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C*-algebras and Elliptic Theory II

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Editors' Introduction

The conference "C^*-algebras and elliptic theory, II" was held at the Stefan Banach International Mathematical Center in Będlewo, Poland, in January 2006, one of a series of meetings in Poland and Russia. This volume is a collection of original and refereed research and expository papers related to the meeting. Although centered on the K-theory of operator algebras, a broad range of topics is covered including geometric, L^2- and spectral invariants, such as the analytic torsion, signature and index, of differential and pseudo-differential operators on spaces which are possibly singular, foliated or non-commutative. This material should be of interest to researchers in Mathematical Physics, Differential Topology and Analysis.

The series of conferences including this one originated with an idea of Professor Bogdan Bojarski, namely, to strengthen collaboration between mathematicians from Poland and Russia on the basis of common scientific interests, particularly in the field of Non-commutative Geometry. This led to the first meeting, in 2004, which brought together about 60 mathematicians not only from Russia and Poland, but from other leading centers. It was supported by the European program "Geometric Analysis Research Training Network". Since then there have been annual meetings alternating between Będlewo and Moscow. The second conference was organized in Moscow in 2005 and was dedicated to the memory of Yu.P. Solovyov. The proceedings will appear in the *Journal of K-Theory.* The conference on which this volume is based was the third conference in the overall series with the fourth being held in Moscow in 2007. A further meeting in Będlewo is planned for 2009.

D. Burghelea, R.B. Melrose, A. Mishchenko, E. Troitsky

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Pseudo-differential operators

In two papers *"Dual manifolds and pseudo-differential operators"* and *"Homotopy classification and K-homology"* **V. Nazaykinskiy, A. Savin and B. Sternin** examine index questions and the homotopy classification of pseudo-differential operators on manifolds with corners.

The paper *"Dixmier traceability for general pseudo-differential operators"* by **F. Nicola and L. Rodino** generalizes previous results about the finiteness of the Dixmier trace of pseudo-differential operators.

In *"Boundaries, Eta invariant and the determinant bundle"*, **R. Melrose and F. Rochon** show that the exponentiated η invariant gives a section of the determinant bundle over the boundary for cusp pseudo-differential operators, generalizing a theorem of Dai and Freed in the Dirac setting.

K-theory

The paper *"K-theory of twisted group algebras"* by **S. Echterhoff** presents applications of the Baum-Connes conjecture to the study of the K-theory of twisted group algebras.

A geometric formulation of the description of the dual of a finite group is extended to discrete infinite groups in the paper *"Twisted Burnside theorem for two-step torsion-free nilpotent groups"* by **A. Felshtyn, F. Indukaev and E. Troitsky**.

The paper *"Group bundle duality, invariants for certain C^*-algebras, and twisted equivariant K-theory"* by **E. Vasselli** describes a general duality for Lie group bundles and its the relation with twisted K-theory.

In the paper *"Topological invariants of bifurcation"*, **J. Pejsachowicz** uses the J-functor in K-theory to describe bifurcation for some nonlinear Fredholm operator families.

Torsion and determinants

"Torsion, as a function on the space of representations" is a survey by **D. Burghelea and S. Haller** of their results on three complex-valued invariants of a smooth closed manifold arising from combinatorial topology, from regularized determinants and from the counting instantons and closed trajectories.

The Ihara zeta function for infinite periodic simple graphs, involving a "determinant" in the setting of von Neumann linear algebra, is defined and studied in the paper *"Ihara zeta function for periodic simple graphs"* by **D. Guido, T. Isola and M. Lapidus**.

Operator algebras

Ch. Wahl, in *"A new topology on the space of unbounded selfadjoint operators and the spectral flow"*, revisits the relationship between the space of Fredholm operators and the classical K^1 and K^0 functors.

In the paper *"L^2-invariants and rank metric"*, **A. Thom** gives results about L^2-Betti numbers for tracial algebras.

A positive answer to a conjecture on non-commutative spheres, is provided by **U. Krähmer** in *"On the non-standard Podleś spheres"*.

The paper *"Modified Hochschild and periodic cyclic homology"* by **N. Teleman** proposes a modification in the definition of these two homologies to better relate them to the Alexander-Spanier homology.

Foliated manifolds

Lefschetz theory associated to a "transverse" action of a Lie group on a foliated manifold is examined in the paper *"Lefschetz distribution of Lie foliation"* by **J. Alvarez Lopez and Yu. Kordyukov**.

The paper *"Adiabatic limits and the spectrum of the Laplacian on foliated manifolds"* by **Yu. Kordyukov and A. Yakovlev** presents results on the spectrum of the Laplacian on differential forms as the Riemannian metric is expanded normal to the leaves.

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Lefschetz Distribution of Lie Foliations

Jesús A. Álvarez López and Yuri A. Kordyukov

Abstract. Let $\mathcal{F}$ be a Lie foliation on a closed manifold M with structural Lie group G. Its transverse Lie structure can be considered as a transverse action Φ of G on $(M, \mathcal{F})$; *i.e.*, an "action" which is defined up to leafwise homotopies. This Φ induces an action Φ^* of G on the reduced leafwise cohomology $\overline{H}(\mathcal{F})$. By using leafwise Hodge theory, the supertrace of Φ^* can be defined as a distribution $L_{\text{dis}}(\mathcal{F})$ on G called the Lefschetz distribution of $\mathcal{F}$. A distributional version of the Gauss-Bonett theorem is proved, which describes $L_{\text{dis}}(\mathcal{F})$ around the identity element. On any small enough open subset of G, $L_{\text{dis}}(\mathcal{F})$ is described by a distributional version of the Lefschetz trace formula.

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Keywords. Lie foliation, Riemannian foliation, leafwise reduced cohomology, distributional trace, Lefschetz distribution, Λ-Euler characteristic, Λ-Lefschetz number, Lefschetz trace formula.

Contents

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1. Introduction

Let $\mathcal{F}$ be a C^∞ foliation on a manifold M. Let $\operatorname{Diff}(M,\mathcal{F})$ be the group of foliated diffeomorphisms $(M,\mathcal{F}) \to (M,\mathcal{F})$. The elements of $\operatorname{Diff}(M,\mathcal{F})$ that are C^∞ leafwisely homotopic to id_M form a normal subgroup $\operatorname{Diff}_0(\mathcal{F})$, and let $\overline{\operatorname{Diff}}(M,\mathcal{F})$ denote the corresponding quotient group. A *right transverse action* of a group G on $(M,\mathcal{F})$ is an anti-homomorphism $\Phi : G \to \overline{\operatorname{Diff}}(M,\mathcal{F})$. A *local representation* of Φ on some open subset $O \subset G$ is a map $\phi : M \times O \to M$ such that $\phi_g = \phi(\cdot, g)$ is a foliated diffeomorphism representing Φ_g for all $g \in G$. Then Φ is said to be of *class* C^∞ if it has a C^∞ local representation on each small enough open subset of G.

Recall that the leafwise de Rham complex $(\Omega(\mathcal{F}), d_{\mathcal{F}})$ consists of the differential forms on the leaves which are C^∞ on M, endowed with the de Rham derivative of the leaves. Its cohomology $H(\mathcal{F})$ is called the leafwise cohomology. This becomes a topological vector space with the topology induced by the C^∞ topology, and its maximal Hausdorff quotient is the reduced leafwise cohomology $\overline{H}(\mathcal{F})$.

Consider the canonical right action of $\operatorname{Diff}(M,\mathcal{F})$ on $\overline{H}(\mathcal{F})$ defined by pulling-back leafwise differential forms. Since $\operatorname{Diff}_0(\mathcal{F})$ acts trivially, we get a canonical right action of $\overline{\operatorname{Diff}}(M,\mathcal{F})$ on $\overline{H}(\mathcal{F})$. Then any right transverse action Φ of a group G on $(M,\mathcal{F})$ induces a left action Φ^* of G on $\overline{H}(\mathcal{F})$.

Suppose from now on that $\mathcal{F}$ is a Lie foliation and the manifold M is closed. It is shown that its transverse Lie structure can be described as a right transverse action Φ of its structural Lie group G on $(M,\mathcal{F})$. Consider the induced left action Φ^* of G on $\overline{H}(\mathcal{F})$. For each $g \in G$, we would like to define the supertrace $\operatorname{Tr}^s \Phi_g^*$, which could be called the *leafwise Lefschetz number* $L(\Phi_g)$ of Φ_g. This can be achieved when $\overline{H}(\mathcal{F})$ is of finite dimension, obtaining a C^∞ function $L(\mathcal{F})$ on G defined by $L(\mathcal{F})(g) = L(\Phi_g)$; the value of $L(\mathcal{F})$ at the identity element e of G is the Euler characteristic $\chi(\mathcal{F})$ of $\overline{H}(\mathcal{F})$, which can be called the *leafwise Euler* *characteristic* of $\mathcal{F}$. But $\overline{H}(\mathcal{F})$ may be of infinite dimension, even when the leaves are dense [1], and thus $L(\mathcal{F})$ is not defined in general.

The first goal of this paper is to show that, in general, the role of the function $L(\mathcal{F})$ can be played by a distribution $L_{\mathrm{dis}}(\mathcal{F})$ on G, called the *Lefschetz distribution* of $\mathcal{F}$, whose singularities are motivated by the infinite dimension of $\overline{H}(\mathcal{F})$.

The first ingredient to define $L_{\mathrm{dis}}(\mathcal{F})$ is the leafwise Hodge theory studied in [2] for Riemannian foliations; recall that Lie foliations form a specially important class of Riemannian foliations [19]. Fix a bundle-like metric on M whose transverse part is induced by a left invariant Riemannian metric on G. For the induced Riemannian structure on the leaves, let $\Delta_{\mathcal{F}}$ be the Laplacian of the leaves operating in $\Omega(\mathcal{F})$. The kernel $\mathcal{H}(\mathcal{F})$ of Δ_F is the space of harmonic forms on the leaves that are C^∞ on M. The metric induces an L^2 inner product on $\Omega(\mathcal{F})$, obtaining a Hilbert space $\mathbf{\Omega}(\mathcal{F})$. Then $\Delta_{\mathcal{F}}$ is an essentially self-adjoint operator in $\mathbf{\Omega}(\mathcal{F})$ whose closure is denoted by $\mathbf{\Delta}_{\mathcal{F}}$. The kernel of $\mathbf{\Delta}_{\mathcal{F}}$ is denoted by $\boldsymbol{\mathcal{H}}(\mathcal{F})$, and let $\mathbf{\Pi} : \mathbf{\Omega}(\mathcal{F}) \to \boldsymbol{\mathcal{H}}(\mathcal{F})$ denote the orthogonal projection. In [2], it is proved that $\mathbf{\Pi}$ has a restriction $\Pi : \Omega(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$ that induces an isomorphism $\overline{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F})$, which can be called the *leafwise Hodge isomorphism*.

Let Λ be the volume form of G, and let $\phi : M \times O \to M$ be a C^∞ local representation of Φ. For each $f \in C^\infty_c(O)$, consider the operator

$$P_f = \int_G \phi_g^* \cdot f(g)\,\Lambda(g) \circ \Pi$$

in $\Omega(\mathcal{F})$. Our first main result is the following.

