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Răzvan Diaconescu

Institution-independent Model Theory

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Author:

Răzvan Diaconescu
Institute of Mathematics „Simion Stoilow“
Calea Griviței 21
București 010702
Romania
Email: razvan.diaconescu@imar.ro

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To my parents, Elena and Ştefan

Preface

This is a book about doing model theory without an underlying logical system. It teaches us how to live without concrete models, sentences, satisfaction and so on. Our approach is based upon the theory of institutions, which has witnessed a vigorous and systematic development over the past two decades and which provides an ideal framework for true abstract model theory. The concept of institution formalizes the intuitive notion of logical system into a mathematical object. Thus our model theory without underlying logical systems and based upon institution theory may be called 'institution-independent model theory'.

Institution-independent model theory has several advantages. One is its generality, since it can be easily applied to a multitude of logical systems, conventional or less conventional, many of the latter kind getting a proper model theory for the first time through this approach. This is important especially in the context of the recent high proliferation of logics in computing science, especially in the area of formal specification. Then there is the advantage of illuminating the model theoretic phenomena and its subtle network of causality relationships, thus leading to a deeper understanding which produces new fundamental insights and results even in well worked traditional areas of model theory.

In this way we study well established topics in model theory but also some newly emerged important topics. The former category includes methods (in fact much of model theory can be regarded as a collection of sometimes overlapping methods) such as (elementary) diagrams, ultraproducts, saturated models and studies about preservation, axiomatizability, interpolation, definability, and possible worlds semantics. The latter category includes methods of doing model theory 'by translation', and Grothendieck institutions, which is a recent successful model theoretic framework for multi-logic heterogeneous environments. The last two chapters (14 and 15) digress from the main topic of the book in that they present some applications of institution-independent model theory to specification and programming and Chap. 13 shows how to integrate proof theoretic concepts to institution-independent model theory (including a general approach to completeness).

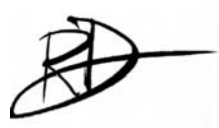
This book is far from being a complete encyclopedia of institution-independent model theory. While several important concepts and results have not been treated here, we believe they can be approached successfully with institutions in the style promoted by our work. Most of all, this book shows *how* to do things rather than provides an exhaustive account of all model theory that can be done institution-independently. It can be used by any working user of model theory but also as a resource for learning model theory.

From the philosophical viewpoint, the institution-independent approach to model theory is based upon a non-essentialist, groundless, perspective on logic and model theory, directly influenced by the doctrine of *śunyata* of the Madhyamaka Prasangika school within Mahayana Buddhism. The interested reader may find more about this connection in the essay [54]. This has been developed mainly at Nalanda monastic university about 2000 years ago by Arya Nāgārjuna and its successors and has been continued to our days by all traditions of Tibetan Buddhism. The relationship between Madhyamaka Prasangika thinking and various branches of modern science is surveyed in [176].

I am grateful to a number of people who supported in various ways the project of institution-independent model theory in general and the writing of this book in particular. I was extremely fortunate to be first the student and later a close friend and collaborator of late Professor Joseph Goguen who together with Rod Burstall introduced institutions. He strongly influenced this work in many ways and at many levels, from philosophical to technical aspects, and was one of the greatest promoters of the non-essentialist approach to science. Andrzej Tarlecki was the true pioneer of doing model theory in an abstract institutional setting. Till Mossakowski made a lot of useful comments on several preliminary drafts of this book and supported this activity in many other ways too. Grigore Roşu and Marc Aiguier made valuable contributions to this area. Lutz Schröder made several comments and gave some useful suggestions. Achim Blumensath read very carefully a preliminary draft of this book and helped to correct a series of errors. I am indebted to Hans-Jürgen Hoenhke for encouragement and managerial support. Special thanks go to the former students of the Informatics Department of "Şcoala Normală Superioară" of Bucharest, namely Marius Petria, Daniel Găină, Andrei Popescu, Mihai Codescu, Traian Şerbănuţă and Cristian Cucu. They started as patient students of institution-independent model theory only to become important contributors to this area. Finally, Jean-Yves Béziau greatly supported the publication and dissemination of this book. I acknowledge financial support for writing this book from the CNCSIS grants GR202/2006 and GR54/2007.

Ploieşti,

December 2007 *Răzvan Diaconescu*



Contents

Chapter 1

Introduction

Model theory is in essence the mathematical study of semantics, or meaning, of logic systems. As it has a multitude of applications to various areas of classical mathematics, and of logic, but also to many areas of informatics and computing science, there are various perspectives on model theory which differ slightly. A rather classical viewpoint is formulated in [32]:

Model theory = logic + universal algebra.

A rather different and more radical perspective which reflects the success of model theoretic methods in some areas of classical mathematics is given in [99]:

Model theory = algebraic geometry - fields.

From a formal specification viewpoint, in a similar tone, one may say that

Model theory = logical semantics - specification.

Each such viewpoint implies a specific way in developing the key concepts and the main model theory methods; it also puts different emphasis on results. For example while forcing is a very important method for the applications of model theory to conventional logic, it plays a very little role in computing science. On the other hand, formal specification theory requires a much more abstract view on model theory than the conventional one. The institution theory of Goguen and Burstall [30, 75] arose out of this necessity.

Institutions. The theory of institutions is a categorical abstract model theory which formalizes the intuitive notion of a logical system, including syntax, semantics, and the satisfaction relation between them. Institutions constitute a model-oriented meta-theory on logics similarly to how the theory of rings and modules constitute a meta-theory for classical linear algebra. Another analogy can be made with universal algebra versus particular algebraic structures such as groups, rings, modules, etc., or with mathematical analysis over Banach spaces versus real analysis.

The notion of institution was introduced by Goguen and Burstall in the late 1970s [30] (with the seminal journal paper [75] being printed rather late) in response to the population explosion of specification logics with the original intention of providing a proper abstract framework for specification of, and reasoning about, software systems. Since then institutions have become a major tool in development of the theory of specification, mainly because they provide a language-independent framework applicable to a wide variety of particular specification logics. It became standard in the field to have a logic system captured as the institution underlying a particular language or system, such that all language/system constructs and features can be rigorously explained as mathematical entities and to separate all aspects that depend on the details of the particular logic system from those that are general and independent of this logic system by basing the latter on an arbitrary institution. All well-designed specification formalisms follow this path, including for example CASL [10] and CafeOBJ [57].

Recently institutions have also been applied to computing science fields other than formal specification; these include ontologies and cognitive semantics [73], concurrency [138], and quantum computing [31].

Institution-independent model theory. This means the development of model theory in the very abstract setting of arbitrary institutions, free of any commitment to a particular logic system. In this way we gain another level of abstraction and generality and a deeper understanding of model theoretic phenomena, not hindered by the largely irrelevant details of a particular logic system, but guided by structurally clean causality. The latter aspect is based upon the fact that concepts come naturally as presumed features that "a logic" might exhibit or not and are defined at the most appropriate level of abstraction; hypotheses are kept as general as possible and introduced on a by-need basis, and thus results and proofs are modular and easy to track down regardless of their depth. Access to highly non-trivial results is also considerably facilitated, which is contrary to the impression of some people that such general abstract approaches produce results that are trivial. As Béziau explains in [20]:

> "This impression is generally due to the fact that these people have a concrete-oriented mind, and that something which is not specified [n.a. concretely] has no meaning for them, and therefore universal logic [n.a. institution-independent model theory in our case] appears as a logical abstract nonsense. They are like someone who understands perfectly what is Felix, his cat, but for whom the concept of cat is a meaningless abstraction. This psychological limitation is in fact a strong defect because, ... [n.a. as this book shows], what is trivial is generally the specific part, not the universal one [n.a. the institution-independent one] which requires what is the fundamental capacity of human thought: abstraction."

