Operator Theory: Advances and Applications
Vol. 179

Recent Advances in Matrix and Operator Theory

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Birkhäuser
Basel · Boston · Berlin

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2000 Mathematics Subject Classification: 15Axx, 28C20, 30A38, 30E05, 34A55, 34D20, 34D45, 46E22, 46J20, 46L40, 47Axx, 47B35, 47J10, 49N45, 60G20, 60H40, 65F05, 65F10, 65F15, 65H17, 70G30, 90C22

Library of Congress Control Number: 2007937322

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 978-3-7643-8538-5 Birkhäuser Verlag AG, Basel - Boston - Berlin

Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Cover design: Heinz Hiltbrunner, Basel
Printed in Germany

ISBN 978-3-7643-8538-5 e-ISBN 978-3-7643-8539-2

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

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Editorial Introduction

This volume contains the proceedings of the International Workshop on Operator Theory and Applications (IWOTA) which was held at the University of Connecticut, Storrs, USA, July 24–27, 2005. This was the sixteenth IWOTA; in fact, the workshop was held biannually since 1981, and annually in recent years (starting in 2002) rotating among ten countries on three continents. Here is the list of the fifteen workshops:

IWOTA'1981: Santa Monica, California, USA (J.W. Helton, Chair)
IWOTA'1983: Rehovot, Israel (H. Dym, Chair)
IWOTA'1985: Amsterdam, The Netherlands (M.A. Kaashoek, Chair)
IWOTA'1987: Mesa, Arizona, USA (L. Rodman, Chair)
IWOTA'1989: Rotterdam, The Netherlands (H. Bart, Chair)
IWOTA'1991: Sapporo, Hokkaido, Japan (T. Ando, Chair)
IWOTA'1993: Vienna, Austria (H. Langer, Chair)
IWOTA'1995: Regensburg, Germany (R. Mennicken, Chair)
IWOTA'1996: Bloomington, Indiana, USA (H. Bercovici, C. Foias, Co-chairs)
IWOTA'1998: Groningen, The Netherlands (A. Dijksma, Chair)
IWOTA'2000: Faro, Portugal (A.F. dos Santos, Chair)
IWOTA'2002: Blacksburg, Virginia, USA (J. Ball, Chair)
IWOTA'2003: Cagliari, Italy (S. Seatzu, C. van der Mee, Co-Chairs)
IWOTA'2004: Newcastle upon Tyne, UK (M.A. Dritschel, Chair)
IWOTA'2005: Storrs, Connecticut, USA (V. Olshevsky, Chair)

The aim of the 2005 IWOTA was to review recent advances in operator theory and its applications to several areas including mathematical systems theory and control theory.

Among the main topics of the workshop was the study of structured matrices, their applications, and their role in the design of fast and numerically reliable algorithms. This topic had already received a considerable attention at IWOTA'2002 and IWOTA'2003 when the main focus was mostly on the structures of Toeplitz, Hankel and Pick types. In the year 2005 the interest shifted towards matrices with quasiseparable structure.

The IWOTA'2005 was made possible through the generous financial support of National Science Foundation (award : 0536873) as well as thanks to the funds of the College of Arts and Sciences and of the Research Foundation of the University of Connecticut. All this support is acknowledged with a gratitude.

Joseph Ball, Yuli Eidelman, William Helton,
Vadim Olshevsky, and James Rovnyak (Editors)

Operator Theory:
Advances and Applications, Vol. 179, 1–24

Inverse Problems for First-Order Discrete Systems

Daniel Alpay and Israel Gohberg

Abstract. We study inverse problems associated to first-order discrete systems in the rational case. We show in particular that every rational function strictly positive on the unit circle is the spectral function of such a system. Formulas for the coefficients of the system are given in terms of realizations of the spectral function or in terms of a realization of a spectral factor. The inverse problems associated to the scattering function and to the reflection coefficient function are also studied. An important role in the arguments is played by the state space method. We obtain formulas which are very similar to the formulas we have obtained earlier in the continuous case in our study of inverse problems associated to canonical differential expressions.

