}$$

This is the analogue of formula (1.12) in the previous section.

1.3 Singular integral equations and factorization

In this section we review the factorization method that is used to solve systems of singular integral equations [48]. Consider the *singular integral equation*

$$a(t)\phi(t) + b(t)\frac{1}{\pi i} \int_\Gamma \frac{\phi(\tau)}{\tau - t}\, d\tau = f(t), \qquad t \in \Gamma, \tag{1.25}$$

with integration taken over a Cauchy contour Γ. (For the definition of the latter notion see the final paragraphs of Chapter 0 dealing with terminology and notation.) We write F_+ for the interior domain of Γ, and F_- for the exterior domain (i.e., the complement of $\overline{F}_+$ in the Riemann sphere $\mathbb{C}\cup\{\infty\}$). The functions a and b in (1.25) are given continuous $m \times m$ matrix functions defined on Γ, and f is a given function from $L_p^m(\Gamma)$, p fixed, $1 < p < \infty$. As usual in the theory of singular integral equations, it is assumed that the interior domain F_+ of Γ is connected and contains 0; the exterior domain F_- of Γ contains ∞. The problem is to find $\phi \in L_p^m(\Gamma)$ such that(1.25) is satisfied.

For ϕ a rational function without poles on Γ we put

$$(S\phi)(t) \;=\; \frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t}\,d\tau \;=\; f(t), \qquad t \in \Gamma, \tag{1.26}$$

where the integral is taken in the sense of the Cauchy principal value. The operator S defined in this way can be extended by continuity to a bounded linear operator, again denoted by S, on all of $L_p^m(\Gamma)$. Equation (1.25) can now be written as

$$aI\phi + bS\phi \;=\; f, \tag{1.27}$$

where I is the identity operator on $L_p^m(\Gamma)$. In other words, the study of the equation (1.25) reduces to that of the operator $aI + bS$. Here a and b are viewed as multiplication operators. Equation (1.25) has a unique solution $\phi \in L_p^m(\Gamma)$ for each choice of $f \in L_p^m(\Gamma)$ if and only if the operator $aI + bS$ is invertible as an operator on $L_p^m(\Gamma)$. In the remainder of this section we shall discuss a necessary and sufficient condition for this to happen, and we shall give formulas for the inverse $(aI + bS)^{-1}$.

The operator S enjoys the property $S^2 = I$. Hence the operators

$$P_\Gamma = \frac{1}{2}(I + S), \qquad Q_\Gamma = \frac{1}{2}(I - S)$$

are complementary projections on $L_p^m(\Gamma)$. The image of P_Γ consists of all functions in $L_p^m(\Gamma)$ that admit an analytic continuation into F_+. Similarly, the image of Q_Γ is the set of all functions in $L_p^m(\Gamma)$ that admit an analytic continuation into F_- vanishing at ∞. Putting $c = a+b$ and $d = a-b$, one can write the equation (1.27) in the form $cP_\Gamma\phi + dQ_\Gamma\phi \;=\; f$.

The following is known (see [62] for the case when the coefficients a and b are scalar functions and [48] for the matrix-valued case). The operator $aI + bS = cP_\Gamma + dQ_\Gamma$ is invertible if and only if the matrices $c(\lambda)$ and $d(\lambda)$ are invertible for each $\lambda \in \Gamma$ and the function w given by $w(\lambda) = d(\lambda)^{-1}c(\lambda)$ admits a *right canonical factorization with respect to* Γ . By this we mean a factorization

$$w(\lambda) = w_-(\lambda)w_+(\lambda), \qquad \lambda \in \Gamma, \tag{1.28}$$

where w_- and w_+ are $m\times m$ matrix functions, analytic and taking invertible values on an open neighborhood of $\overline{F}_-$ and $\overline{F}_+$, respectively. With the help of (1.28), the operator $aI+bS = cP_\Gamma+dQ_\Gamma$ can be rewritten as $aI+bS = dw_-(w_+P_\Gamma+w_-^{-1}Q_\Gamma)$, and its inverse is given by

$$\begin{aligned}(aI+bS)^{-1} &= (w_+^{-1}P_\Gamma + w_- Q_\Gamma)w_-^{-1}d^{-1}\\ &= w_+^{-1}P_\Gamma w_-^{-1}d^{-1} + w_- Q_\Gamma w_-^{-1}d^{-1}.\end{aligned} \tag{1.29}$$

Replacing P_Γ and Q_Γ by $\frac{1}{2}(I+S)$ and $\frac{1}{2}(I-S)$, respectively, one gets

$$\begin{aligned}(aI+bS)^{-1} &= \frac{1}{2}(c^{-1}+d^{-1})I + \frac{1}{2}(w_+^{-1} - w_-)Sw_-^{-1}d^{-1}\\ &= \frac{1}{2}[(a+b)^{-1}+(a-b)^{-1}]I + \frac{1}{2}(w_+^{-1} - w_-)Sw_-^{-1}(a-b)^{-1}\\ &= (a+b)^{-1}a(a-b)^{-1}I + \frac{1}{2}(w_+^{-1} - w_-)Sw_-^{-1}(a-b)^{-1}.\end{aligned}$$

Summarizing we get the following theorem.