The continuous interplay between the specific and the general in institution-independent model theory brings a large array of new results for particular non-conventional logics, unifies several known results, produces new results in well-studied conventional areas, reveals previously unknown causality relations, and dismantles some which are usually assumed as natural.

Institution-independent model theory also provides a clear and efficient framework for doing logic and model theory 'by translation (or borrowing)' via a general theory of mappings (homomorphisms) between institutions. For example, a certain property P which holds in an institution I' can be also established in another institution I provided that we can define a mapping $I \to I'$ which 'respects' P.

Institution-independent model theory can be regarded as a form of 'universal model theory', part of the so-called 'universal logic', a recent trend in logic promoted by Bèziau and others [21].

Other abstract model theories. Only two major abstract approaches to logic have a model theoretic nature and are therefore comparable to the institution-independent model theory.

The so-called "abstract model theory" developed by Barwise and others [12, 13] however keeps a strong commitment to conventional concrete systems of logic by explicitly extending them and retaining many of their features, hence one may call this framework "half-abstract model theory". In this context even the remarkable Lindström characterization of first order logic by some of its properties should be rather considered as a first order logic result rather than as a true abstract model theoretic one.

Another framework is given by the so-called "categorical model theory" best represented by the works on sketches [63, 88, 181] or on satisfaction as cone injectivity [5, 6, 7, 120, 118, 116]. The former just develops another language for expressing (possibly infinitary) first order logic realities. While the latter considers models as objects of abstract categories, it lacks the multi-signature aspect of institutions given by the signature morphism and the model reducts, which leads to severe methodological limitations. Moreover in these categorical model theory frameworks, the satisfaction of sentences by the models is usually defined rather than being axiomatized.

By contrast to the two abstract model theoretic approaches mentioned above, institutions capture directly the essence of logic systems by axiomatizing the satisfaction relationship between models and sentences without any initial commitment to a particular logic system and by emphasizing propertly the multi-signature aspect of logics.

Book content. The book consists of four parts.

In the first part we introduce the basic institution theory including the concept of institution and institution morphisms, and several model theoretic fundamental concepts such as model amalgamation, (elementary) diagrams, inclusion systems, and free models. We develop an 'internal logic' for abstract institutions, which includes a semantic treatment to Boolean connectives, quantifiers, atomic sentences, substitutions, and elementary homomorphisms, all of them in an institution-independent setting.

The second part is the core of our institution-independent model theoretic study because it develops the main model theory methods and results in an institution-independent setting.

The first method considered in this part is that of ultraproducts. Based upon the well-established concept of categorical filtered products, we develop an ultraproduct fundamental theorem in an institution-independent setting and explore some of its immediate consequences, such as ultrapower embeddings and compactness.

The chapter on saturated models starts by developing sufficient conditions for directed co-limits of homomorphisms to retain the elementarity. This rather general version of Tarski's elementary chain theorem is a prerequisite for a general result about existence of saturated models, later used for developing other important results. We also develop the complementary result on uniqueness of saturated models. Here the necessary concept of cardinality of a model is handled categorically with the help of elementary extensions, a concept given by the method of diagrams. We develop an important application for the uniqueness of saturated models, namely a generalized version of the remarkable Keisler-Shelah result in first order model theory, "two models are elementarily equivalent if and only if they have isomorphic ultrapowers".

A good application of the existence result for saturated models is seen in the preservation results, such as "a theory has a set of universal axioms if and only if its class of models is closed under 'sub-models'". We develop a generic preservation-by-saturation theorem. Such preservation results might lead us straight to their axiomatizability versions. One way is to assume the Keisler-Shelah property for the institution and to use a direct consequence of the fundamental ultraproducts theorem which may concisely read as "a class of models is elementary if and only if it is closed under elementary equivalence and ultraproducts".

Another method to reach an important class of axiomatizability results is by expressing the satisfaction of Horn sentences as categorical injectivity. This leads to general quasi-variety theorems such as "a class of models is closed under products and 'sub-models' if and only if it is axiomatizable by a set of (universal) Horn sentences" and variety theorems such as "a class of models is closed under products and 'sub-models' and 'homomorphic images' if and only if it is axiomatizable by a set of (universal) 'atoms'".

All axiomatizability results presented here are collected under the abstract concept of 'Birkhoff institution'.

The next topic is interpolation. The institution-independent approach brings several significant upgrades to the conventional formulation. We develop here three main methods for obtaining the interpolation property, the first two having rather complementary application domains. The first one is based upon a semantic approach to interpolation and exploits the Birkhoff-style axiomatizability properties of the institution (captured by the above mentioned concept of Birkhoff institution), while the second, inspired by the conventional methods of first order logic, is via Robinson consistency. The third one is a borrowing method across institutions.

We next treat definability, again with rather two complementary methods, via Birkhoff-style axiomatizability and via interpolation. While the latter represents a generalization of Beth's theorem of conventional first order model theory, the former reveals a causality relationship between axiomatizability and definability.

The final chapter of the second part of the book is devoted to possible worlds (Kripke) semantics and to extensions of the satisfaction relation of abstract institutions to modal satisfaction. By applying the general ultraproducts method to possible worlds semantics, we develop the preservation of modal satisfaction by ultraproducts together with its semantic compactness consequence.

The third part of the book is devoted to special modern topics in institution theory, such as Grothendieck constructions on systems of institutions with applications to heterogeneous multi-logic frameworks, and an extension of institutions with proof theoretic concepts. For the Grothendieck institutions we develop a systematic study of lifting of important properties such as theory co-limits, model amalgamation, and interpolation, from the level of the 'local' institutions to the 'global' Grothendieck institution. We present a rather striking application of the interpolation result for Grothendieck institutions, which leads for example to a quite surprising interpolation property in the Horn fragment of conventional first order logic. The chapter on proof theory for institutions introduces the concept of proof in a simple way that suits the model theory, explores proof theoretic versions of compactness and internal logic, and presents general soundness results for institutions with proofs. The final part of this chapter develops a general sound and complete Birkhoff-style proof system with applications significantly wider than that of the Horn institutions.

The last part presents a few of the multitude of applications of institution-independent model theory to computing science, especially in the areas of formal specification and logic programming. This includes structured specifications over arbitrary institutions, the lifting of a complete calculus from the base institution to structured specifications, Herbrand theorems and modularization for logic programming, and semantics of logic programming with pre-defined types.

The concepts introduced and the results obtained are systematically illustrated in the main text by their applications to the model theory of conventional logic (which includes first order logic but also fragments and extensions of it). There are only two reasons for doing this. The first is to build a bridge between our approach and the conventional model theory culture. The second reason has to do with keeping the material within reasonable size. Otherwise, while conventional (first-order) model theory has been historically the framework for the development of the main concepts and methods of model theory, one of the main messages of this book is that these do not depend on that framework. Any other concrete logic or model theory could be used as a benchmark example in this book, and in fact we do this systematically in the exercise sections with several less conventional logics.

How to use this book. The material of this book can be used in various ways by various audiences both from logic and computing science. Students and researchers of logic can use material of the first two parts (up to Chap. 11 included) as an institution-independent introduction to model theory. Working logicians and model theorists will find in this monograph a novel view and a new methodological approach to model theory. Computer scientists may use the material of the first part as an introduction to institution theory, and material from the third and the fourth parts for an advanced approach to topics from the semantics of formal specification and logic programming. Also, institution-independent model theory constitutes a powerful tool for workers in formal specification to perform a systematic model theoretic analysis of the logic underlying the particular system they employ.