Mathematics Subject Classification (2000). Primary: 34A55, 49N45, 70G30; Secondary: 93B15, 47B35.

Keywords. Inverse problems, spectral function, scattering function, Schur parameters, state space method.

1. Introduction

Here we continue to study first-order discrete systems. We defined the characteristic spectral functions associated to a first-order discrete in [7] and studied the corresponding inverse problems in [8] for scalar systems. In the matrix-valued case, see [3], a system of equations of the form

$$X_n(z) = \begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* \begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix} X_{n-1}(z), \quad n = 1, 2, \ldots, \tag{1.1}$$

is called a canonical discrete first-order one-sided system. The sequence of matrices (α_n, β_n) is not arbitrary, but has the following property: there exists a sequence Δ

Daniel Alpay wishes to thank the Earl Katz family for endowing the chair which supported his research.

of strictly positive block diagonal matrices in $\mathbb{C}^{2p\times 2p}$ such that

$$\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix} J\Delta_n \begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* = J\Delta_{n-1}, \quad n = 1, 2, \ldots, \tag{1.2}$$

where

$$J = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}.$$

The sequence is then called Δ-admissible. In the scalar case (that is, when $p = 1$) condition (1.2) forces $\alpha_n = \beta_n^*$ (see [3]). Still for $p = 1$ these systems arise as the discretization of the telegrapher equation; see [7] for a discussion and references. An *a posteriori* motivation for the study of such systems is the fact that we obtain formulas very close to the ones we proved in the continuous case in our study of inverse problems associated to canonical differential expressions. To be more precise we need to present our setting in greater details. We first gather the main results from [3] needed in the sequel. Let

$$Z = \begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix} \quad \text{and} \quad F_n = \begin{pmatrix} 0 & \beta_n^* \\ \alpha_n^* & 0 \end{pmatrix}, \quad n = 1, 2, \ldots.$$

Under the hypothesis

$$\sum_{n=1}^{\infty} (\|\alpha_n\| + \|\beta_n\|) < \infty, \tag{1.3}$$

the infinite product

$$Y(z) = \left(\prod_{n=1}^{\overset{\curvearrowleft}{\infty}} (I_{2p} + Z^{-n} F_n Z^n) \right) \tag{1.4}$$

converges absolutely and uniformly on the unit circle, and the functions

$$X_n(z) = Z^n \left((I_{2p} + Z^{-n} F_n Z^n) \cdots (I_{2p} + Z^{-1} F_1 Z) \right) Y(z)^{-1}, \quad n = 1, 2, \ldots,$$

define the unique $\mathbb{C}^{2p\times 2p}$-valued solution to the system (1.1) with the property that

$$\lim_{n\to\infty} \begin{pmatrix} z^{-n} I_p & 0 \\ 0 & I_p \end{pmatrix} X_n(z) = I_{2p}, \quad |z| = 1. \tag{1.5}$$

See [3, Section 2.1]. This solution is called the *fundamental solution* of the first-order discrete system (1.1). The function $Y(z)^{-1}$ is called the *asymptotic equivalence matrix function*; see [3, Section 2.2]. Under the supplementary hypothesis

$$\lim_{n\longrightarrow\infty} \Delta_n > 0 \tag{1.6}$$

the function $Y(z)$ allows to define the characteristic spectral functions of the system (1.1). We note that when (1.6) is not in force the situation seems to be much more involved, and leads to degenerate cases. Furthermore, conditions such as (1.2) and (1.6) seem to be specific of the discrete case; no counterpart of these conditions is needed in the continuous case.