Proposition 1.1. *P_f is of trace class, and the functional $f \mapsto \mathrm{Tr}^s P_f$ defines a distribution on O.*

It can be easily seen that $\mathrm{Tr}^s P_f$ is independent of the choice of ϕ, and thus the distributions given by Proposition 1.1 can be combined to define a distribution $L_{\mathrm{dis}}(\mathcal{F})$ on G; this is the *Lefschetz distribution* of $\mathcal{F}$.

Observe that $L_{\mathrm{dis}}(\mathcal{F}) \equiv L(\mathcal{F}) \cdot \Lambda$ when $\overline{H}(\mathcal{F})$ is of finite dimension. This justifies the consideration of $L_{\mathrm{dis}}(\mathcal{F})$ as a generalization of $L(\mathcal{F})$; in particular, the germ of $L_{\mathrm{dis}}(\mathcal{F})$ at e generalizes $\chi(\mathcal{F})$.

If the operators P_f are restricted to $\Omega^i(\mathcal{F})$ for each degree i, its trace defines a distribution $\mathrm{Tr}^i_{\mathrm{dis}}(\mathcal{F})$, called *distributional trace*, whose germ at e generalizes the *leafwise Betti number* $\beta^i(\mathcal{F}) = \dim \overline{H}^i(\mathcal{F})$.

The distributions $L_{\mathrm{dis}}(\mathcal{F})$ and $\mathrm{Tr}^i_{\mathrm{dis}}(\mathcal{F})$ depend on Λ and $\mathcal{F}$, endowed with the transverse Lie structure. If the leaves are dense, then the transverse Lie structure is determined by the foliation, and thus these distributions depend only on Λ and the foliation. On the other hand, the dependence on Λ can be avoided by using top-dimensional currents instead of distributions, in the obvious way.

Our second goal is to prove a distributional version of the Gauss-Bonett theorem, which describes $L_{\mathrm{dis}}(\mathcal{F})$ around e. Let $R_{\mathcal{F}}$ be the curvature of the leafwise metric. Suppose for simplicity that $\mathcal{F}$ is oriented. Then $\mathrm{Pf}(R_{\mathcal{F}}/2\pi) \in \Omega^p(\mathcal{F})$ ($p = \dim \mathcal{F}$) can be called the *leafwise Euler form*. This form can be paired with Λ, considered as a transverse invariant measure, to give a differential form $\omega_\Lambda \wedge \mathrm{Pf}(R_{\mathcal{F}}/2\pi)$ of top degree on M. In particular, if $\dim\mathcal{F} = 2$, then

$$\omega_\Lambda \wedge \mathrm{Pf}(R_{\mathcal{F}}/2\pi) = \frac{1}{2\pi} K_{\mathcal{F}}\, \omega_M ,$$

where $K_{\mathcal{F}}$ is the Gauss curvature of the leaves and ω_M is the volume form of M. Let δ_e denote the Dirac measure at e.

Theorem 1.2 (Distributional Gauss-Bonett theorem). *We have*

$$L_{\mathrm{dis}}(\mathcal{F}) = \int_M \omega_\Lambda \wedge \mathrm{Pf}(R_{\mathcal{F}}/2\pi) \cdot \delta_e$$

on some neighborhood of e.

To prove Theorem 1.2, we really prove that

$$L_{\mathrm{dis}}(\mathcal{F}) = \chi_\Lambda(\mathcal{F}) \cdot \delta_e \tag{1.1}$$

around e, where Λ is considered as a transverse invariant measure of $\mathcal{F}$, and $\chi_\Lambda(\mathcal{F})$ is the Λ-Euler characteristic of $\mathcal{F}$ introduced by Connes [9]. Then Theorem 1.2 follows from the index theorem of [9].

The third goal is to prove a distributional version of the Lefschetz trace formula, which describes $L_{\mathrm{dis}}(\mathcal{F})$ on any small enough open subset of G. For a C^∞ local representation $\phi : M \times O \to M$ of Φ, let $\phi' : M \times O \to M \times O$ be the map defined by $\phi'(x, g) = (\phi_g(x), g)$. The fixed point set of ϕ', $\mathrm{Fix}(\phi')$, consists of the points (x, g) such that $\phi_g(x) = x$. A point $(x, g) \in \mathrm{Fix}(\phi')$ is said to be *leafwise simple* when $\phi_{g*} - \mathrm{id} : T_x\mathcal{F} \to T_x\mathcal{F}$ is an isomorphism; in this case, the sign of the determinant of this isomorphism is denoted by $\epsilon(x, g)$. The set of leafwise simple fixed points of ϕ' is denoted by $\mathrm{Fix}_0(\phi')$. Let $\mathrm{pr}_1 : M \times O \to M$ and $\mathrm{pr}_2 : M \times O \to O$ be the factor projections. It is proved that $\mathrm{Fix}_0(\phi')$ is a C^∞ manifold of dimension equal to $\operatorname{codim}\mathcal{F}$. Moreover the restriction $\mathrm{pr}_1 : \mathrm{Fix}_0(\phi') \to M$ is a local embedding transverse to $\mathcal{F}$. So Λ defines a measure $\Lambda'_{\mathrm{Fix}_0(\phi')}$ on $\mathrm{Fix}_0(\phi')$. Observe that $\mathrm{pr}_2 : \mathrm{Fix}(\phi') \to O$ is a proper map.

Theorem 1.3 (Distributional Lefschetz trace formula). *Suppose that every fixed point of* ϕ' *is leafwise simple. Then*

$$L_{\mathrm{dis}}(\mathcal{F}) = \mathrm{pr}_{2*}(\epsilon \cdot \Lambda'_{\mathrm{Fix}(\phi')})$$

on O.

To prove Theorem 1.3, we consider certain submanifold $M'_1 \subset M \times O$ endowed with a foliation $\mathcal{F}'_1$, whose leaves are of the form $L \times \{g\}$, where L is a leaf of $\mathcal{F}$ and $g \in G$. It is proved that $\mathrm{pr}_2(M'_1)$ is open in some orbit of the adjoint action of G on itself, $\mathrm{pr}_1 : M'_1 \to M$ is a local diffeomorphism, and $\mathcal{F}'_1 = \mathrm{pr}_1^* \mathcal{F}$. So Λ lifts to a transverse invariant measure Λ'_1 of $\mathcal{F}'_1$. Moreover the restriction ϕ'_1 of ϕ' to M'_1 is defined and maps each leaf of $\mathcal{F}'_1$ to itself. For each $f \in C^\infty_c(O)$ supported in an appropriate open subset $O_1 \subset O$, the transverse invariant measure $\Lambda'_{1,f} = \mathrm{pr}_2^* f \cdot \Lambda'_1$ is compactly supported. Then the $\Lambda'_{1,f}$-Lefschetz number $L_{\Lambda'_{1,f}}(\phi'_1)$ is defined according to [14]. Without assuming any condition on the fixed point set, we show that

$$\langle L_{\mathrm{dis}}(\mathcal{F}), f\rangle = L_{\Lambda'_{1,f}}(\phi'_1)\ . \tag{1.2}$$

We have that $\mathrm{Fix}(\phi'_1)$ is a C^∞ local transversal of $\mathcal{F}'_1$. Hence Theorem 1.3 follows from (1.2) and the foliation Lefschetz theorem of [14, 24].

The numbers $\chi_\Lambda(\mathcal{F})$ and $L_{\Lambda'_{1,f}}(\phi'_1)$ are defined by using L^2 differential forms on the leaves, whilst $L_{\mathrm{dis}}(\mathcal{F})$ is defined by using leafwise differential forms that are C^∞ on M. These are sharply different conditions when the leaves are not compact. So (1.1) and (1.2) are surprising relations.

By (1.2), $L_{\mathrm{dis}}(\mathcal{F})$ is supported in the union of a discrete set of orbits of the adjoint action. Therefore, when $\operatorname{codim}\mathcal{F} > 0$, $L_{\mathrm{dis}}(\mathcal{F})$ is C^∞ just when it is trivial, obtaining the following.

Corollary 1.4. *If $\overline{H}(\mathcal{F})$ is of finite dimension and* $\operatorname{codim}\mathcal{F} > 0$*, then* $L_{\mathrm{dis}}(\mathcal{F}) \equiv L(\mathcal{F}) = 0$.

By Corollary 1.4, $\chi(\mathcal{F})$ is useless: it vanishes just when it can be defined. Moreover $\chi_\Lambda(\mathcal{F}) = 0$ in this case by (1.1). So, when $\operatorname{codim}\mathcal{F} > 0$, the condition $\chi_\Lambda(\mathcal{F}) \neq 0$ yields $\dim\overline{H}(\mathcal{F}) = \infty$. More precise results of this type would be desirable.

Let $\dim\mathcal{F} = p$. When the leaves are dense, $\beta^0(\mathcal{F})$ and $\beta^p(\mathcal{F})$ are finite, and thus $\mathrm{Tr}^0_{\mathrm{dis}}(\mathcal{F})$ and $\mathrm{Tr}^p_{\mathrm{dis}}(\mathcal{F})$ are C^∞. On the other hand, when the leaves are not compact, the Λ-Betti numbers of [9] satisfy $\beta^0_\Lambda(\mathcal{F}) = \beta^p_\Lambda(\mathcal{F}) = 0$. Then the following result follows from (1.1) and Corollary 1.4.

Corollary 1.5. *If* $\operatorname{codim}\mathcal{F} > 0$, $\dim\mathcal{F} = 2$ *and the leaves are dense, then* $\mathrm{Tr}^1_{\mathrm{dis}}(\mathcal{F}) - \beta^1_\Lambda(\mathcal{F})\cdot\delta_e$ *is C^∞ around e.*

In Corollary 1.5, we could say that $\beta^1_\Lambda(\mathcal{F})\cdot\delta_e$ is the "singular part" of $\mathrm{Tr}^1_{\mathrm{dis}}(\mathcal{F})$ around e.

Corollary 1.6. *Suppose that* $\operatorname{codim}\mathcal{F} > 0$ *and* $\dim\mathcal{F} = 2$*. If there is a nontrivial harmonic L^2 differential form of degree one on some leaf, then* $\dim\overline{H}^1(\mathcal{F}) = \infty$.

It would be nice to generalize Corollary 1.6 for arbitrary dimension. Thus we conjecture the following.

Conjecture 1.7. If $\operatorname{codim}\mathcal{F} > 0$ and the leaves are dense, then $\mathrm{Tr}^i_{\mathrm{dis}}(\mathcal{F}) - \beta^i_\Lambda(\mathcal{F})\cdot\delta_e$ is C^∞ around e for each degree i.

The main results were proved in [3] for the case of codimension one. Our results also overlap the corresponding results of [20].

We hope to prove elsewhere another version of Theorem 1.3 with a more general condition on the fixed points, always satisfied by some local representation ϕ of Φ defined around any point of G. By (1.2), what is needed is another version of the Lefschetz theorem of [14], which holds for more general fixed point sets when the transverse measure is C^∞.