Each section comes with a number of exercises. While some of them are meant to help the reader accommodate the concepts introduced, others contain quite important results and applications. In fact, in order to keep the book within a reasonable size, much of the knowledge had to be exiled to the exercise sections.

Chapter 2

Categories

Institution-independent model theory as a categorical abstract model theory relies heavily on category theory. This preliminary chapter gives a brief overview of the categorical concepts and results used by this book. The reader without enough familiarity with category theory is advised to use one of the textbooks on category theory available in the literature. [111] and [26] are among standard references for category theory. A reference for indexed categories discussing many examples from the model theory of algebraic specification is [174], while [101] contains a rather compact presentation of fibred category theory.

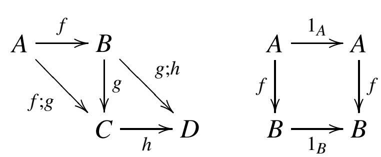
2.1 Basic Concepts

Categories

A *category* $\mathbb{C}$ consists of

- a class $|\mathbb{C}|$ of *objects*,
- a class of *arrows* (sometimes also called 'morphisms' or 'homomorphisms'), denoted just as $\mathbb{C}$,
- two maps $dom, cod : \mathbb{C} \to |\mathbb{C}|$ giving the *domain* and *codomain* of each arrow such that for each pair of objects A and B, $\mathbb{C}(A,B) = \{f \in \mathbb{C} \mid dom(f) = A, cod(f) = B\}$ is a *set*,
- for all objects A,B,C, a *composition* map $\_;\_ : \mathbb{C}(A,B) \times \mathbb{C}(B,C) \to \mathbb{C}(A,C)$,
- an *identity* arrow map $1 : |\mathbb{C}| \to \mathbb{C}$ such that $1_A \in \mathbb{C}(A,A)$ for each $A \in |\mathbb{C}|$,

such that the (arrow) composition $\_;\_$ is associative and with identity arrows as left and right identities.



Notice that we prefer to use the diagrammatic notation $f;g$ for composition of arrows in categories, rather than the alternative set theoretic one $g \circ f$ used in many category theory works.

Categories arise everywhere in mathematics. A most typical example is that of sets (as objects) and functions (as arrows) with the usual (functional) composition. We denote this category by $\mathbb{S}et$. Notice that $|\mathbb{S}et|$, the collection of all sets, is *not* a set, it is a proper *class*.

The arrows of a category in general reflect the structure of objects in the sense of preserving that structure. However, obviously this should not always be the case. One can go further by saying that, in reality, a particular category is determined only by its arrows, the objects being a derived rather than a primary concept.

A category $\mathbb{C}$ is *small* when its class of objects $|\mathbb{C}|$ is a set. Note that this implies that $\mathbb{C}$, the class of arrows, is also a set.

$\mathbb{C}$ is *connected* when there exists only one equivalence class for the equivalence generated by the relation on objects given by "there exists an arrow $A \to B$".

Isomorphisms. An arrow $f: A \to B$ is an *isomorphism* when there exists an arrow $g: B \to A$ such that $f;g = 1_A$ and $g;f = 1_B$. The *inverse* g is denoted as f^{-1}. Two objects A and B are *isomorphic*, and we denote this by $A \cong B$, when there exists an isomorphism $f: A \to B$. Isomorphisms in $\mathbb{S}et$ are precisely the bijective (injective and surjective) functions. However this is not true in general; structure preserving mappings that are bijective are not necessarily isomorphisms. A simple counterexample is given by the category of partial orders (objects) with order-preserving functions as arrows.

Monoids are exactly the categories with only one object. Then *groups* are exactly the monoids for which all elements (arrows) are isomorphisms.

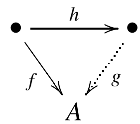
Being isomorphic is an equivalence relation on objects; the equivalence classes of $\cong$ are called *isomorphism classes*.

Epis and monos. A family of arrows $\{f_i: A \to B\}_{i \in I}$ is *epimorphic* when for each pair of parallel arrows $g_1, g_2: B \to C$, $f_i;g_1 = f_i;g_2$ for each $i \in I$ implies $g_1 = g_2$, and it is *monomorphic* when for each pair of parallel arrows $g_1, g_2: C \to A$, $g_1;f_i = g_2;f_i$ for each $i \in I$ implies $g_1 = g_2$. An arrow $f: A \to B$ is *epi/mono* when it is epimorphic/monomorphic as a (singleton) family, i.e., $\{f\}$ is epimorphic/monomorphic.

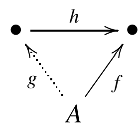
In $\mathbb{S}et$ epis are exactly the surjective functions and the monos are exactly the injective ones. Note that while, in general, whenever arrows appear as functions with additional structure, the injectivity (respectively surjectivity) of the underlying function is a sufficient condition for a function to be mono (respectively epi), the converse is not true. For example, the inclusion $\mathbb{Z} \to \mathbb{Q}$ of integers into the rationals is epi in the category of rings but it is not surjective. This is also an example of an arrow which is both epi and mono but is not an isomorphism.

An arrow $f: A \to B$ is a *retract* to $g: B \to A$ when $g;f = 1_B$. Notice that each retract is epi. The converse, which is not true in general, is one of the categorical formulations of the Axiom of Choice. Note that $\mathbb{S}et$ has the Axiom of Choice in this sense.

An object A is *injective* with respect to an arrow h when for each arrow $f: dom(h) \to A$ there exists an arrow g such that $h;g = f$. A is simply *injective* when it is injective with respect to all mono arrows.



Dually, an object A is *projective* with respect to an arrow h when for each arrow $f: A \to cod(h)$ there exists an arrow g such that $g;h = f$. A is simply *projective* when it is projective with respect to all epi arrows. Note that in $\mathbb{S}et$ all objects (sets) are both injective and projective.



Functors

A *functor* $\mathcal{U}: \mathbb{C} \to \mathbb{C}'$ between categories $\mathbb{C}$ and $\mathbb{C}'$ maps

- objects to objects, $|\mathcal{U}|: |\mathbb{C}| \to |\mathbb{C}'|$, and
- arrows to arrows, $\mathcal{U}_{A,B}: \mathbb{C}(A,B) \to \mathbb{C}'(\mathcal{U}(A), \mathcal{U}(B))$ for all objects $A, B \in |\mathbb{C}|$

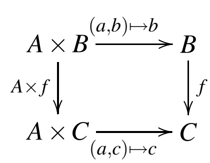
such that

- $\mathcal{U}(1_A) = 1_{\mathcal{U}(A)}$ for each object $A \in |\mathbb{C}|$, and
- $\mathcal{U}(f;g) = \mathcal{U}(f); \mathcal{U}(g)$ for all composable arrows $f, g \in \mathbb{C}$.

Most of the time we will denote $|\mathcal{U}|$ and $\mathcal{U}_{A,B}$ simply by $\mathcal{U}$. The application of functors (to either objects or arrows) can also be written in a "diagrammatic" way as $f\mathcal{U}$ rather than the more classical $\mathcal{U}(f)$. Sometimes it is even convenient to use subscripts or superscripts for the application of functors to objects or arrows.

A simple example is the power-set functor $\mathcal{P}: \mathbb{S}et \to \mathbb{S}et$ which maps each set S to the set of its subsets $\{X \mid X \subseteq S\}$ and maps each function $f: S \to S'$ to the function $\mathcal{P}(f): \mathcal{P}(S) \to \mathcal{P}(S')$ such that $\mathcal{P}(f)(X) = f(X) = \{f(x) \mid x \in X\}$.