Let

$$\lim_{n\to\infty} \Delta_n = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}. \tag{1.7}$$

The function

$$\begin{aligned} W(z) &= \left((Y_{21}+Y_{22})(1/z)\right)^{-1} \delta_2^{-1} \left((Y_{21}+Y_{22})(1/z)\right)^{-*} \\ &= \left((Y_{11}+Y_{12})(1/z)\right)^{-1} \delta_1^{-1} \left((Y_{11}+Y_{12})(1/z)\right)^{-*} \end{aligned}$$

is called the *spectral function*. The Weyl function is the uniquely defined function $N(z)$ analytic in the closed unit disk such that $N(0) = iI_p$ and

$$W(z) = \mathrm{Im}\ N(z), \quad |z| = 1.$$

Associated to N is the reproducing kernel space of functions with reproducing kernel $\frac{N(z)-N(w)^*}{z-w^*}$ and denoted by $\mathcal{L}(N)$. The function $W(z)$ is the spectral function of the unitary operator U defined in $\mathcal{L}(N)$ by

$$(U - \alpha I)^{-1} f(z) = \frac{f(z) - f(\alpha)}{z - \alpha}, \quad |\alpha| \neq 1.$$

See [11].

From (1.5) follows that there exists a $\mathbb{C}^{2p\times p}$-valued solution $B_n(z)$ to (1.1) with the following properties:

(a) $\begin{pmatrix} I_p & -I_p \end{pmatrix} B_0(z) = 0$, and
(b) $\begin{pmatrix} 0 & I_p \end{pmatrix} B_n(z) = I_p + o(n), \quad |z| = 1$.

It then holds that

$$\begin{pmatrix} I_p & 0 \end{pmatrix} B_n(z) = z^n S(z) + o(n)$$

where

$$S(z) = (Y_{11}(z) + Y_{12}(z))(Y_{21}(z) + Y_{22}(z))^{-1}. \tag{1.8}$$

The function (1.8) is called the *scattering matrix function* associated to the discrete system. The scattering matrix function has the following properties: it is in the Wiener algebra $\mathcal{W}^{p\times p}$ (see the end of the section for the definition), admits a Wiener–Hopf factorization and is such that

$$S(z)^* \delta_1 S(z) = \delta_2, \quad |z| = 1. \tag{1.9}$$

See [3, Section 2.3]. The inverse scattering problem considered in this paper is defined as follows: given a function $S(z)$ which admits a Wiener–Hopf factorization and satisfies moreover the condition (1.9) for some matrices δ_1 and δ_2, is $S(z)$ the scattering function of a first-order discrete system?

Some preliminary notation and remarks are needed to define the *reflection coefficient function*. First, for

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in \mathbb{C}^{2p\times 2p} \quad \text{and} \quad X \in \mathbb{C}^{p\times p}$$

we define the linear fractional transformation $T_M(X)$:

$$T_M(X) = (M_{11}X + M_{12})(M_{21}X + M_{22})^{-1}.$$

Recall that the semi-group property

$$T_{\Theta_1\Theta_2}(X) = T_{\Theta_1}(T_{\Theta_2}(X))$$

holds when the three matrices $T_{\Theta_2}(X)$, $T_{\Theta_1}(T_{\Theta_2})(X)$ and $T_{\Theta_1\Theta_2}(X)$ are well defined. Next, it follows from (1.2) that the matrices

$$C_n = \Delta_n^{1/2}\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* \Delta_{n-1}^{-1/2} \tag{1.10}$$

are J-unitary: $C_n^* J C_n = J$. Moreover, for every $n \in \mathbb{N}$ the solution $\Psi_n(z)$ of the system

$$\Psi_n(z) = \Psi_{n-1}(z)C_n^*\begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix}, \quad n = 1, 2, \dots \quad \text{and} \quad \Psi_0(z) = I_{2p} \tag{1.11}$$

is a matrix-valued function whose entries are polynomials of degree at most n and which is J-inner:

$$J - \Theta(z)J\Theta(z)^* \begin{cases} \leq 0, & |z| < 1, \\ = 0, & |z| = 1. \end{cases} \tag{1.12}$$