The idea of using such type of trace class operators to define distributional spectral invariants is due to Atiyah and Singer [5, 30]. They consider transversally elliptic operators with respect to compact Lie group actions. Further generalizations to foliations and non-compact Lie group actions were given in [21, 10, 15, 17]. In our case, $\Delta_{\mathcal{F}}$ is not transversally elliptic with respect to any Lie group action or any foliation, but it can be considered as being "transversely elliptic" with respect to the structural transverse action; this simply means that it is elliptic along the leaves of $\mathcal{F}$.

2. Transverse actions

Recall that a foliation $\mathcal{F}$ on a manifold M can be described by a *foliated cocycle*, which is a collection $\{U_i, f_i\}$, where $\{U_i\}$ is an open cover of X and each f_i is a topological submersion of U_i onto some manifold T_i whose fibers are connected open subsets of $\mathbb{R}^n$, such that the following *compatibility condition* is satisfied: for every $x \in U_i \cap U_j$, there is an open neighborhood $U^x_{i,j}$ of x in $U_i \cap U_j$ and a homeomorphism $h^x_{i,j} : f_i(U^x_{i,j}) \to f_j(U^x_{i,j})$ such that $f_j = h^x_{i,j} \circ f_i$ on $U^x_{i,j}$. Two foliated cocycles describe the same foliation $\mathcal{F}$ when their union is a foliated cocycle. The *leaf topology* on M is the topology with a base given by the open sets of the fibers of all the submersions f_i. The *leaves* of $\mathcal{F}$ are the connected components of M with the leaf topology. The leaf through each point $x \in M$ is denoted by L_x. The pseudogroup on $\bigsqcup_i T_i$ generated by the maps $h^x_{i,j}$, given by the compatibility condition, is called (a representative of) the *holonomy pseudogroup* of $\mathcal{F}$, and describes the "transverse dynamics" of $\mathcal{F}$. Different foliated cocycles of $\mathcal{F}$ induce equivalent pseudogroups in the sense of [12, 13].

Another representative of the holonomy pseudogroup is defined on any transversal of $\mathcal{F}$ that meets every leaf. It is generated by "sliding" small open subsets (local transversals) along the leaves; its precise definition is given in [12].

When M is a C^∞ manifold, it is said that $\mathcal{F}$ is C^∞ if it is described by a foliated cocycle $\{U_i, f_i\}$ which is C^∞ in the sense that each f_i is a C^∞ submersion to some C^∞ manifold.

Let Γ be a group of homeomorphisms of a manifold T. A foliated cocycle (U_i, f_i) of $\mathcal{F}$, with $f_i : U_i \to T_i$, is said to be (T, Γ)-*valued* when each T_i is an open subset of T, and the maps $h^x_{i,j}$, given by the compatibility condition, are restrictions of maps in Γ. A *transverse* (T, Γ)-*structure* of $\mathcal{F}$ is given by a (T, Γ)-valued foliated cocycle, and two (T, Γ)-valued foliated cocycles define the same transverse (T, Γ)-structure when their union is a (T, Γ)-valued foliated cocycle. When $\mathcal{F}$ is endowed with a transverse (T, Γ)-structure, it is called a (T, Γ)-*foliation*.

Let $\mathcal{F}$ and $\mathcal{G}$ be foliations on manifolds M and N, respectively. Recall the following concepts. A *foliated map* $f : (M, \mathcal{F}) \to (N, \mathcal{G})$ is a map $f : M \to N$ that maps each leaf of $\mathcal{F}$ to a leaf of $\mathcal{G}$; the simpler notation $f : \mathcal{F} \to \mathcal{G}$ will be also used. A *leafwise homotopy* (or *integrable homotopy*) between two continuous foliated maps $f, f' : (M, \mathcal{F}) \to (N, \mathcal{G})$ is a continuous map $H : M \times I \to N$

($I = [0,1]$) such that the path $H(x,\cdot) : I \to N$ lies in a leaf of $\mathcal{G}$ for each $x \in M$; in this case, it is said that f and f' are *leafwisely homotopic* (or *integrably homotopic*).

Suppose from now on that $\mathcal{F}$ and $\mathcal{G}$ are C^∞. Two C^∞ foliated maps are said to be C^∞ *leafwisely homotopic* when there is a C^∞ leafwise homotopy between them. As usual, $T\mathcal{F} \subset TM$ denotes the subbundle of vectors tangent to the leaves of $\mathcal{F}$, $\mathfrak{X}(M,\mathcal{F})$ denotes the Lie algebra of infinitesimal transformations of $(M,\mathcal{F})$, and $\mathfrak{X}(\mathcal{F}) \subset \mathfrak{X}(M,\mathcal{F})$ is the normal Lie subalgebra of vector fields tangent to the leaves of $\mathcal{F}$ (C^∞ sections of $T\mathcal{F} \to M$). Then we can consider the quotient Lie algebra $\overline{\mathfrak{X}}(M,\mathcal{F}) = \mathfrak{X}(M,\mathcal{F})/\mathfrak{X}(\mathcal{F})$, whose elements are called *transverse vector fields.* Observe that, for each $x \in M$, the evaluation map $\mathrm{ev}_x : \mathfrak{X}(M,\mathcal{F}) \to T_xM$ induces a map $\overline{\mathrm{ev}}_x : \overline{\mathfrak{X}}(M,\mathcal{F}) \to T_xM/T_x\mathcal{F}$, which can be also called *evaluation map*. For any Lie algebra $\mathfrak{g}$, a homomorphism $\mathfrak{g} \to \overline{\mathfrak{X}}(M,\mathcal{F})$ is called an *infinitesimal transverse action* of $\mathfrak{g}$ on $(M,\mathcal{F})$. In particular, we have a canonical infinitesimal transverse action of $\overline{\mathfrak{X}}(M,\mathcal{F})$ on $(M,\mathcal{F})$.

Let $\mathrm{Diff}(M,\mathcal{F})$ be the group of C^∞ foliated diffeomorphisms $(M,\mathcal{F}) \to (M,\mathcal{F})$ with the operation of composition, let $\mathrm{Diff}(\mathcal{F}) \subset \mathrm{Diff}(M,\mathcal{F})$ be the normal subgroup C^∞ foliated diffeomorphisms that preserve each leaf of $\mathcal{F}$, and let $\mathrm{Diff}_0(\mathcal{F}) \subset \mathrm{Diff}(\mathcal{F})$ be the normal subgroup of C^∞ foliated diffeomorphisms that are C^∞ leafwisely homotopic to the identity map. Then we can consider the quotient group $\overline{\mathrm{Diff}}(M,\mathcal{F}) = \mathrm{Diff}(M,\mathcal{F})/\mathrm{Diff}_0(\mathcal{F})$, whose operation is also denoted by "$\circ$". The elements of $\overline{\mathrm{Diff}}(M,\mathcal{F})$ can be called *transverse transformations* of $(M,\mathcal{F})$. For any group G, an anti-homomorphism $\Phi : G \to \overline{\mathrm{Diff}}(M,\mathcal{F})$, $g \mapsto \Phi_g$, is called a *right transverse action* of G on $(M,\mathcal{F})$. For an open subset $O \subset G$, a map $\phi : M \times O \to M$ is called a *local representation* of Φ on O if $\phi_g = \phi(\cdot, g) \in \Phi_g$ for all $g \in O$. For any leaf L of $\mathcal{F}$ and any $g \in O$, the leaf $\phi_g(L)$ is independent of the local representative ϕ, and thus it will be denoted by $\Phi_g(L)$. When G is a Lie group, Φ is said to be of *class* C^∞ if it has a C^∞ local representation around each element of G.

Somehow, we can think of $\overline{\mathrm{Diff}}(M,\mathcal{F})$ as a Lie group whose Lie algebra is $\overline{\mathfrak{X}}(M,\mathcal{F})$; indeed, it will be proved elsewhere that, if G is a simply connected Lie group and $\mathfrak{g}$ is its Lie algebra of left invariant vector fields, then there is a canonical bijection between infinitesimal transverse actions of $\mathfrak{g}$ on $(M,\mathcal{F})$ and C^∞ right transverse actions of G on $(M,\mathcal{F})$.

The *leafwise de Rham complex* $(\Omega(\mathcal{F}), d_\mathcal{F})$ is the space of differential forms on the leaves smooth on M (C^∞ sections of $\bigwedge T\mathcal{F}^* \to M$) endowed with the leafwise de Rham differential. It is also a topological vector space with the C^∞ topology, and $d_\mathcal{F}$ is continuous. The cohomology $H(\mathcal{F})$ of $(\Omega(\mathcal{F}), d_\mathcal{F})$ is called the *leafwise cohomology* of $\mathcal{F}$, which is a topological vector space with the induced topology. Its maximal Hausdorff quotient $\overline{H}(\mathcal{F}) = H(\mathcal{F})/\overline{0}$ is called the *reduced leafwise cohomology*.

By pulling back leafwise differential forms, any C^∞ foliated map $f : (M,\mathcal{F}) \to (N,\mathcal{G})$ induces a continuous homomorphism of complexes, $f^* : \Omega(\mathcal{G}) \to \Omega(\mathcal{F})$, obtaining a continuous homomorphism $f^* : \overline{H}(\mathcal{G}) \to \overline{H}(\mathcal{F})$. Moreover, if f is

C^∞ leafwisely homotopic to another C^∞ foliated map $f' : (M,\mathcal{F}) \to (M,\mathcal{F})$, then $f^* = f'^* : \overline{H}(\mathcal{G}) \to \overline{H}(\mathcal{F})$ by standard arguments [7]. Therefore, for any $F \in \overline{\mathrm{Diff}}(M,\mathcal{F})$ and any $f \in F$, the endomorphism f^* of $\overline{H}(\mathcal{F})$ can be denoted by F^*. So any right transverse action Φ of a group G on $(M,\mathcal{F})$ induces a left action Φ^* of G on $\overline{H}(\mathcal{F})$ given by $(g,\xi) \mapsto \Phi_g^*\xi$.

3. Lie foliations

Let $\mathcal{F}$ be a C^∞ foliation of codimension q on a C^∞ closed manifold M. Let G be a simply connected Lie group of dimension q, and $\mathfrak{g}$ its Lie algebra of left invariant vector fields. A *transverse Lie structure* of $\mathcal{F}$, with *structural Lie group* G and *structural Lie algebra* $\mathfrak{g}$, can be described with any of the following objects that determine each other [11, 19]:

(L.1) A transverse (G,G)-structure of $\mathcal{F}$, where G is identified with the group of its left translations.

(L.2) A $\mathfrak{g}$-valued 1-form ω on M such that $\omega_x : T_xM \to \mathfrak{g}$ is surjective with kernel $T_x\mathcal{F}$ for every $x \in M$, and

$$d\omega + \frac{1}{2}[\omega,\omega] = 0\ .$$

(L.3) A homomorphism $\theta : \mathfrak{g} \to \overline{\mathfrak{X}}(M,\mathcal{F})$ such that the composite

$$\mathfrak{g} \xrightarrow{\ \theta\ } \overline{\mathfrak{X}}(M,\mathcal{F}) \xrightarrow{\ \overline{\mathrm{ev}}_x\ } T_xM/T_x\mathcal{F}$$

is an isomorphism for every $x \in M$.