Another example is given by 'cartesian product with A'. For any fixed set A, let $A \times -: \mathbb{S}et \to \mathbb{S}et$ be the functor mapping each set B to $A \times B = \{(a,b) \mid a \in A, b \in B\}$ and each function $f: B \to C$ to $(A \times f): A \times B \to A \times C$ defined by $(A \times f)(a,b) = (a, f(b))$.



A third example is that of 'hom-functors'. For any category $\mathbb{C}$ and any object $A \in |\mathbb{C}|$, the hom-functor $\mathbb{C}(A,-): \mathbb{C} \to \mathbb{S}et$ maps any object $B \in |\mathbb{C}|$ to the set of arrows $\mathbb{C}(A,B)$ and each arrow $f: B \to B'$ to the function $\mathbb{C}(A,f): \mathbb{C}(A,B) \to \mathbb{C}(A,B')$ defined by $\mathbb{C}(A,f)(g) = g;f$.

Each preorder-preserving function between two preorders $(P,\leq) \to (Q,\leq)$ is another example of a functor. In fact, functors between preorders are precisely the monotonic functions.

A functor $\mathcal{U}: \mathbb{C} \to \mathbb{C}'$ is *full* when for each objects A and B, the mapping on arrows $\mathcal{U}_{A,B}: \mathbb{C}(A,B) \to \mathbb{C}'(\mathcal{U}(A),\mathcal{U}(B))$ is surjective and is *faithful* when $\mathcal{U}_{A,B}$ is injective. Note that both functors of the first and of the second example are faithful but not full.

Functors can be composed in the obvious way and each category has an identity functor with respect to functor composition. By discarding the foundational issues (for the interested reader we recommend [95] or [111]), let $\mathbb{C}at$ be the 'quasi-category' of categories (as objects) and functors (as arrows).

$\mathbb{C} \subseteq \mathbb{C}'$ is a *subcategory* (of $\mathbb{C}'$) when $|\mathbb{C}| \subseteq |\mathbb{C}'|$, $\mathbb{C}(A,B) \subseteq \mathbb{C}'(A,B)$ for all $A,B \in |\mathbb{C}|$, and the composition in $\mathbb{C}$ is a restriction of the composition in $\mathbb{C}'$. A subcategory $\mathbb{C} \subseteq \mathbb{C}'$ is *broad* when $|\mathbb{C}| = |\mathbb{C}'|$.

Natural transformations

Fixing categories $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{C}at(\mathbb{A},\mathbb{B})$ can be regarded as a category with functors as objects and *natural transformations* as arrows. A natural transformation $\tau: \mathcal{S} \Rightarrow \mathcal{T}$ between functors $\mathcal{S},\mathcal{T}: \mathbb{A} \to \mathbb{B}$ is a map $|\mathbb{A}| \to \mathbb{B}$ such that $\tau(A) \in \mathbb{B}(\mathcal{S}(A),\mathcal{T}(A))$ for each $A \in |\mathbb{A}|$ and the following diagram commutes (in $\mathbb{B}$)

$$\begin{array}{ccc} \mathcal{S}(A) & \xrightarrow{\tau(A)} & \mathcal{T}(A) \\ {\scriptstyle \mathcal{S}(f)}\downarrow & & \downarrow{\scriptstyle \mathcal{T}(f)} \\ \mathcal{S}(B) & \xrightarrow[\tau(B)]{} & \mathcal{T}(B) \end{array}$$

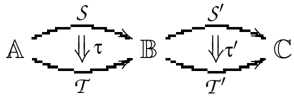
for each arrow $f \in \mathbb{A}(A,B)$. The classical notation for the component $\tau(A)$ is τ_A, however the diagrammatic notation $A\tau$ is also frequently used.

A simple example is generated by considering a function $A \xrightarrow{f} A'$ which determines a natural transformation $nt(f): (A \times -) \Rightarrow (A' \times -)$ given by $nt(f)_B = f \times 1_B$ for each set B, where $(f \times 1_B)(a,b) = (f(a),b)$ for each $(a,b) \in A \times B$.

An additional example is given by the natural transformation $\mathbb{C}(f,-): \mathbb{C}(A,-) \Rightarrow \mathbb{C}(B,-)$ for each arrow $B \xrightarrow{f} A$ in a category $\mathbb{C}$. For each $D \in |\mathbb{C}|$, $\mathbb{C}(f,-)_D = \mathbb{C}(f,D): \mathbb{C}(A,D) \to \mathbb{C}(B,D)$ where $\mathbb{C}(f,D)(g) = f;g$.

The composition of natural transformations is defined component-wise, i.e., $A(\sigma;\tau) = A\sigma;A\tau$ where $\sigma: \mathcal{R} \Rightarrow \mathcal{S}: \mathbb{A} \to \mathbb{B}$ and $\tau: \mathcal{S} \Rightarrow \mathcal{T}: \mathbb{A} \to \mathbb{B}$. This is called the *'vertical' composition* of natural transformations.

Given the natural transformations $\tau: \mathcal{S} \Rightarrow \mathcal{T}: \mathbb{A} \to \mathbb{B}$ and $\tau': \mathcal{S}' \Rightarrow \mathcal{T}': \mathbb{B} \to \mathbb{C}$



we may define their 'horizontal' composition $\tau\tau' : \ \mathcal{S};\mathcal{S}' \Rightarrow \mathcal{T};\mathcal{T}'$ by

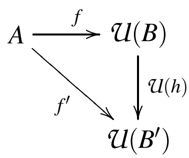
$$A(\tau\tau') = (A\mathcal{S})\tau';(A\tau)\mathcal{T}' = (A\tau)\mathcal{S}';(A\mathcal{T})\tau'.$$

When τ, respectively τ', is an identity natural transformation we may replace it in notation by $\mathcal{S}$, respectively $\mathcal{S}'$.

Basic categorical constructions

The *opposite* $\mathbb{C}^{\mathrm{op}}$ of a category $\mathbb{C}$ is just reversing the arrows and the arrow composition. This means $|\mathbb{C}^{\mathrm{op}}| = |\mathbb{C}|$, $\mathbb{C}^{\mathrm{op}}(A,B) = \mathbb{C}(B,A)$. Identities in $|\mathbb{C}^{\mathrm{op}}|$ are the same as in $\mathbb{C}$.

Given a functor $\mathcal{U} : \ \mathbb{C}' \to \mathbb{C}$, for any object $A \in |\mathbb{C}|$, the *comma category* $A/\mathcal{U}$ has arrows $f : \ A \to \mathcal{U}(B)$ as objects (sometimes denoted as (f,B)) and $h \in \mathbb{C}'(B,B')$ with $f;\mathcal{U}(h) = f'$ as arrows $(f,B) \to (f',B')$.



When $\mathbb{C} = \mathbb{C}'$ and $\mathcal{U}$ is the identity functor, the category $A/\mathcal{U}$ is denoted by $A/\mathbb{C}$. $\mathbb{C}/A$ is just $(A/\mathbb{C}^{\mathrm{op}})^{\mathrm{op}}$.

Given a class $\mathcal{D} \subseteq \mathbb{C}$ of arrows of a category $\mathbb{C}$ we say that $\mathbb{C}$ is *$\mathcal{D}$-well-powered* when for each object $A \in |\mathbb{C}|$ the isomorphism classes of $\{(B,f) \in |\mathbb{C}/A| \mid f \in \mathcal{D}\}$ form a set (rather than a proper class). Dually, $\mathbb{C}$ is *$\mathcal{D}$-co-well-powered* when for each $A \in |\mathbb{C}|$ the isomorphism classes of $\{(f,B) \in |A/\mathbb{C}| \mid f \in \mathcal{D}\}$ form a set.