Theorem 1.3. *The singular integral operator $T = aI + bS$ on $L_p^m(\Gamma)$ is invertible if and only if the matrices $a(\lambda)+b(\lambda)$ and $a(\lambda)-b(\lambda)$ are invertible for each $\lambda \in \Gamma$ and the function w given by*

$$w(\lambda) = \big(a(\lambda)+b(\lambda)\big)^{-1}\big(a(\lambda)+b(\lambda)\big)$$

admits a right canonical factorization with respect to Γ. Furthermore, if (1.28) *is such a factorization of w, then the inverse of T is given by*

$$T^{-1} = (a+b)^{-1}a(a-b)^{-1}I + \frac{1}{2}(w_+^{-1} - w_-)Sw_-^{-1}(a-b)^{-1}. \tag{1.30}$$

Thus, as before for Wiener-Hopf and block Toeplitz operators, canonical factorization is a useful method for inverting singular integral operators too.

Notes

The material in this chapter is standard, and can be found in much more detail and greater generality in various monographs and papers, for instance, see the books [29] and [50]. A first introduction to the theory of Wiener-Hopf integral equations and the theory of (block) Toeplitz operators can be found in Chapters XII and XIII of [51] and Chapters XXIII–XXV of [52], respectively. More information can be found in the monographs [37], [62], [63], [64] and [24]. For an extensive review (with many additional references) of the factorization theory of matrix functions with respect to a curve and its applications to inversion of singular integral operators of different types, including Wiener-Hopf and block Toeplitz operators, the reader is referred to the recent survey paper [59].

Chapter 2

The state space method and factorization

This chapter describes in detail the elements of the state space method that are used throughout this book. The central notion is that of a realization of a matrix or operator function. The chapter consists of six sections. Section 2.1 presents preliminaries on realization, including the relevant definitions and the connection with systems theory. In the next two sections the realization problem is discussed. First for rational matrix functions in Section 2.2, and then for analytic operator functions in a possibly infinite dimensional setting in Section 2.3. The last three sections are devoted to the main operations on realizations that are needed in this book: inversion (Section 2.4), taking products (Section 2.5), and factorization (Section 2.6).

2.1 Preliminaries on realization

Let W be a rational matrix function which is also *proper*, that is, W has no pole at infinity. As is well-known such a function can always be represented (see the next section for an explicit construction) in the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B. \tag{2.1}$$

Here λ is a complex variable, A is a square matrix, I is the identity matrix of the same size as A, and B and C are matrices of appropriate sizes. Since A, B, C and D are matrices, it is immediate from Cramer's rule that the right-hand side of (2.1) is also a proper rational matrix function. We shall understand the equality in (2.1) as an equality between rational matrix functions, and we shall refer to (2.1) as a *matrix-valued realization* of W. Sometimes we simply say that the quadruple of matrices (A, B, C, D) is a *realization* of W. A rational matrix function has many different realizations. Of particular interest are those matrix-valued realizations of W of which the order of the matrix A is as small as possible. These realizations are called *minimal*; we shall describe their properties in Chapter 8.

For operator-valued functions W, expressions of the type (2.1) are important too but have to be considered with some care. Let W be an $\mathcal{L}(U, Y)$-valued function on a subset Ω of $\mathbb{C}$. Here U and Y are possibly infinite dimensional complex Banach spaces. We say that W admits a *realization on* Ω whenever W can be written as

$$W(\lambda) = D + C(\lambda I_X - A)^{-1}B, \qquad \lambda \in \Omega. \tag{2.2}$$

Here A is a bounded linear operator on a complex Banach space X such that Ω is a subset of $\rho(A)$, the *resolvent set* of A. Furthermore, I_X is the identity operator on X, and

$$B \in \mathcal{L}(U, X), \qquad C \in \mathcal{L}(X, Y), \qquad D \in \mathcal{L}(U, Y),$$

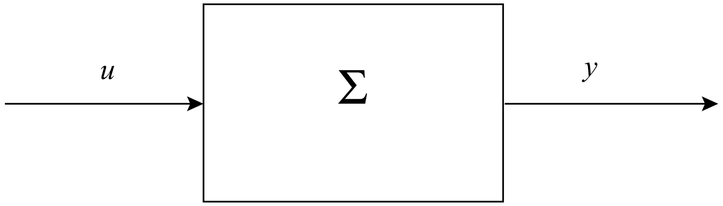
that is $B : U \to X$, $C : X \to Y$, and $D : U \to Y$, are bounded linear operators. The fact that $\Omega \subset \rho(A)$ implies that the right-hand side of (2.2) is a well-defined bounded linear operator which maps U into Y for each $\lambda \in \Omega$. Also, $W(\lambda)$ is a bounded linear operator mapping U into Y for each $\lambda \in \Omega$. Note that (2.2) requires these operators to be equal for each $\lambda \in \Omega$. When Ω is open, an obvious necessary condition for W to admit a realization on Ω is that W be analytic on Ω. When Ω is a punctured open neighborhood of ∞, then (2.2) implies $\lim_{\lambda \to \infty} W(\lambda) = D$ and so W is proper.

Often the identity matrix I in (2.1) and the identity operator I_X in (2.2) will be suppressed, and we simply write $(\lambda - A)^{-1}$ in place of $(\lambda I - A)^{-1}$ or $(\lambda I_X - A)^{-1}$.

When X and Y are both finite dimensional, then the realization (2.2) is called *finite dimensional*. In that case $W(\lambda)$, A, B, C and D can be identified in the usual way with matrices.

In the next two sections we shall address the realization problem, i.e., the question under what conditions a given matrix or operator function admits a realization. First however, we sketch a connection with systems theory which reflects itself in some terminology to be introduced at the end of the present section.

A system Σ can be considered as a physical object which produces an output in response to an input. Schematically:



where u denotes the input and y denotes the output. Mathematically, the input u and the output y are vector-valued functions of a parameter t. The input can