In (L.1), the elements of G whose corresponding left translations are involved in the definition of the transverse (G,G)-structure form a subgroup Γ, which is called the *holonomy group* of $\mathcal{F}$. So the transverse (G,G)-structure is a transverse (G,Γ)-structure. In (L.2) and (L.3), ω and θ can be respectively called the *structural form* and the *structural infinitesimal transverse action*.

A C^∞ foliation endowed with a transverse Lie structure is called a *Lie foliation*; the terms *Lie G-foliation* or *Lie $\mathfrak{g}$-foliation* are used too. If the leaves are dense, then the transverse Lie structure is unique, and thus it is determined by the foliation.

A Lie G-foliation $\mathcal{F}$ on a C^∞ closed manifold M has the following description due to Fedida [11, 19]. There exists a regular covering $\pi : \widetilde{M} \to M$, a fibre bundle $D : \widetilde{M} \to G$ and an injective homomorphism $h : \mathrm{Aut}(\pi) \to G$ such that the leaves of $\widetilde{\mathcal{F}} = \pi^*\mathcal{F}$ are the fibres of D, and D is h-equivariant; *i.e.*,

$$D \circ \sigma(\tilde{x}) = h(\sigma) \cdot D(\tilde{x})$$

for all $\tilde{x} \in \widetilde{M}$ and $\sigma \in \mathrm{Aut}(\pi)$. This h is called the *holonomy homomorphism*. By using the covering space $\ker(h)\backslash\widetilde{M}$ of M if necessary, we can assume that h is injective, and thus π restricts to diffeomorphisms of the leaves of $\widetilde{\mathcal{F}}$ to the leaves of $\mathcal{F}$. The leaf of $\widetilde{\mathcal{F}}$ through each point $\tilde{x} \in \widetilde{M}$ will be denoted by $\widetilde{L}_{\tilde{x}}$.

Given a (G, G)-valued foliated cocycle $\{U_i, f_i\}$ defining the transverse Lie structure according to (L.1), the $\mathfrak{g}$-valued 1-form ω of (L.2) and the infinitesimal transverse action θ of (L.3) can be defined as follows. For $x \in U_i$ and $v \in T_xM$, $\omega_x(v)$ is the left invariant vector field on G whose value at $f_i(x)$ is $f_{i*}(v)$. To define θ, fix an auxiliary vector subbundle $\nu \subset TM$ complementary of $T\mathcal{F}$ ($TM = \nu \oplus T\mathcal{F}$). Each $X \in \mathfrak{g}$ defines a C^∞ vector field $X^\nu \in \mathfrak{X}(M, \mathcal{F})$ by the conditions $X^\nu(x) \in \nu_x$ and $f_{i*}(X^\nu(x)) = X(f_i(x))$ if $x \in U_i$. Then $\theta(X)$ is the class of X^ν in $\overline{\mathfrak{X}}(M, \mathcal{F})$, which is independent of the choice of ν.

By using Fedida's geometric description of $\mathcal{F}$, the definitions of ω and X^ν can be better understood:

- Let ω_G be the canonical $\mathfrak{g}$-valued 1-form on G defined by $\omega_G(X(g)) = X$ for any $X \in \mathfrak{g}$ and any $g \in G$. Then ω is determined by the condition $\pi^*\omega = D^*\omega_G$.
- Let $\tilde{\nu} = \pi_*^{-1}(\nu) \subset T\widetilde{M}$, which is a vector subbundle complementary of $T\widetilde{\mathcal{F}}$. Then, for any $X \in \mathfrak{g}$, there is a unique $\widetilde{X}^\nu \in \mathfrak{X}(\widetilde{M}, \widetilde{\mathcal{F}})$ which is a section of $\tilde{\nu}$ and satisfies $D_* \circ \widetilde{X}^\nu = X \circ D$. Since D is h-equivariant, $\widetilde{X}^\nu$ is $\mathrm{Aut}(\pi)$-invariant. Then X^ν is the projection of $\widetilde{X}^\nu$ to M.

4. Structural transverse action

Let G be a simply connected Lie group, and let $\mathcal{F}$ be a Lie G-foliation on a closed manifold M. According to Section 2, the structural infinitesimal transverse action corresponds to a unique right transverse action of G on $(M, \mathcal{F})$, obtaining another description of the transverse Lie structure:

(L.4) A C^∞ right transverse action Φ of G on $(M, \mathcal{F})$ which has a C^∞ local representation ϕ around the identity element e of G such that the composite

$$T_eG \xrightarrow{\phi_*^x} T_xM \longrightarrow T_xM/T_x\mathcal{F}$$

is an isomorphism for all $x \in M$, where $\phi^x = \phi(x, \cdot)$ and the second map is the canonical projection. This condition is independent of the choice of ϕ. This Φ is called the *structural transverse action*.

To describe Φ, consider Fedida's geometric description of $\mathcal{F}$ (Section 3). For any $g \in G$, take a continuous, piecewise C^∞ path $c : I \to G$ with $c(0) = e$ and $c(1) = g$. For any $\tilde{x} \in \widetilde{M}$, there exists a unique continuous piecewise C^∞ path $\tilde{c}^\nu_{\tilde{x}} : I \to \widetilde{M}$ such that

- $\tilde{c}^\nu_{\tilde{x}}(0) = \tilde{x}$,
- $\tilde{c}^\nu_{\tilde{x}}$ is tangent to $\tilde{\nu}$ at every $t \in I$ where it is C^∞, and
- $D \circ \tilde{c}^\nu_{\tilde{x}}(t) = D(\tilde{x}) \cdot c(t)$ for any $t \in I$.

It is easy to see that such a $\tilde{c}^\nu_{\tilde{x}}$ depends smoothly on $\tilde{x}$.

Lemma 4.1. *We have* $\sigma \circ \tilde{c}^\nu_{\tilde{x}} = \tilde{c}^\nu_{\sigma(\tilde{x})}$ *for* $\tilde{x} \in \widetilde{M}$ *and* $\sigma \in \mathrm{Aut}(\pi)$.

Proof. This is a direct consequence of the h-equivariance of D and the unicity of the paths $\tilde{c}^{\nu}_{\tilde{x}}$. □

For each $g \in G$, let $\tilde{\phi}_g : (\widetilde{M}, \widetilde{\mathcal{F}}) \to (\widetilde{M}, \widetilde{\mathcal{F}})$ be the C^∞ foliated diffeomorphism given by $\tilde{\phi}_g(\tilde{x}) = \tilde{c}^{\nu}_{\tilde{x}}(1)$. For any $\tilde{x} \in \widetilde{M}$ and $\sigma \in \mathrm{Aut}(\pi)$, we have

$$\sigma \circ \tilde{\phi}_g(\tilde{x}) = \sigma \circ \tilde{c}^{\nu}_{\tilde{x}}(1) = \tilde{c}^{\nu}_{\sigma(\tilde{x})}(1) = \tilde{\phi}_g \circ \sigma(\tilde{x})$$

by Lemma 4.1, yielding $\sigma \circ \tilde{\phi}_g = \tilde{\phi}_g \circ \sigma$. Therefore, there exists a unique C^∞ foliated diffeomorphism $\phi_g : (M, \mathcal{F}) \to (M, \mathcal{F})$ such that $\pi \circ \tilde{\phi}_g = \phi_g \circ \pi$.

Lemma 4.2. *The C^∞ leafwise homotopy class of ϕ_g is independent of the choice of c.*

Proof. Let $d : I \to G$ be another continuous and piecewise smooth path with $d(0) = e$ and $d(1) = g$, which defines a C^∞ foliated map $\varphi_g : (M, \mathcal{F}) \to (M, \mathcal{F})$ as above. Since G is simply connected, there exists a family of continuous and piecewise smooth paths $c_s : I \to G$, depending smoothly on $s \in I$, with $c_s(0) = e$, $c_s(1) = g$, $c_0 = c$ and $c_1 = d$. The paths c_s induce a family of C^∞ foliated maps $\phi_{g,s} : (M, \mathcal{F}) \to (M, \mathcal{F})$ as above, defining a C^∞ leafwise homotopy between ϕ_g and φ_g. □

Lemma 4.3. *The C^∞ leafwise homotopy class of ϕ_g is independent of the choice of ν.*

Proof. Let $\nu' \subset TM$ be another vector subbundle complementary of $T\mathcal{F}$, which can be used to define a C^∞ foliated map ϕ'_g as above. It is easy to find a C^∞ deformation of vector subbundles of $\nu_s \subset TM$ complementary of $T\mathcal{F}$, $s \in I$, with $\nu_0 = \nu$ and $\nu_1 = \nu'$. Then the foliated maps $\phi_{g,s}$, induced by the vector bundles ν_s as above, define a C^∞ leafwise homotopy between ϕ_g and ϕ'_g. □

Therefore, for each g, the C^∞ leafwise homotopy class Φ_g of ϕ_g depends only on g, $\mathcal{F}$ and its transverse Lie structure. So a map $\Phi : G \to \overline{\mathrm{Diff}}(M, \mathcal{F})$ is given by $g \mapsto \Phi_g$.

Lemma 4.4. *Φ is a right transverse action of G in $(M, \mathcal{F})$.*

Proof. Given $g_1, g_2 \in G$, let $c_1, c_2 : I \to G$ be continuous, piecewise smooth paths such that $c_1(0) = c_2(0) = e$, $c_1(1) = g_1$ and $c_2(1) = g_2$, which are used to define ϕ_{g_1} and ϕ_{g_2} as above. Let $c : I \to G$ be the path product of c_1 and $L_{g_1} \circ c_2$, where L_{g_1} denotes the left translation by g_1. We have $c(0) = e$ and $c(1) = g_1 g_2$. We can use this c to define $\phi_{g_1 g_2}$, obtaining $\phi_{g_1 g_2} = \phi_{g_2} \circ \phi_{g_1}$, and thus $\Phi_{g_1 g_2} = \Phi_{g_2} \circ \Phi_{g_1}$. □

Lemma 4.5. *Φ is C^∞.*

Proof. It is easy to prove that each element of G has a neighbourhood O such that there is a C^∞ map $c : I \times O \to G$ so that each $c_g = c(\cdot, g)$ is a path from e to g. The corresponding foliated diffeomorphisms ϕ_g form a C^∞ representation of Φ on O. □

This construction defines the structural transverse action Φ. According to Section 2, Φ induces a left action Φ^* of G on $\overline{H}(\mathcal{F})$.

Lemma 4.6. *There is a local representation $\varphi : M \times O \to M$ of Φ around the identity element e such that $\varphi_e = \mathrm{id}_M$.*

Proof. Construct ϕ like in the proof of Lemma 4.5 such that $e \in O$ and c_e is the constant path at e. □

Let $\varphi : M \times O \to M$ be a local representation of Φ. A map $\tilde{\varphi} : \widetilde{M} \times O \to \widetilde{M}$ is called a *lift* of φ if $\pi \circ \tilde{\varphi}_g = \varphi_g \circ \pi$ for all $g \in O$, where $\tilde{\varphi}_g = \tilde{\varphi}(\cdot, g)$. In particular, the above construction of ϕ also gives a lift $\tilde{\phi}$. Let $R_g : G \to G$ denote the right translation by any $g \in G$.