2.2 Limits and Co-limits

An object 0 is *initial* in a category $\mathbb{C}$ when for each object $A \in |\mathbb{C}|$ there exists a unique arrow in $\mathbb{C}(0,A)$. Dually, an object 1 is *final* in $\mathbb{C}$ when it is initial in $\mathbb{C}^{\mathrm{op}}$, which means that for each object $A \in |\mathbb{C}|$ there exists a unique arrow in $\mathbb{C}(A,1)$.

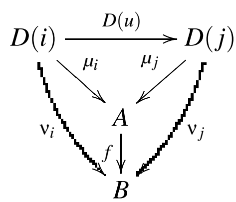
In $\mathbb{S}et$, the empty set $\emptyset$ is initial and each singleton set $\{*\}$ is final. In $\mathbb{G}rp$, the category of groups, the trivial groups (with only one element) are both initial and final.

Given a functor $\mathcal{U} : \ \mathbb{A} \to \mathbb{X}$, for each $X \in |\mathbb{X}|$, a *universal arrow from X to $\mathcal{U}$* is just an initial object in the comma category $X/\mathcal{U}$. Notice that universal arrows are unique up to isomorphism.

For any categories J and $\mathbb{C}$, the *diagonal* functor $\Delta : \ \mathbb{C} \to \mathbb{C}at(J,\mathbb{C})$ maps any $A \in |\mathbb{C}|$ to the functor $A\Delta : \ J \to \mathbb{C}$ such that $(A\Delta)(j) = A$ for each object $j \in |J|$ and

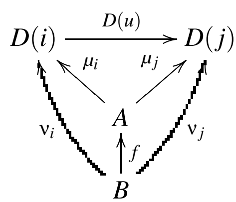
$(A\Delta)(u) = 1_A$ for each arrow $u \in J$, and maps any $f \in \mathbb{C}(A,B)$ to the natural transformation $f\Delta\colon\ A\Delta \Rightarrow B\Delta$ with $(f\Delta)_j = f$ for each $j \in |J|$.

Co-limits. For any functor $D\colon\ J \to \mathbb{C}$, a *co-cone* to D is just an object of the comma category D/Δ, while a *co-limit* of D is a universal arrow from D to the diagonal functor Δ. As universal arrows, co-limits of functors are unique up to isomorphism. A co-limit $\mu\colon\ D \Rightarrow A\Delta$ of D may be therefore denoted as $\mu\colon\ D \Rightarrow A$ (by omitting the diagonal functor from the notation). More explicitly, a co-limit of D consists of a family of arrows $\{\mu_i\}_{i\in|J|}$ such that $\mu_i = D(u);\mu_j$ for each $u \in J(i,j)$ which behaves like a lowest upper bound for D, i.e., for any family $\{\nu_i\}_{i\in|J|}$ such that $\nu_i = D(u);\nu_j$ for each $u \in J(i,j)$, there exists a unique arrow f such that $\mu_i;f = \nu_i$ for each $i \in |J|$.



We may denote the vertex A by $Colim(D)$.

Limits. Limits are dual to co-limits. For any functor $D\colon\ J \to \mathbb{C}$, a *limit* $\mu\colon\ A \Rightarrow D$ of D is the ‘greatest lower bound’ of the *cones* over D, i.e. $\mu = \{\mu_i\}_{i\in|J|}$ such that $\mu_i;D(u) = \mu_j$ for each $u \in J(i,j)$ and for any family $\{\nu_i\}_{i\in|J|}$ with the same property, there exists a unique arrow f such that $f;\mu_i = \nu_i$ for each $i \in |J|$.



We may denote the vertex A by $Lim(D)$.

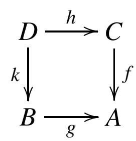
Diagrams as functors. The functors $D\colon\ J \to \mathbb{C}$ for which we have considered limits and co-limits are often called *categorical diagrams* (in $\mathbb{C}$), or just *diagrams* for short. Such a diagram D may be denoted $\left(D(i) \xrightarrow{D(u)} D(j)\right)_{(i \xrightarrow{u} j)\in J}$. Note that the meaning of the functoriality of D, that $D(u;u') = D(u);D(u')$, is the commutativity of D regarded as a diagram in $\mathbb{C}$.

Products and co-products. When J is discrete (has no arrows except the identities), J-limits are called *products* and J-co-limits are called *co-products*; when J is a finite set then the corresponding products or co-products are referred to as finite. The product of two objects A and B is denoted by $A \times B$ and their co-product by $A + B$. Notice that when $J = \emptyset$, then the products are the final objects and the co-products are initial objects. The product of a family $\{A_i\}_{i \in I}$ of objects is denoted by $\prod_{i \in I} A_i$.

In $\mathbb{S}et$ the categorical products are just cartesian products, while co-products $A + B$ are disjoint unions $A \uplus B$ which can be defined as $\{(a,1) \mid a \in A\} \cup \{(b,2) \mid b \in B\}$.

Pullbacks. When J is the category $\bullet \longrightarrow \bullet \longleftarrow \bullet$ with three objects and two non-identity arrows, J-limits are called *pullbacks*.

In $\mathbb{S}et$, the pullback square



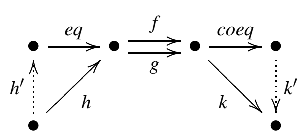
of $C \xrightarrow{f} A \xleftarrow{g} B$ can be defined by $D = \{(b,c) \in B \times C \mid g(b) = f(c)\}$, $k(b,c) = b$, and $h(b,c) = c$.

For any arrow f, the pullback of a span $\bullet \xrightarrow{f} \bullet \xleftarrow{f} \bullet$ is called the *kernel of* f. The kernel of any function $f:\ A \to B$ is $\{(a,a') \in A \times A \mid f(a) = f(a')\}$.

Pushouts. When J is the category $\bullet \longleftarrow \bullet \longrightarrow \bullet$ with three objects and two non-identity arrows, J-co-limits are called *pushouts*.

In $\mathbb{S}et$, the pushout of any *span* of functions $B \xleftarrow{f} A \xrightarrow{g} C$ always exists and is given by the quotient of the disjoint union $B \uplus C$ which identifies all the elements $f(a)$ and $g(a)$ for each $a \in A$.

Equalizers and co-equalizers. When J is the category with two objects and a pair of parallel arrows between these objects, then J-limits are called *equalizers* and J-co-limits are called *co-equalizers*.



In $\mathbb{S}et$, the equalizer of any pair of parallel arrows $f,g:\ A \to B$ is just the subset inclusion $\{a \mid f(a) = g(a)\} \subseteq A$. The co-equalizer k is the quotient of B by the equivalence generated by $\{(f(a),g(a)) \mid a \in A\}$.

Directed co-limits. When J is a directed partially ordered set (i.e., for each $i, i' \in |J|$ there exists $j \in |J|$ such that $i \leq j$ and $i' \leq j$), then J-co-limits are called *directed* co-limits. For the special case when J is a total order, the J-co-limits are called *inductive* co-limits.

In $\mathbb{S}et$, directed co-limits can be thought of as a generalized kind of union. For any directed diagram of sets $(A_i \xrightarrow{f_{i,j}} A_j)_{(i\leq j)\in(J,\leq)}$ its co-limit is given by the quotient of the disjoint union $\uplus\{A_i \mid i \in |J|\}$ which identifies the elements a_i and $f_{i,j}(a_i)$.