The *reflection coefficient function* is defined to be

$$R(z) = \lim_{n\to\infty} T_{\Psi_n(z)}(0).$$

We proved in [3, Section 2.4] that $R(z)$ belongs to the Wiener algebra $\mathcal{W}_+^{p\times p}$ and takes strictly contractive values on the unit circle. We also proved in [3, Section 2.4] that

$$R(z) = \frac{1}{z}Y_{21}(\overline{z})^*(Y_{22}(\overline{z}))^{-*} = \frac{1}{z}(Y_{11}(1/z))^{-1}Y_{12}(1/z), \quad |z| = 1,$$

and that the reflection coefficient function and the Weyl function are related by the formula

$$N(z) = i(I_p - zR(z))(I_p + zR(z))^{-1}. \tag{1.13}$$

This paper presents the solution of the inverse spectral problem in the rational case. We also briefly discuss how to recover the system using the scattering function or the reflection coefficient function. In the paper [8], where we considered the scalar case, a key role was played by the description of the solutions of an underlying Nehari problem which are unitary and admit a Wiener–Hopf factorization. The point of view in the present paper is different. A key tool is a certain uniqueness result in the factorization of J-inner polynomial functions (see Theorem 2.4).

We would like to mention that the formulas we obtain in Theorems 4.2 and 4.3 (that is, when one is given a minimal realization of the spectral function or a minimal realization of a spectral factor, respectively) are very similar to the formulas which we obtained earlier in the continuous case, in our study of inverse problems associated to canonical differential expressions with rational spectral data; see in particular formulas (4.7) and (4.12), which are the counterparts of [6, (3.1) p. 9] and [6, Theorem 3.5 p. 9], respectively.

The paper consists of five sections besides the introduction and its outline is as follows. In the second section we review part of the theory of certain finite dimensional reproducing kernel Hilbert spaces (called $\mathcal{H}(\Theta)$ spaces) which will be needed in the sequel. The inverse spectral problem is studied in Section 3 and the inverse scattering problem in Section 4. In the fifth and last section we consider the inverse problem associated to the reflection coefficient function.

We note that another kind of discrete systems have been studied in [18].

We will denote by $\mathbb{D}$ the open unit disk and by $\mathbb{T}$ the unit circle. The Wiener algebra of Fourier series $\sum_\ell z^\ell w_\ell$ with absolutely summable coefficients:

$$\sum_\ell |w_\ell| < \infty$$

will be denoted by $\mathcal{W}$. By $\mathcal{W}_+$ (resp. $\mathcal{W}_-$) we denote the sub-algebra of elements of $\mathcal{W}$ for which $w_\ell = 0$ for $\ell < 0$ (resp. $\ell > 0$). We denote by $\mathcal{W}^{p\times p}$ (resp. $\mathcal{W}_+^{p\times p}$, resp. $\mathcal{W}_-^{p\times p}$) the algebra of matrices with entries in $\mathcal{W}$ (resp. in $\mathcal{W}_+$, resp. in $\mathcal{W}_-$).

Finally, we denote by $\mathbb{C}_J$ the space $\mathbb{C}^{2p}$ endowed with the indefinite inner product

$$\langle f, g\rangle_{\mathbb{C}_J} = g^* J f, \quad f, g \in \mathbb{C}^{2p}. \tag{1.14}$$

2. Reproducing kernel Hilbert spaces

First recall that a Hilbert space $\mathcal{H}$ of $\mathbb{C}^k$-valued functions defined on a set Ω is called a *reproducing kernel Hilbert space* if there is a $\mathbb{C}^{k\times k}$-valued function $K(z, w)$ defined on $\Omega \times \Omega$ and with the following properties:

(i) For every $w \in \Omega$ and every $c \in \mathbb{C}^k$ the function $z \mapsto K(z, w)c$ belongs to $\mathcal{H}$.

(ii) It holds that

$$\langle f(z), K(z, w)c\rangle_{\mathcal{H}} = c^* f(w).$$

The function $K(z, w)$ is called the reproducing kernel of the space; it is positive in the sense that for every $\ell \in \mathbb{N}^*$ and every $w_1, \ldots, w_\ell \in \Omega$ the block matrix with ij block entry $K(w_i, w_j)$ is non-negative. Conversely, to any positive function corresponds a uniquely defined reproducing kernel Hilbert space with reproducing kernel the given positive function; see [9], [19], [1].