Lemma 4.7. *Any C^∞ lift $\tilde{\varphi} : \widetilde{M} \times O \to \widetilde{M}$ of each C^∞ local representation $\varphi : M \times O \to M$ of Φ, such that O is connected, satisfies $D \circ \tilde{\varphi}_g = R_g \circ D$ for all $g \in O$.*

Proof. It is enough to prove the result when O is as small as desired. It is clear that the property of the statement is satisfied by the maps $\tilde{\phi}$ constructed above for connected O.

For an arbitrary φ, if O is small enough and connected, there is some $\phi : M \times O \to M$ defined by the above construction and some homotopy $H : M \times O \times I \to M$ between φ and ϕ such that each path $t \mapsto H(x, g, t)$ is contained in a leaf of $\mathcal{F}$. This H lifts to a homotopy $\widetilde{H} : \widetilde{M} \times O \times I \to \widetilde{M}$ between $\tilde{\varphi}$ and $\tilde{\phi}$ so that each path $t \mapsto \widetilde{H}(\tilde{x}, g, t)$ is contained in a leaf of $\widetilde{\mathcal{F}}$. Then $D \circ \tilde{\varphi} = D \circ \tilde{\phi}$, completing the proof. □

Corollary 4.8. *$\tilde{\varphi} : \widetilde{L} \times O \to \widetilde{M}$ is a C^∞ embedding for each leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$.*

The transverse Lie structure of $\mathcal{F}$ lifts to a transverse Lie structure of $\widetilde{\mathcal{F}}$, whose structural right transverse action is locally represented by the C^∞ lifts of C^∞ local representations of Φ.

5. The Hodge isomorphism

Recall that any Lie foliation is Riemannian [23]. Then fix a bundle-like metric on M [23], and equip the leaves of $\mathcal{F}$ with the induced Riemannian metric. Let $\delta_{\mathcal{F}}$ denote the leafwise coderivative on the leaves operating in $\Omega(\mathcal{F})$, and set $D_{\mathcal{F}} = d_{\mathcal{F}} + \delta_{\mathcal{F}}$. Then $\Delta_{\mathcal{F}} = D_{\mathcal{F}}^2 = d_{\mathcal{F}} \circ d_{\mathcal{F}} + d_{\mathcal{F}} \circ \delta_{\mathcal{F}}$ is the leafwise Laplacian operating in $\Omega(\mathcal{F})$. Let $\mathcal{H}(\mathcal{F}) = \ker \Delta_{\mathcal{F}}$ (the space of leafwise harmonic forms which are smooth on M). Since the metric is bundle-like, the transverse volume element is holonomy invariant, which implies that $D_{\mathcal{F}}$ and $\Delta_{\mathcal{F}}$ are symmetric, and thus they have the same kernel.

Let $\mathbf{\Omega}(\mathcal{F})$ be the Hilbert space of square integrable leafwise differential forms on M. The metric of M induces a Hilbert structure in $\mathbf{\Omega}(\mathcal{F})$. For any C^∞ foliated map $f : (M, \mathcal{F}) \to (M, \mathcal{F})$, the endomorphism f^* of $\Omega(\mathcal{F})$ is obviously L^2-bounded, and thus extends to a bounded operator f^* in $\boldsymbol{\Omega}(\mathcal{F})$. Consider $D_{\mathcal{F}}$ and $\Delta_{\mathcal{F}}$ as unbounded operators in $\boldsymbol{\Omega}(\mathcal{F})$, which are essentially self-adjoint [8], and whose closures are denoted by $\mathbf{D}_{\mathcal{F}}$ and $\boldsymbol{\Delta}_{\mathcal{F}}$ (see, *e.g.*, [4, 16]). By [2], $\boldsymbol{\mathcal{H}}(\mathcal{F}) = \ker \boldsymbol{\Delta}_{\mathcal{F}}$ is the closure of $\mathcal{H}(\mathcal{F})$ in $\boldsymbol{\Omega}(\mathcal{F})$, and the orthogonal projection $\boldsymbol{\Pi} : \boldsymbol{\Omega}(\mathcal{F}) \to \boldsymbol{\mathcal{H}}(\mathcal{F})$ has a restriction $\Pi : \Omega(\mathcal{F}) \to \mathcal{H}(\mathcal{F})$, which induces a leafwise Hodge isomorphism

$$\overline{H}(\mathcal{F}) \cong \mathcal{H}(\mathcal{F}) \; .$$

For any C^∞ foliated map $f : (M, \mathcal{F}) \to (M, \mathcal{F})$, the homomorphism $f^* : \overline{H}(\mathcal{F}) \to \overline{H}(\mathcal{F})$ corresponds to the operator $\Pi \circ f^*$ in $\mathcal{H}(\mathcal{F})$ via the Hodge isomorphism. So the left G-action on $\overline{H}(\mathcal{F})$, defined in Section 4, corresponds to the left G-action on $\mathcal{H}(\mathcal{F})$ given by $(g, \alpha) \mapsto \Pi \circ \phi_g^* \alpha$ for any $\phi_g \in \Phi_g$.

Since the left action of G on $\mathcal{H}(\mathcal{F})$ is L^2-continuous, we get an extended left action of G on $\boldsymbol{\mathcal{H}}(\mathcal{F})$ given by $(g, \alpha) \mapsto \boldsymbol{\Pi} \circ \phi_g^* \alpha$ for any $\phi_g \in \Phi_g$.

These actions on $\mathcal{H}(\mathcal{F})$ and $\boldsymbol{\mathcal{H}}(\mathcal{F})$ are continuous on G since Φ is C^∞.

6. A class of smoothing operators

6.1. Preliminaries on smoothing and trace class operators

Let ω_M denote the volume forms of M. A *smoothing operator* in $\Omega(\mathcal{F})$ is a linear map $P : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$, continuous with respect to the C^∞ topology, given by

$$(P\alpha)(x) = \int_M k(x, y)\, \alpha(y)\, \omega_M(y)$$

for some C^∞ section k of $\bigwedge T\mathcal{F}^* \boxtimes \bigwedge T\mathcal{F}$ over $M \times M$; thus

$$k(x, y) \in \bigwedge T\mathcal{F}_x^* \otimes \bigwedge T\mathcal{F}_y \equiv \operatorname{Hom}(\bigwedge T\mathcal{F}_y^*, \bigwedge T\mathcal{F}_x^*)$$

for any $x, y \in M$. This k is called the *smoothing kernel* or *Schwartz kernel* of P. Such a P defines a trace class operator in $\boldsymbol{\Omega}(\mathcal{F})$, and we have

$$\operatorname{Tr} P = \int_M \operatorname{Tr} k(x, x)\, \omega_M(x) \; .$$

The supertrace formalism will be also used. For any homogeneous operator T in $\boldsymbol{\Omega}(\mathcal{F})$ or in $\bigwedge T_x\mathcal{F}^*$, let $T^\pm$ denote its restriction to the even and odd degree part, and let $T^{(i)}$ denote its restriction to the part of degree i. If T is of trace class, then its supertrace is

$$\operatorname{Tr}^s T = \operatorname{Tr} T^+ - \operatorname{Tr} T^- = \sum_i (-1)^i \operatorname{Tr} T^{(i)} \; .$$

Thus

$$\operatorname{Tr}^s P = \int_M \operatorname{Tr}^s k(x, x)\, \omega_M(x) \; .$$

Let $W^k\Omega(\mathcal{F})$ denote the Sobolev space of order k of leafwise differential forms on M, and let $\|\cdot\|_k$ denote a norm of $W^k\Omega(\mathcal{F})$. A continuous operator P in $\Omega(\mathcal{F})$ is smoothing if and only if P extends to a bounded operator $P : W^k\Omega(\mathcal{F}) \to W^l\Omega(\mathcal{F})$ for any k and l.

If an operator P in $\Omega(\mathcal{F})$ has an extension $P : W^k\Omega(\mathcal{F}) \to W^\ell\Omega(\mathcal{F})$, then $\|P\|_{k,\ell}$ denotes the norm of this extension; the notation $\|P\|_k$ is used when $k = \ell$. By the Sobolev embedding theorem, the trace of a smoothing operator P in $\Omega(\mathcal{F})$ can be estimated in the following way: for any $k > \dim M$, there is some $C > 0$ independent of P such that

$$|\operatorname{Tr} P| \leq C\,\|P\|_{0,k}\,. \tag{6.1}$$

6.2. The class $\mathcal{D}$

Let $\mathcal{A}$ be the set of all functions $\psi : \mathbb{R} \to \mathbb{C}$, extending to an entire function ψ on $\mathbb{C}$ such that, for each compact set $K \subset \mathbb{R}$, the set of functions $\{(x \mapsto \psi(x+iy)) \mid y \in K\}$ is bounded in the Schwartz space $\mathcal{S}(\mathbb{R})$. This $\mathcal{A}$ has a structure of Fréchet algebra, and, in fact, it is a module over $\mathbb{C}[z]$. This algebra contains all functions with compactly supported Fourier transform, and the functions $x \mapsto e^{-tx^2}$ with $t > 0$.

By [25, Proposition 4.1], there exists a "functional calculus map" $\mathcal{A} \to \operatorname{End}(\Omega(\mathcal{F}))$, $\psi \mapsto \psi(D_\mathcal{F})$, which is a continuous homomorphism of $\mathbb{C}[z]$-modules and of algebras. Any operator $\psi(D_\mathcal{F})$, $\psi \in \mathcal{A}$, extends to a bounded operator in $W^k\Omega(\mathcal{F})$ for any k with the following estimate for its norm: there is some $C > 0$, independent of ψ, such that

$$\|\psi(D_\mathcal{F})\|_k \leq \int |\hat{\psi}(\xi)|\, e^{C\,|\xi|}\, d\xi\,, \tag{6.2}$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. Therefore, for any natural N, the operator $(\mathrm{id} + \Delta_\mathcal{F})^N \psi(D_\mathcal{F})$ extends to a bounded operator in $W^k\Omega(\mathcal{F})$ for any k whose norm can be estimated as follows: there is some $C > 0$, independent of ψ, such that

$$\|(\mathrm{id} + \Delta_\mathcal{F})^N \psi(D_\mathcal{F})\|_k \leq \int |(\mathrm{id} - \partial_\xi^2)^N \hat{\psi}(\xi)|\, e^{C\,|\xi|}\, d\xi\,. \tag{6.3}$$

Fix a left-invariant Riemannian metric on G, and let Λ denote its volume form. We can assume that the metrics on M and G agree in the sense that the maps f_i of (L.1) are Riemannian submersions (Section 3). Thus $D : \widetilde{M} \to G$ is a Riemannian submersion with respect to the lift of the bundle-like metric to $\widetilde{M}$.