A category that has all J-(co-)limits is called *J-(co-)complete*. Also, by *small* (co-)-limits we mean all J-(co-)limits for all J that are small categories.

Theorem 2.1. *In any category the following conditions are equivalent:*

1. *the category has finite (co-)limits,*
2. *the category has finite (co-)products and (co-)equalizers, and*
3. *the category has a final (initial) object and pullbacks (pushouts).*

Lifting, creation, preservation, reflection of (co-)limits

Limits and co-limits, respectively, in base categories determine 'pointwise' limits and co-limits, respectively, in corresponding functor categories.

Proposition 2.2. *If the category $\mathbb{B}$ has J-(co-)limits, then for any category $\mathbb{A}$, the category $\mathbb{C}at(\mathbb{A},\mathbb{B})$ of functors $\mathbb{A} \to \mathbb{B}$ has small J-(co-)limits (which can be calculated separately in $\mathbb{B}$ for each object $A \in |\mathbb{A}|$).*

A functor $\mathcal{U}\colon \mathbb{C} \to \mathbb{C}'$ *preserves* a (co-)limit of a functor $D\colon J \to \mathbb{C}$ when $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$. Note that in $\mathbb{S}et$ the 'product with A', $A \times -$, preserves all co-limits.

The functor $\mathcal{U}$ *lifts (uniquely)* a (co-)limit μ' of $D;\mathcal{U}$ for any functor $D\colon J \to \mathbb{C}$, if there exists a (unique) (co-)limit μ of D such that $\mu\mathcal{U} = \mu'$. Notice that if $\mathcal{U}$ lifts J-(co-)limits and $\mathbb{C}'$ has J-(co-)limits, then $\mathbb{C}$ has J-(co-)limits which are preserved by $\mathcal{U}$.

Stronger than lifting is the following notion. The functor $\mathcal{U}$ *creates* a (co-)limit μ' of $D;\mathcal{U}$, when there exists a unique (co-)cone μ of D such that that $\mu\mathcal{U} = \mu'$ and such that μ is a (co-)limit. For example the forgetful functor $\mathbb{G}rp \to \mathbb{S}et$ creates all limits.

Proposition 2.3. *If the functor $\mathcal{U}\colon \mathbb{C}' \to \mathbb{C}$ preserves J-(co-)limits, then for each object $A \in |\mathbb{C}|$, the forgetful functor $A/\mathcal{U} \to \mathbb{C}'$ creates J-(co-)limits.*

The functor $\mathcal{U}$ *reflects* (co-)limits of a functor $D\colon J \to \mathbb{C}$ if μ is a (co-)limit of D whenever $\mu\mathcal{U}$ is a (co-)limit of $D;\mathcal{U}$.

Co-limits of final functors

A functor $L\colon J' \to J$ is called *final* if for each object $j \in |J|$ the comma category j/L is non-empty and connected. Consequently, a subcategory $J' \subseteq J$ is final when the corresponding inclusion functor is final.

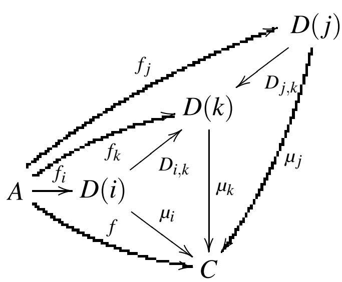
For example, for each natural number n, $(n \to n+1 \to n+2 \to \dots)$ is a final subcategory of $\omega = (0 \to 1 \to 2 \to \dots)$. More generally, for each directed poset $(P, \leq)$ and each $p \in P$, $\{p' \in P \mid p \leq p'\}$ is final in $(P, \leq)$.

Theorem 2.4. *For each final functor $L\colon\ J' \to J$ and each functor $D\colon\ J \to \mathbb{C}$ when a co-limit $\mu'\colon\ L;D \Rightarrow Colim(L;D)$ exists, there exists a co-limit $\mu\colon\ D \Rightarrow Colim(D)$ and the canonical arrow $h\colon\ Colim(L;D) \to Colim(D)$ (given by the universal property of the co-limit of $L;D$) is an isomorphism.*

Finitely presented objects

An object A in a category $\mathbb{C}$ is *finitely presented* if and only if the hom-functor $\mathbb{C}(A,-)\colon \mathbb{C} \to \mathbb{S}et$ preserves directed co-limits. This is equivalent to the following condition:

- for any arrow $f\colon\ A \to C$ to the vertex of a co-limiting co-cone $\mu\colon\ D \Rightarrow C$ of a directed diagram $D\colon\ (J, \leq) \to \mathbb{C}$, there exists $i \in J$ and an arrow $f_i\colon\ A \to D(i)$ such that $f = f_i;\mu_i$, and
- for any two arrows f_i and f_j as above, there exists $k > i, j$ such that $f_i;D_{i,k} = f_j;D_{j,k}$.

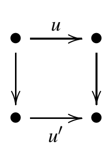


In $\mathbb{S}et$ the finitely presented objects are precisely the finite sets. In the category of groups $\mathbb{G}rp$, the finitely presented groups are exactly the quotients of finitely generated groups by finitely generated congruences.

A category is *locally presentable* when each object is a directed co-limit of finitely presented objects. $\mathbb{S}et$ is locally presentable because each set is the (directed) co-limit of its finite subsets.

Stability under pushouts/pullbacks

A class of arrows $\mathcal{S} \subseteq \mathbb{C}$ in a category $\mathbb{C}$ is *stable under pushouts* if for any pushout square in $\mathbb{C}$



$u' \in \mathcal{S}$ whenever $u \in \mathcal{S}$. *Stability under pullbacks* in $\mathbb{C}$ is stability under pushouts in $\mathbb{C}^{\mathrm{op}}$.

In general, epis are stable under pushouts and monos under pullbacks. In $\mathbb{S}et$, monos (injective functions) are stable under pushouts too. Injective functions $f: A \to B$ such that $B \setminus f(A)$ is finite are also stable under pushouts.

Weak limits and co-limits

These are weaker variants of the concepts of limits and co-limits, respectively, obtained by dropping the uniqueness requirement from the universal property of the limits and co-limits, respectively. For example, in $\mathbb{S}et$ for any two sets A and B, any super-set C of their disjoint union, i.e., $A \uplus B \subseteq C$, is a weak co-product for A and B. Obviously, weak limits and co-limits, respectively, are no longer unique up to isomorphism.

2.3 Adjunctions

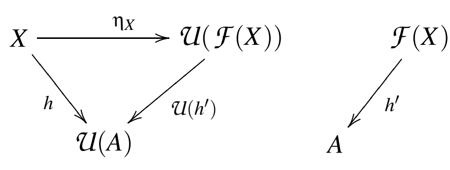
Adjoint functors are a core concept of category theory. Mathematical practice abounds with examples of adjoint functors.

Proposition 2.5. *For any functor $\mathcal{U}: \mathbb{A} \to \mathbb{X}$ the following conditions are equivalent:*

1. *For each object $X \in \mathbb{X}$ there exists a universal arrow from X to $\mathcal{U}$.*
2. *There exists a functor $\mathcal{F}: \mathbb{X} \to \mathbb{A}$ and a bijection $\varphi_{X,A}: \mathbb{A}(\mathcal{F}(X),A) \to \mathbb{X}(X, \mathcal{U}(A))$ indexed by $|\mathbb{X}| \times |\mathbb{A}|$ and natural in X and A.*
3. *There exists a functor $\mathcal{F}: \mathbb{X} \to \mathbb{A}$ and natural transformations $\eta: 1_{\mathbb{X}} \Rightarrow \mathcal{F};\mathcal{U}$ (called the* unit*) and $\varepsilon: \mathcal{U};\mathcal{F} \Rightarrow 1_{\mathbb{A}}$ (called the* co-unit*) such that the following* triangular equations *hold: $\eta\mathcal{F};\mathcal{F}\varepsilon = 1_{\mathcal{F}}$ and $\mathcal{U}\eta;\varepsilon\mathcal{U} = 1_{\mathcal{U}}$.*

If the conditions above hold, then $\mathcal{U}$ is called a *right adjoint*, and the functor $\mathcal{F}$ is called a *left adjoint* to $\mathcal{U}$. The tuple $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$ is called an *adjunction* from (the category) $\mathbb{X}$ to (the category) $\mathbb{A}$.