A *leafwise differential* operator in $\Omega(\mathcal{F})$ is a differential operator which involves only leafwise derivatives; for instance, $d_\mathcal{F}$, $\delta_\mathcal{F}$, $D_\mathcal{F}$ and $\Delta_\mathcal{F}$ are leafwise differential operators. A family of leafwise differential operators in $\Omega(\mathcal{F})$, $A = \{A_v \mid v \in V\}$, is said to be *smooth* when V is a C^∞ manifold and, with respect to C^∞ local coordinates, the local coefficients of each A_v depend smoothly on v in the C^∞-topology. We also say that A is *compactly supported* when there is some compact subset $K \subset V$ such that $A_v = 0$ if $v \notin K$. Given another smooth family of leafwise differential operators in $\Omega(\mathcal{F})$ with the same parameter manifold, $B = \{B_v \mid v \in V\}$, the *composite* $A \circ B$ is the family defined by $(A \circ B)_v = A_v \circ B_v$. Similarly, we can define the *sum* $A + B$ and the *product* $\lambda \cdot A$ for some $\lambda \in \mathbb{R}$.

We introduce the class $\mathcal{D}$ of operators $P : \Omega(\mathcal{F}) \to \Omega(\mathcal{F})$ of the form

$$P = \int_O \phi_g^* \circ A_g \, \Lambda(g) \circ \psi(D_{\mathcal{F}}) \ ,$$

where O is some open subset of G, $\phi : M \times O \to M$ is a C^∞ local representation of Φ, $A = \{A_g \mid g \in O\}$ is a smooth compactly supported family of leafwise differential operators in $\Omega(\mathcal{F})$, and $\psi \in \mathcal{A}$.

Proposition 6.1. *Any operator $P \in \mathcal{D}$ is a smoothing operator in $\Omega(\mathcal{F})$.*

Proof. Let $P \in \mathcal{D}$ as above. By (6.3) and since the operator ϕ_g^* preserves any Sobolev space, P defines a bounded operator in $W^k\Omega(\mathcal{F})$ for any k.

Let $\varphi : M \times O_0 \to M$ be a C^∞ local representation of Φ on some open neighborhood O_0 of the identity element e; we can assume that $\varphi_e = \mathrm{id}_M$ by Corollary 4.8. For any $Y \in \mathfrak{g}$, let $\widehat{Y}$ be the first-order differential operator in $\Omega(\mathcal{F})$ defined by

$$\widehat{Y} u = \frac{d}{dt} \varphi_{\exp tY}^* u \bigg|_{t=0} \ ,$$

which makes sense because $\exp tY \in O_0$ for any $t > 0$ small enough.

Fix a base $Y_1, \dots, Y_q$ of $\mathfrak{g}$. Then the second-order differential operator $L = -\sum_{j=1}^q \widehat{Y}_j^2$ in $\Omega(\mathcal{F})$ is transversely elliptic. Moreover $\Delta_{\mathcal{F}}$ is leafwise elliptic. By the elliptic regularity theorem, it suffices to prove that $L^N \circ P$ and $\Delta_{\mathcal{F}}^N \circ P$ belong to $\mathcal{D}$ for any natural N. In turn, this follows by showing that $Q \circ P$ and $\widehat{Y} \circ P$ are in $\mathcal{D}$ for any leafwise differential operator Q and any $Y \in \mathfrak{g}$.

We have

$$Q \circ P = \int_O \phi_g^* \circ B_g \, \Lambda(g) \circ \psi(D_{\mathcal{F}}) \ ,$$

where $B_g = (\phi_g^*)^{-1} \circ Q \circ \phi_g^* \circ A_g$. Since ϕ_g is a foliated map, it follows that $\{B_g \mid g \in O\}$ is a smooth family of leafwise differential operators, yielding $Q \circ P \in \mathcal{D}$.

For $g \in O$ and $a \in O_0$ close enough to e, let

$$F_{a,g} = \phi_{ag} \circ \varphi_a \circ \phi_g^{-1} \ .$$

Observe that $F_{e,g} = \mathrm{id}_M$ because $\varphi_e = \mathrm{id}_M$. For each $Y \in \mathfrak{g}$, we get a smooth family $V_Y = \{V_{Y,g} \mid g \in O\}$ of first-order leafwise differential operators in $\Omega(\mathcal{F})$ given by

$$V_{Y,g} u = \frac{d}{dt} F_{\exp tY, g}^* u \bigg|_{t=0} \ .$$

Let also $L_Y A = \{(L_Y A)_g \mid g \in O\}$ be the smooth family of leafwise differential operators given by

$$(L_Y A)_g u = \frac{d}{dt} A_{\exp(-tY) \cdot g} u \bigg|_{t=0} \ .$$

In particular, if A_g is given by multiplication by $f(g)$ for some $f \in C_c^\infty(G)$, then $(L_Y A)_g$ is given by multiplication by $(Yf)(g)$.

We proceed as follows:

$$\begin{aligned}\int_O \varphi^*_{\exp tY}\circ\phi^*_g\circ A_g\,\Lambda(g) &= \int_O \phi^*_{\exp tY\cdot g}\circ F^*_{\exp tY,\exp(-tY)\cdot g}\circ A_g\,\Lambda(g)\\ &= \int_O \phi^*_g\circ F^*_{\exp tY,g}\circ A_{\exp tY\cdot g}\,\Lambda(g)\;,\end{aligned}$$

yielding

$$\begin{aligned}\widehat{Y}\circ P &= \lim_{t\to 0}\frac{1}{t}\left(\int_O \varphi^*_{\exp tY}\circ\phi^*_g\circ A_g\,dg - \int_O \phi^*_g\circ A_g\,dg\right)\circ\psi(D_{\mathcal F})\\ &= \lim_{t\to 0}\frac{1}{t}\left(\int_O \phi^*_g\circ F^*_{\exp tY,g}\circ A_{\exp tY\cdot g}\,dg - \int_O \phi^*_g\circ A_g\,dg\right)\circ\psi(D_{\mathcal F})\\ &= \int_O \phi^*_g\circ(V_Y\circ A + L_Y A)_g\,dg\circ\psi(D_{\mathcal F})\;.\end{aligned}$$

So $\widehat{Y}\circ P\in\mathcal{D}$. □

With the above notation, by the proof of Proposition 6.1 and (6.3), it can be easily seen that, for integers $k\le\ell$, there are some $C, C'>0$ and some natural N such that

$$\|P\|_{k,\ell}\le C'\int |(\mathrm{id}-\partial_\xi^2)^N\hat\psi(\xi)|\,e^{C|\xi|}\,d\xi\;. \tag{6.4}$$

Here, C depends on k and ℓ, and C' depends on k, ℓ and A.

6.3. A norm estimate

Let

$$P=\int_O \phi^*_g\cdot f(g)\,\Lambda(g)\circ\psi(D_{\mathcal F})\in\mathcal{D}\;,$$

where ϕ and ψ are like in Section 6.2, and $f\in C^\infty_c(O)$. In this case, (6.4) is improved by the following result, where Δ_G denotes the Laplacian of G.

Proposition 6.2. *Let $K\subset O$ be a compact subset containing* $\operatorname{supp} f$. *For naturals $k\le\ell$, there are some $C, C''>0$ and some natural N, depending only on K, k and ℓ, such that*

$$\|P\|_{k,\ell}\le C''\max_{g\in K}|(\mathrm{id}+\Delta_G)^N f(g)|\int |(\mathrm{id}-\partial_\xi^2)^N\hat\psi(\xi)|\,e^{C|\xi|}\,d\xi\;.$$

Proof. Fix an orthonormal frame $Y_1,\dots,Y_q$ of $\mathfrak{g}$. Consider any multi-index $J=(j_1,\dots,j_k)$ with $j_1,\dots,j_k\in\{1,\dots,q\}$. We use the standard notation $|J|=k$, and, with the notation of the proof of Proposition 6.1, let:

- $Y_J=Y_{j_1}\circ\cdots\circ Y_{j_k}$ (operating in $C^\infty(G)$);
- $\widehat{Y}_J=\widehat{Y_{j_1}}\circ\cdots\circ\widehat{Y_{j_k}}$;
- $V_J=V_{Y_{j_1}}\circ\cdots\circ V_{Y_{j_k}}$; and
- $L_JA=L_{Y_{j_1}}\cdots L_{Y_{j_k}}A$ for any smooth family A of leafwise differential operators in $\Omega(\mathcal{F})$.

Consider the empty multi-index $\emptyset$ too, with $|\emptyset|=0$, and define:

- $Y_\emptyset = \mathrm{id}_{C^\infty(G)}$;
- $\widehat{Y}_\emptyset = \mathrm{id}_{\Omega(\mathcal{F})}$;
- $V_{\emptyset,g} = \mathrm{id}_{\Omega(\mathcal{F})}$ for all $g \in O$, defining a smooth family $V_\emptyset$; and
- $L_\emptyset A = A$ for any smooth family A of leafwise differential operators in $\Omega(\mathcal{F})$.

Given any natural N, there is some $C_1 > 0$ such that

$$\|\phi_g^*\|_k \le C_1 \;, \qquad \|(L_J V_{J'})_g\| \le C_1 \;,$$
$$\|(Y_J f)(g)\| \le C_1 \max_{g\in K} |(\mathrm{id} + \Delta_G)^N f(g)| \;,$$
$$\|(\mathrm{id} + \phi_g^{*-1} \circ \Delta_{\mathcal{F}} \circ \phi_g^*)^N \circ \psi(\Delta_{\mathcal{F}})\|_k \le C_1 \,\|(\mathrm{id} + \Delta_{\mathcal{F}})^N \circ \psi(D_{\mathcal{F}})\|_k$$

for all $g \in K$ and all multi-indices J and J' with $|J|, |J'| \le N$.

For any multi-index J, we have

$$\widehat{Y}_J \circ P = \int_O \phi_g^* \circ A_{J,g}\, \Lambda(g) \circ \psi(D_{\mathcal{F}}) \;,$$

where $A_J = \{A_{J,g} \mid g \in G\}$ is the smooth family of leafwise differential operators inductively defined by setting

$$A_{\emptyset,g} = \mathrm{id}_{\Omega(\mathcal{F})} \cdot f(g) \;,$$
$$A_{(j,J)} = V_j \circ A_J + L_j A_J \;.$$

By induction on $|J|$, we easily get that A_J is a sum of smooth families of leafwise differential operators of the form

$$L_{J_1} V_{J_1'} \circ \cdots \circ L_{J_\ell} V_{J_\ell'} \cdot Y_{J''} f \;,$$

where $J_1, J_1', \dots, J_\ell, J_\ell', J''$ are possibly empty multi-indices satisfying

$$|J_1| + |J_1'| + \cdots + |J_\ell| + |J_\ell'| + |J''| = |J| \;.$$

So there is some $C_2 > 0$ such that

$$\|A_{J,g}\|_k \le C_2 \max_{g\in K} |(\mathrm{id} + \Delta_G)^N f(g)|$$

for all $g \in K$ and every multi-index J with $|J| \le N$. Hence

$$\begin{aligned}\|\widehat{Y}_J \circ P\|_k &\le \int_O \|\phi_g^*\|_k \|A_{J,g}\|_k \, dg \, \|\psi(D_{\mathcal{F}})\|_k \\ &\le C_1 C_2 \max_{g\in K} |(\mathrm{id} + \Delta_G)^N f(g)| \int |\hat{\psi}(\xi)|\, e^{C|\xi|}\, d\xi\end{aligned}$$

for some $C > 0$ by (6.2). On the other hand,

$$\begin{aligned}\|(\mathrm{id} + \Delta_{\mathcal{F}})^N \circ P\|_k &\le \int_O \|(\mathrm{id} + \phi_g^{*-1} \circ \Delta_{\mathcal{F}} \circ \phi_g^*)^N \circ \psi(\Delta_{\mathcal{F}})\|_k \, |f(g)|\, \Lambda(g) \\ &\le C_1 \int_O \|(\mathrm{id} + \Delta_{\mathcal{F}})^N \circ \psi(\Delta_{\mathcal{F}})\|_k \, |f(g)|\, \Lambda(g) \\ &\le C_1 \max_{g\in K} |f(g)| \int |(\mathrm{id} - \partial_\xi^2)^N \hat{\psi}(\xi)|\, e^{C|\xi|}\, d\xi\end{aligned}$$

for some $C > 0$ by (6.3). Now, the result follows because $-\sum_{j=1}^{q} \widehat{Y_j}^2$ is transversely elliptic, and $\Delta_{\mathcal{F}}$ is leafwise elliptic. $\square$

6.4. Parameter independence of the supertrace

Choose an even function in $\mathcal{A}$, which can be written as $x \mapsto \psi(x^2)$. Take also a C^∞ local representation $\phi : M \times O \to M$ of Φ and some $f \in C_c^\infty(O)$. Then consider the one parameter family of operators $P_t \in \mathcal{D}$, $t > 0$, defined by

$$P_t = \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \psi(t\Delta_{\mathcal{F}})^2 \ .$$

Lemma 6.3. $\operatorname{Tr}^s P_t$ *is independent of* t.