Very often the notion of adjunction is used in the following "freeness" form. Given an adjunction $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$, for any object $X \in |\mathbb{X}|$ there exists an object $\mathcal{F}(X)$, called $\mathcal{U}$-*free over* A and an arrow $\eta_X: X \to \mathcal{U}(\mathcal{F}(X))$ such that for each object $A \in |\mathbb{A}|$ and arrow $h: X \to \mathcal{U}(A)$, there exists a unique arrow $h': \mathcal{F}(X) \to A$ such that $h = \eta_X;\mathcal{U}(h')$.



When a category $\mathbb{C}$ has J-(co-)limits, then these are adjoints to the diagonal functor $\Delta: \mathbb{C} \to \mathbb{C}at(J,\mathbb{C})$. More precisely, *Lim* is a right adjoint to Δ, while *Colim* is a left adjoint to Δ.

The forgetful functor $\mathbb{G}rp \to \mathbb{S}et$ is right adjoint, its left adjoint constructing the groups freely generated by sets.

Galois connections. Let $(P,\leq)$ and $(Q,\leq)$ be preorders. Two preorder preserving functions $L\colon\ (P,\leq) \to (Q,\leq)^{\mathrm{op}}$ and $R\colon\ (Q,\leq)^{\mathrm{op}} \to (P,\leq)$ constitute an adjunction when $L(p) \geq q$ if and only if $p \leq R(q)$ for all $p \in P$ and $q \in Q$. Notice that triangular equations mean $L(p) \geq L(R(L(p))) \geq L(p)$ and $R(q) \leq R(L(R(q))) \leq R(q)$. The pair (L,R) is called a *Galois connection* between $(P,\leq)$ and $(Q,\leq)$.

Persistent adjunctions. Given an adjunction $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$, the object $\mathcal{F}(X)$ is called *persistently* $\mathcal{U}$-free when the unit component η_X is an isomorphism, and is called *strongly persistently* $\mathcal{U}$-free when η_X is identity. We can easily see that an object of $\mathbb{A}$ is persistently free if and only if it is strongly persistently free. An adjunction such that for each object X of $\mathbb{X}$, $\mathcal{F}(X)$ is [strongly] persistently $\mathcal{U}$-free, is called a *[strongly] persistent adjunction* .

Composition of adjunctions. Given two adjunctions $(\mathcal{U},\mathcal{F},\eta,\varepsilon)$ from $\mathbb{X}$ to $\mathbb{A}$, and $(\mathcal{U}',\mathcal{F}',\eta',\varepsilon')$ from $\mathbb{A}$ to $\mathbb{A}'$, note that $(\mathcal{U}';\mathcal{U},\mathcal{F};\mathcal{F}',\eta;\mathcal{F}\eta'\mathcal{U},\mathcal{U}'\varepsilon\mathcal{F}';\varepsilon')$ is an adjunction from $\mathbb{X}$ to $\mathbb{A}'$. This is called the *composition* of the two adjunctions. Adjunctions thus form a 'quasi-category' $\mathbb{A}dj$ with categories as objects and adjunctions as arrows.

The following is one of the most useful properties of adjoint functors.

Proposition 2.6. *Right adjoints preserve all limits and, dually, left adjoints preserve all co-limits.*

Special adjunctions

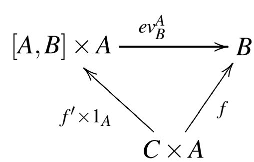
Categorical equivalences. The following equivalent conditions define a functor $\mathcal{U}\colon\ \mathbb{X} \to \mathbb{X}'$ as an *equivalence of categories*:

Proposition 2.7. *For any functor $\mathcal{U}\colon\ \mathbb{X} \to \mathbb{X}'$ the following conditions are equivalent:*

- *$\mathcal{U}$ belongs to an adjunction with unit and co-unit being natural isomorphisms, and*
- *$\mathcal{U}$ is full and faithful and each object $A' \in |\mathbb{X}'|$ is isomorphic to $\mathcal{U}(A)$ for some object $A \in |\mathbb{X}|$.*

We say that $\mathbb{X}$ is a *skeleton* of $\mathbb{X}'$ when all isomorphisms in $\mathbb{X}$ are identities.

Cartesian closed categories. A category $\mathbb{C}$ is *cartesian closed* when it has all finite products, denoted $-\times-$, and for each object A the product functor $-\times A\colon\ \mathbb{C} \to \mathbb{C}$ has a right adjoint $[A,-]$. If we denote the co-unit of this adjunction by ev^A, it means that for each pair of objects A and B, and for each arrow $f\colon\ C \times A \to B$, there exists a unique arrow $f'\colon\ C \to [A,B]$ such that $f = (f' \times 1_A);ev^A_B$,

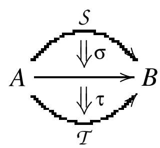


In examples the co-unit components ev_B^A play the role of 'evaluation maps'. We have that $\mathbb{S}et$ is cartesian closed where $[A,B]$ is the set of all functions $A \to B$, and $ev_B^A(f,a) = f(a)$. $\mathbb{C}at$ is also cartesian closed with $[A,B]$ being the category $\mathbb{C}at(A,B)$ of the functors $A \to B$ and with the natural transformations between them as arrows.

2.4 2-categories

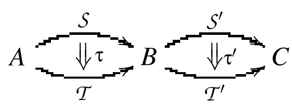
A *2-category* $\mathbb{C}$ is an ordinary category whose objects are called *0-cells*, whose arrows are called *1-cells*, and in addition to ordinary objects and arrows, for each pair of 1-cells $\mathcal{S}, \mathcal{T}$ there is a set $\mathbb{C}(\mathcal{S}, \mathcal{T})$ of *2-cells* (denoted by $\mathcal{S} \Rightarrow \mathcal{T}$) together with two compositions for the 2-cells:

- a 'vertical' one $\sigma;\tau : \ \mathcal{S} \Rightarrow \mathcal{T}$

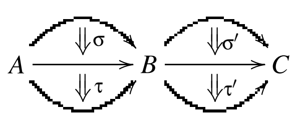


and

- a 'horizontal' one (denoted by simple juxtaposition) $\tau\tau' : \ \mathcal{S};\mathcal{S}' \Rightarrow \mathcal{T};\mathcal{T}'$



such that every identity arrow for the first composite is also an identity for the second composition, $1_{\mathcal{S};\mathcal{T}} = 1_{\mathcal{S}} 1_{\mathcal{T}}$ for all composable 1-cells $\mathcal{S}$ and $\mathcal{T}$, and such that the following *Interchange Law* holds: given three categories and four natural transformations



the 'vertical' compositions and the 'horizontal' compositions are related by

$$(\sigma;\tau)(\sigma';\tau') = (\sigma\sigma');(\tau\tau').$$

Evidently any category is trivially a 2-category without proper 2-cells. The typical non-trivial example of a 2-category is $\mathbb{C}at$ with categories as 0-cells, functors as 1-cells, and natural transformations as 2-cells.