Proof. The proof is similar to the proof of the corresponding result in the heat equation proof of the Lefschetz trace formula (see, *e.g.*, [28]). We have

$$\begin{aligned}
\frac{d}{dt}\operatorname{Tr}^s P_t &= 2\operatorname{Tr}^s \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \Delta_{\mathcal{F}} \circ \psi'(t\Delta_{\mathcal{F}}) \circ \psi(t\Delta_{\mathcal{F}}) \\
&= 2\operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ d_{\mathcal{F}}^- \circ \delta_{\mathcal{F}}^+ \circ \psi'(t\Delta_{\mathcal{F}}^+) \circ \psi(t\Delta_{\mathcal{F}}^+) \\
&\quad - 2\operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ d_{\mathcal{F}}^+ \circ \delta_{\mathcal{F}}^- \circ \psi'(t\Delta_{\mathcal{F}}^-) \circ \psi(t\Delta_{\mathcal{F}}^-) \\
&\quad + 2\operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \delta_{\mathcal{F}}^- \circ d_{\mathcal{F}}^+ \circ \psi'(t\Delta_{\mathcal{F}}^+) \circ \psi(t\Delta_{\mathcal{F}}^+) \\
&\quad - 2\operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \delta_{\mathcal{F}}^+ \circ d_{\mathcal{F}}^- \circ \psi'(t\Delta_{\mathcal{F}}^-) \circ \psi(t\Delta_{\mathcal{F}}^-) \ .
\end{aligned}$$

On the other hand, since the function $x \mapsto \psi'(x^2)$ is in $\mathcal{A}$, we have

$$\begin{aligned}
&\operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ d_{\mathcal{F}}^\mp \circ \delta_{\mathcal{F}}^\pm \circ \psi'(t\Delta_{\mathcal{F}}^\pm) \circ \psi(t\Delta_{\mathcal{F}}^\pm) \\
&\qquad = \operatorname{Tr} d_{\mathcal{F}}^\mp \circ \int_O \phi_t^* \cdot f(g)\,\Lambda(g) \circ \psi'(t\Delta_{\mathcal{F}}^\pm) \circ \psi(t\Delta_{\mathcal{F}}^\pm) \circ \delta_{\mathcal{F}}^\pm \\
&\qquad = \operatorname{Tr} \psi(t\Delta_{\mathcal{F}}^\pm) \circ \delta_{\mathcal{F}}^\pm \circ d_{\mathcal{F}}^\mp \circ \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \psi'(t\Delta_{\mathcal{F}}^\pm) \\
&\qquad = \operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \psi'(t\Delta_{\mathcal{F}}^\pm) \circ \psi(t\Delta_{\mathcal{F}}^\pm) \circ \delta_{\mathcal{F}}^\pm \circ d_{\mathcal{F}}^\mp \\
&\qquad = \operatorname{Tr} \int_O \phi_g^* \cdot f(g)\,\Lambda(g) \circ \delta_{\mathcal{F}}^\pm \circ d_{\mathcal{F}}^\mp \circ \psi'(t\Delta_{\mathcal{F}}^\pm) \circ \psi(t\Delta_{\mathcal{F}}^\pm) \ ,
\end{aligned}$$

where we have used the well-known fact that, if A is a trace class operator and B is bounded, then AB and BA are trace class operators with the same trace. Therefore $\frac{d}{dt}\operatorname{Tr}^s P_t = 0$ as desired. $\square$

6.5. The global action on the leafwise complex

Let $\mathfrak{G}$ be the holonomy groupoid of $\mathcal{F}$. Since the leaves of Lie foliations have trivial holonomy groups, we have

$$\mathfrak{G} \equiv \{(x,y) \in M \times M \mid x \text{ and } y \text{ lie in the same leaf of } \mathcal{F}\} .$$

This is a C^∞ submanifold of $M \times M$ which contains the diagonal Δ_M. Let $d_{\mathcal{F}}$ be the distance function of the leaves of $\mathcal{F}$. For each $r > 0$, the *r-penumbra* of Δ_M in $\mathfrak{G}$ is defined by

$$\mathrm{Pen}_{\mathfrak{G}}(\Delta_M, r) = \{(x,y) \in \mathfrak{G} \mid d_{\mathcal{F}}(x,y) < r\} .$$

Observe that a subset of $\mathfrak{G}$ has compact closure if and only if it is contained in some penumbra of Δ_M. The product of two elements $(x_1,y_1),(x_2,y_2) \in \mathfrak{G}$ is defined when $y_1 = x_2$, and it is equal to (x_1,y_2). The space of units of $\mathfrak{G}$ is $\Delta_M \equiv M$. The source and target projections $s, r : \mathfrak{G} \to M$ are the restrictions of the first and second factor projections $M \times M \to M$; thus

$$r^{-1}(x) = L_x \times \{x\} , \quad s^{-1}(x) = \{x\} \times L_x$$

for each $x \in M$.

Let S denote the C^∞ vector bundle

$$s^* \bigwedge T\mathcal{F}^* \otimes r^* \bigwedge T\mathcal{F}$$

over $\mathfrak{G}$; thus

$$S_{(x,y)} \equiv \bigwedge T_x\mathcal{F}^* \otimes \bigwedge T_y\mathcal{F} \equiv \mathrm{Hom}(\bigwedge T_y\mathcal{F}^*, \bigwedge T_x\mathcal{F}^*)$$

for each $(x,y) \in \mathfrak{G}$. Let $\omega_{\mathcal{F}}$ be the volume form of the leaves of $\mathcal{F}$ (we assume that $\mathcal{F}$ is oriented). Recall that $C^\infty_c(S)$ is an algebra with the convolution product given by

$$(k_1 \cdot k_2)(x,y) = \int_{L_x} k_1(x,z) \circ k_2(z,y)\, \omega_{\mathcal{F}}(z)$$

for $k_1, k_2 \in C^\infty_c(S)$ and $(x,y) \in \mathfrak{G}$. Recall also that the *global action* of $C^\infty_c(S)$ in $\Omega(\mathcal{F})$ is defined by

$$(k \cdot \alpha)(x) = \int_{L_x} k(x,y)\, \alpha(y)\, \omega_{\mathcal{F}}(y)$$

for $k \in C^\infty_c(S)$, $\alpha \in \Omega(\mathcal{F})$ and $x \in M$.

Consider the lift to $\widetilde{M}$ of the bundle-like metric of M, and its restriction to the leaves of $\widetilde{\mathcal{F}}$. Let $U\Omega(\widetilde{\mathcal{F}}) \subset \Omega(\widetilde{\mathcal{F}})$ be the subcomplex of differential forms α whose covariant derivatives $\nabla^r\alpha$ of arbitrary order r are uniformly bounded; this is a Fréchet space with the metric induced by the seminorms

$$|||\alpha|||_r = \sup\{\nabla^r\alpha(\tilde{x}) \mid \tilde{x} \in \widetilde{M}\} .$$

Observe that $\pi^*(\Omega(\mathcal{F})) \subset U\Omega(\widetilde{\mathcal{F}})$.

The holonomy groupoid $\widetilde{\mathfrak{G}}$ of $\widetilde{\mathcal{F}}$ satisfies the same properties as $\mathfrak{G}$, except that, in $\widetilde{\mathfrak{G}}$, the penumbras of the diagonal $\Delta_{\widetilde{M}}$ have compact closure if and only $\widetilde{M}$ is compact.

The map $\pi\times\pi:\widetilde{M}\times\widetilde{M}\to M\times M$ restricts to a covering map $\widetilde{\mathfrak{G}}\to\mathfrak{G}$, whose group of deck transformations is isomorphic to $\mathrm{Aut}(\pi)$: for each $\sigma\in\mathrm{Aut}(\pi)$, the corresponding element in $\mathrm{Aut}(\widetilde{\mathfrak{G}}\to\mathfrak{G})$ is the restriction $\sigma\times\sigma:\widetilde{\mathfrak{G}}\to\widetilde{\mathfrak{G}}$.

Let $\widetilde{S}$ denote the C^∞ vector bundle

$$\tilde{s}^*\bigwedge T\widetilde{\mathcal{F}}^*\otimes\tilde{r}^*\bigwedge T\widetilde{\mathcal{F}}$$

over $\widetilde{\mathfrak{G}}$, and let $C^\infty_\Delta(\widetilde{S})\subset C^\infty(\widetilde{S})$ denote the subspace of sections supported in some penumbra of $\Delta_{\widetilde{M}}$. As above, this set becomes an algebra with the convolution product, and there is a *global action* of $C^\infty_\Delta(\widetilde{S})$ in $U\Omega(\widetilde{\mathcal{F}})$.

Any $k\in C^\infty(S)$ lifts via $\pi\times\pi$ to a section $\tilde{k}\in C^\infty(\widetilde{S})$. Since π restricts to diffeomorphisms of the leaves of $\widetilde{\mathcal{F}}$ to the leaves of $\mathcal{F}$, it follows that $\tilde{k}\in C^\infty_\Delta(\widetilde{S})$ if $k\in C^\infty_c(S)$.

Take any $\psi\in\mathcal{A}$. For each leaf L of $\mathcal{F}$, denoting by Δ_L the Laplacian of L, the spectral theorem defines a smoothing operator $\psi(\Delta_L)$ in $\mathbf{\Omega}(L)$, and the family

$$\{\psi(\Delta_L)\mid L\text{ is a leaf of }\mathcal{F}\}$$

is also denoted by $\psi(\Delta_{\mathcal{F}})$. By [26, Proposition 2.10], the Schwartz kernels k_L of the operators $\psi(\Delta_L)$ can be combined to define a section $k\in C^\infty(S)$, called the *leafwise smoothing kernel* or *leafwise Schwartz kernel* of $\psi(\Delta_{\mathcal{F}})$.