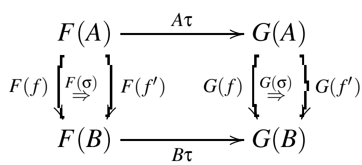
Adjunctions, natural transformations, (co-)limits

The concept of adjunction can be defined abstractly in any 2-category: $(\mathcal{U}, \mathcal{F}, \eta, \varepsilon)$ is an *adjunction* if $\mathcal{U} : A \to X$ and $\mathcal{F} : X \to A$ are 1-cells, $\eta : 1_X \Rightarrow \mathcal{F};\mathcal{U}$ and $\varepsilon : \mathcal{U};\mathcal{F} \Rightarrow 1_{\mathbb{A}}$ are 2-cells such that the *triangular equations* are satisfied:

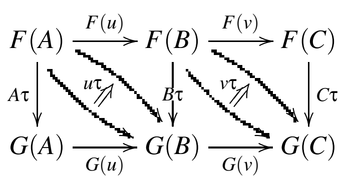
$$\eta\mathcal{F};\mathcal{F}\varepsilon = 1_{\mathcal{F}} \text{ and } \mathcal{U}\eta;\varepsilon\mathcal{U} = 1_{\mathcal{U}}.$$

The proper mappings between 2-categories are 2-functors. A *2-functor* $F : \mathbb{C} \to \mathbb{C}'$ between 2-categories $\mathbb{C}$ and $\mathbb{C}'$ maps 0-cells to 0-cells, 1-cells to 1-cells, and 2-cells to 2-cells, such that $F(\mathcal{S}) : F(A) \to F(B)$ for any 1-cell $\mathcal{S} : A \to B$, and $F(\sigma) : F(\mathcal{S}) \Rightarrow F(\mathcal{T})$ for any 2-cell $\sigma : \mathcal{S} \Rightarrow \mathcal{T}$, and such that it preserves both the 'vertical' and the 'horizontal' compositions.

A *2-natural transformation* $\tau : F \Rightarrow G$ between 2-functors $F, G : \mathbb{A} \to \mathbb{B}$ maps any object A of $|\mathbb{A}|$ to a 1-cell $A\tau : F(A) \to G(A)$ such that $(A\tau)G(\sigma) = F(\sigma)(B\tau)$ for each 2-cell $\sigma : f \Rightarrow f' : A \to B$.



Lax natural transformations relax the commutativity of the natural transformation square above to the existence of 2-cells. Therefore a lax natural transformation τ between 2-functors F and G maps any object $A \in |\mathbb{A}|$ to $A\tau : F(A) \to G(A)$ and any 1-cell $u : A \to B$ to $u\tau : A\tau;G(u) \Rightarrow F(u);B\tau$ such that $(F(\sigma)(B\tau));f'\tau = f\tau;((A\tau)G(\sigma))$ for each 2-cell $\sigma : f \Rightarrow f' : A \to B$ and



$(u;v)\tau = (u\tau)(G(v));F(u)(v\tau)$ for each $u : A \to B$ and $v : B \to C$.

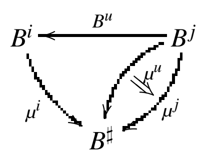
2-categorical limits and co-limits can be defined similarly to the conventional limits and co-limits as universal arrows from/to a diagonal functor. However, in the 2-categorical framework, different concepts of natural transformations determine different concepts of (co-)limits. Therefore, when we employ 2-natural transformations we get the concepts of *2-(co-)limit* as a final (initial) *2-(co-)cone*, and when we employ lax natural transformations we get the concepts of *lax (co-)limit* as a final/initial *lax cone/co-cone*.

2.5 Indexed Categories and Fibrations

An *indexed category* is a functor $B: I^{\mathrm{op}} \to \mathbb{C}at$; sometimes we denote $B(i)$ as B_i (or B^i) for an index $i \in |I|$ and $B(u)$ as B^u for an index morphism $u \in I$. Given an indexed category $B: I^{\mathrm{op}} \to \mathbb{C}at$, let $B^\sharp$ be the *Grothendieck category* having $\langle i, \Sigma \rangle$, with $i \in |I|$ and $\Sigma \in |B^i|$, as objects and $\langle u, \varphi \rangle: \langle i, \Sigma \rangle \to \langle i', \Sigma' \rangle$, with $u \in I(i, i')$ and $\varphi: \Sigma \to B^u(\Sigma')$, as arrows. The composition of arrows in $B^\sharp$ is defined by $\langle u, \varphi \rangle; \langle u', \varphi' \rangle = \langle u; u', \varphi; (B^u(\varphi')) \rangle$.

Proposition 2.8. *The Grothendieck category $B^\sharp$ of an indexed category $B: I^{\mathrm{op}} \to \mathbb{C}at$ is the vertex of the lax co-limit $\mu: B \rightsquigarrow B^\sharp$ of B in $\mathbb{C}at$, where*

- *for each index $i \in |I|$, $\mu^i: B^i \to B^\sharp$ is the canonical inclusion of categories, and*
- *for each index morphism $u \in I(i, j)$, $\mu^u: B^u; \mu^i \Rightarrow \mu^j$ is defined by $\mu^u_b = \langle u, 1_{B^u(b)} \rangle$ for each object $b \in |B^j|$.*



Grothendieck constructions in 2-categories. Prop. 2.8 allows us to internalize the concept of Grothendieck construction to any 2-category. Given a (1-)functor $B: I^{\mathrm{op}} \to V$, where V is an arbitrary 2-category, a *Grothendieck construction* for B is a lax co-limit $\mu: B \rightsquigarrow B^\sharp$. Then the vertex $B^\sharp$ is called the *Grothendieck object* associated to B. We say that a 2-category V *admits Grothendieck constructions* when each (1-)functor $B: I^{\mathrm{op}} \to V$ has a lax co-limit.

Notice also that any 2-functor $B: I^* \to \mathbb{C}at$, where I^* is the *2-dimensional opposite* changing the direction of 2-cells both horizontally and vertically, induces a canonical 2-category structure on the Grothendieck category $B^\sharp$ of the (1)-functor $B: I^{\mathrm{op}} \to \mathbb{C}at$.

Fibrations

Given a functor $p: \mathbb{B} \to I$, an object/arrow $f \in \mathbb{B}$ is said to be *above* an object/arrow $u \in I$ when $p(f) = u$. An arrow above an identity is called *vertical*. Every object $i \in |I|$ determines a *fibre* category $\mathbb{B}_i$ consisting of objects above i and vertical morphisms above 1_i. An arrow $f \in \mathbb{B}(A, C)$ is called *cartesian* over an arrow $u \in I$ when f is above u and every $f' \in \mathbb{B}(A', C)$ with $p(f') = v; u$ uniquely determines a $g \in \mathbb{B}(A', A)$ above v such that $f' = g; f$. p is called a *fibred category* or *fibration* when for every $A \in |\mathbb{B}|$ and $u \in I(i, p(A))$ there is a cartesian arrow (called *cartesian lifting* or *critical lifts* in [1]) with codomain A above u.

Each indexed category $B: I^{\mathrm{op}} \to \mathbb{C}at$ naturally determines a fibration $p: B^\sharp \to I$ as the index projection, i.e., $p(\langle i, \Sigma \rangle) = i$, such that for each index i, the fibre $B^\sharp_i$ is B^i and $\langle u, \varphi \rangle \in B^\sharp$ is cartesian over u when φ is isomorphism. Notice that for each index