Suppose that the Fourier transform $\hat{\psi}$ of ψ is supported in $[-R,R]$ for some $R>0$. Then, according to the proof of Assertion 1 in [25, page 461], k is supported in the R-penumbra of Δ_M, and thus $k\in C^\infty_c(S)$. Moreover the operator $\psi(D_{\mathcal{F}})$ in $\Omega(\mathcal{F})$, defined by the spectral theorem, equals the operator given by the global action of k.

Consider also the lift $\tilde{k}\in C^\infty_\Delta(\widetilde{S})$, whose global action in $U\Omega(\widetilde{\mathcal{F}})$ defines an operator denoted by $\psi(D_{\widetilde{\mathcal{F}}})$. It is clear that the diagram

$$\begin{array}{ccc} U\Omega(\widetilde{\mathcal{F}}) & \xrightarrow{\psi(D_{\widetilde{\mathcal{F}}})} & U\Omega(\widetilde{\mathcal{F}}) \\ {\scriptstyle\pi^*}\uparrow & & \uparrow{\scriptstyle\pi^*} \\ \Omega(\mathcal{F}) & \xrightarrow{\psi(D_{\mathcal{F}})} & \Omega(\mathcal{F}) \end{array} \tag{6.5}$$

commutes.

Any function $\psi\in\mathcal{A}$ with compactly supported Fourier transform can be modified as follows to achieve the condition of being supported in $[-R,R]$. For each $t>0$, let $\psi_t\in\mathcal{A}$ be the function defined by $\psi_t(x)=\psi(tx)$.

Lemma 6.4. *If $\hat{\psi}$ is compactly supported for some $\psi\in\mathcal{A}$, then $\widehat{\psi_t}$ is supported in $[-R,R]$ for t small enough.*

Proof. This holds because $\widehat{\psi_t}(\xi)=\frac{1}{t}\hat{\psi}(\frac{\xi}{t})$. □

6.6. Schwartz kernels

Let ϕ, f, ψ and P be like in Section 6.3 such that $\hat{\psi}$ is compactly supported. Take some $R > 0$ so that $\operatorname{supp}\hat{\psi} \subset [-R, R]$. Let $k \in C^\infty_c(S)$ be the leafwise kernel of $\psi(D_{\mathcal{F}})$, and let $\tilde{k} \in C^\infty_\Delta(\widetilde{S})$ be the lift of k, whose action in $\Omega(\widetilde{\mathcal{F}})$ defines the operator $\psi(D_{\widetilde{\mathcal{F}}})$ (Section 6.5).

Let $\tilde{\phi} : \widetilde{M} \times O \to \widetilde{M}$ be a C^∞ lift of ϕ. Define $\widetilde{P} : U\Omega(\widetilde{\mathcal{F}}) \to U\Omega(\widetilde{\mathcal{F}})$ by

$$\widetilde{P} = \int_O \tilde{\phi}_g^* \cdot f(g)\, \Lambda(g) \circ \psi(D_{\widetilde{\mathcal{F}}}).$$

The commutativity of the diagram

$$\begin{array}{ccc} U\Omega(\widetilde{\mathcal{F}}) & \xrightarrow{\widetilde{P}} & U\Omega(\widetilde{\mathcal{F}}) \\ {\scriptstyle \pi^*}\uparrow & & \uparrow{\scriptstyle \pi^*} \\ \Omega(\mathcal{F}) & \xrightarrow{P} & \Omega(\mathcal{F}) \end{array}$$

follows from the commutativity of (6.5).

Let $\omega_{\widetilde{\mathcal{F}}}$ be the volume form of the leaves of $\widetilde{\mathcal{F}}$, which can be also considered as a differential form on M that vanishes when some vector is orthogonal to the leaves. Thus the volume form of $\widetilde{M}$ is $\omega_{\widetilde{M}} = D^*\Lambda \wedge \omega_{\widetilde{\mathcal{F}}}$ with the right choice of orientations. For $\tilde{x} \in \widetilde{M}$ and $\alpha \in U\Omega(\widetilde{\mathcal{F}})$, we have

$$\begin{aligned} (\widetilde{P}\alpha)(\tilde{x}) &= (\int_O \tilde{\phi}_g^* \cdot f(g)\, \Lambda(g) \circ \psi(D_{\widetilde{\mathcal{F}}})\alpha)(\tilde{x}) \\ &= \int_O \tilde{\phi}_g^*((\psi(D_{\widetilde{\mathcal{F}}})\alpha)(\tilde{\phi}_g(\tilde{x})) \cdot f(g)\, \Lambda(g) \\ &= \int_O \int_{\widetilde{L}_{\tilde{x}}} \tilde{\phi}_g^* \circ \tilde{k}(\tilde{\phi}_g(\tilde{x}), \tilde{y})(\alpha(\tilde{y}))\, \omega_{\widetilde{\mathcal{F}}}(\tilde{y}) \cdot f(g)\, \Lambda(g) \\ &= \int_{\phi(\widetilde{L}_{\tilde{x}} \times O)} \tilde{\phi}_g^* \circ \tilde{k}(\tilde{\phi}_g(\tilde{x}), \tilde{y})(\alpha(\tilde{y})) \cdot f(g)\, \omega_{\widetilde{M}}(\tilde{y}) \end{aligned}$$

by Corollary 4.8, where $g \in O$ is determined by the condition $\tilde{y} \in \tilde{\phi}_g(\widetilde{L}_{\tilde{x}})$, which means $g = D(\tilde{x})^{-1} D(\tilde{y})$ by Lemma 4.7. So we can say that $\widetilde{P}$ is given by the Schwartz kernel $\tilde{p}$ defined by

$$\tilde{p}(\tilde{x}, \tilde{y}) = \begin{cases} \tilde{\phi}_g^* \circ \tilde{k}(\tilde{\phi}_g(\tilde{x}), \tilde{y}) \cdot f(g) & \text{if } \tilde{y} \in \tilde{\phi}(\widetilde{L}_{\tilde{x}} \times O) \\ 0 & \text{otherwise} \end{cases} \tag{6.6}$$

for $g \in O$ as above. It follows that

$$p(x, y) = \sum_{\sigma \in \operatorname{Aut}(\pi)} \tilde{p}(\tilde{x}, \sigma(\tilde{y}))\,, \tag{6.7}$$

where $\tilde{x} \in \pi^{-1}(x)$, $\tilde{y} \in \pi^{-1}(y)$, and we use identifications $T_{\tilde{x}}\widetilde{\mathcal{F}} \equiv T_x\mathcal{F}$ and $T_{\sigma(\tilde{y})}\widetilde{\mathcal{F}} \equiv T_y\mathcal{F}$ given by π_*.

For each $x \in M$, $\tilde{x} \in \widetilde{M}$ and $r > 0$, let $B_{\mathcal{F}}(x,r)$ and $B_{\widetilde{\mathcal{F}}}(\tilde{x},r)$ be the r-balls of centers x and $\tilde{x}$ in L_x and $\widetilde{L}_{\tilde{x}}$, respectively. Let O_1 be an open subset of G whose closure is compact and contained in O. By the compactness of $M \times \overline{O}_1$, there is some $R_1 > 0$ such that

$$B_{\mathcal{F}}(\phi_g(x), R) \subset \phi_g(B_{\mathcal{F}}(x, R_1)) \tag{6.8}$$

for all $x \in M$ and all $g \in O_1$. So

$$B_{\widetilde{\mathcal{F}}}(\tilde{\phi}_g(\tilde{x}), R) \subset \tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1)) \tag{6.9}$$

for all $\tilde{x} \in \widetilde{M}$ and all $g \in O_1$ because π restricts to isometries of the leaves of $\widetilde{\mathcal{F}}$ to the leaves of $\mathcal{F}$.

Lemma 6.5. *Each $g \in O$ has a neighborhood O_1 as above such that*

$$\pi : \tilde{\phi}(\overline{B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1)} \times O_1) \to M$$

is injective for any $\tilde{x} \in \widetilde{M}$.

Proof. Since M is compact, there exists a compact subset $K \subset \widetilde{M}$ with $\pi(K) = M$. Notice that, if the statement holds for some $\tilde{x} \in \widetilde{M}$, then it also holds for all points in the $\operatorname{Aut}(\pi)$-orbit of $\tilde{x}$. So, if the statement fails, there exist sequences $\tilde{x}_i, \tilde{y}_i \in \widetilde{M}$ and $\sigma_i \in \operatorname{Aut}(\pi)$ such that $\tilde{x}_i \in K$, $\sigma_i \neq \mathrm{id}_{\widetilde{M}}$, and

$$d_{\widetilde{M}}(\{\tilde{y}_i, \sigma_i(\tilde{y}_i)\}, \tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}_i, R_1))) \to 0$$

as $i \to \infty$; observe that $D(\tilde{x}_i)^{-1} D(\tilde{y}_i) \to g$ by Lemma 4.7. Since K is compact, we can assume that there exists $\lim_i \tilde{x}_i = \tilde{x} \in \widetilde{M}$, where $d_{\widetilde{M}}$ denotes the distance function of $\widetilde{M}$. Hence $\tilde{y}_i$ and $\sigma_i(\tilde{y}_i)$ approach $\tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1))$. Since $\tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1))$ has compact closure, it follows that $\tilde{y}_i$ and $\sigma_i(\tilde{y}_i)$ lie in some compact neighborhood Q of $\tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1))$ for infinitely many indices i, yielding $\sigma_i(Q) \bigcap Q \neq \emptyset$. So there is some $\sigma \in \operatorname{Aut}(\pi)$ such that $\sigma_i = \sigma$ for infinitely many indices i. In particular, $\sigma \neq \mathrm{id}_{\widetilde{M}}$.

On the other hand, since $\tilde{y}_i$ and $\sigma_i(\tilde{y}_i)$ approach $\tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1))$, which has compact closure, we can assume that there exist $\lim_i \tilde{y}_i = \tilde{y}$ and $\lim_i \sigma_i(\tilde{y}_i) = \sigma(\tilde{y})$ in $\overline{\tilde{\phi}_g(B_{\widetilde{\mathcal{F}}}(\tilde{x}, R_1))}$, which is contained in the leaf $\tilde{\phi}_g(\widetilde{L}_{\tilde{x}})$ (a fiber of D). So

$$D(\tilde{y}) = D(\sigma(\tilde{y})) = h(\sigma) \cdot D(\tilde{y}) \ ,$$

yielding $h(\sigma) = e$, and thus $\sigma = \mathrm{id}_{\widetilde{M}}$ because h is injective. This contradiction concludes the proof. □

From now on, assume that ϕ satisfies (6.8) and the property of the statement of Lemma 6.5 with some fixed open subset $O_1 \subset O$ which contains the support of f.

Corollary 6.6. *The map π is injective on the support of $\tilde{p}(\tilde{x}, \cdot)$ for any $\tilde{x} \in \widetilde{M}$.*