Finite dimensional reproducing kernel spaces with reproducing kernel of the form

$$K_\Theta(z, w) = \frac{J - \Theta(z) J \Theta(w)^*}{1 - zw^*}$$

have been studied in [2] and [4]. They correspond to rational functions which are J-unitary on the unit circle (but they may have singularities on the unit circle). In this work, a special role is played by the class $P(J)$ of $\mathbb{C}^{2p\times 2p}$-valued polynomial functions Θ which are J-inner (see (1.12) for the definition).

For $\Theta \in P(J)$ the function $K_\Theta(z, w)$ defined above is positive (in the sense of reproducing kernels) in $\mathbb{C}$. We denote by $\mathcal{H}(\Theta)$ the associated reproducing kernel Hilbert space and gather in the next theorem the main features of these spaces which will be used in the sequel. In the statement, deg Θ denotes the McMillan degree of Θ and $\mathbf{H}_{2,J}$ denotes the Kreĭn space of pairs of functions $\begin{pmatrix} f(z) \\ g(z) \end{pmatrix}$ with f and g in the Hardy space $\mathbf{H}_2^p$ and indefinite inner product

$$\left[\begin{pmatrix} f(z) \\ g(z) \end{pmatrix}, \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}\right]_{\mathbf{H}_{2,J}} = \left\langle \begin{pmatrix} f(z) \\ g(z) \end{pmatrix}, J \begin{pmatrix} f(z) \\ g(z) \end{pmatrix} \right\rangle_{\mathbf{H}_2^{2p}}.$$

Furthermore, R_0 denotes the backward shift operator

$$R_0 f(z) = \frac{f(z) - f(0)}{z}.$$

Theorem 2.1. *Let $\Theta \in P(J)$.*

(i) *We have that $R_0 \mathcal{H}(\Theta) \subset \mathcal{H}(\Theta)$.*

(ii) dim $\mathcal{H}(\Theta) = \deg \Theta$.

(iii) det $\Theta(z) = c_\Theta z^{\deg \Theta}$ for some $c_\Theta \in \mathbb{T}$.

(iv) *The space $\mathcal{H}(\Theta)$ is spanned by the columns of the matrix functions*

$$R_0^\ell \Theta(z), \quad \ell = 1, 2, \ldots,$$

and in particular the elements of $\mathcal{H}(\Theta)$ are $\mathbb{C}^{2p}$-valued polynomials.

(v)
$$\mathcal{H}(\Theta) = \mathbf{H}_{2,J} \ominus \Theta \mathbf{H}_{2,J}. \tag{2.1}$$

(vi) *The product of any two elements in $P(J)$ is always minimal, and for Θ_1 and Θ_2 in $P(J)$ it holds that*

$$\mathcal{H}(\Theta_1 \Theta_2) = \mathcal{H}(\Theta_1) \oplus \Theta_1 \mathcal{H}(\Theta_2).$$

Proof. For the proofs of items (i), (ii) and (iv) and further references and information we refer to the papers [2] and [4]. These papers deal with the more general case of rational functions J-unitary on the unit circle (or the real line). To prove (iii) we note (see [2]) that Θ is a minimal product of degree one factors in $P(J)$ and that each one of these elementary factors has determinant equal to z. To prove (2.1) one checks that the space $\mathbf{H}_{2,J} \ominus \Theta \mathbf{H}_{2,J}$ has reproducing kernel $K_\Theta(z,w)$. By uniqueness of the reproducing kernel we have the desired equality. Since (property (iii))

$$\begin{aligned} \det\ \Theta_1\Theta_2(z) &= c_{\Theta_1\Theta_2} z^{\deg\ \Theta_1\Theta_2} \\ &= (\det\ \Theta_1)(\det\ \Theta_2) \\ &= c_{\Theta_1} z^{\deg\ \Theta_1} c_{\Theta_2} z^{\deg\ \Theta_2} \\ &= c_{\Theta_1} c_{\Theta_2} z^{\deg\ \Theta_1 + \deg\ \Theta_2} \end{aligned}$$

we have that

$$\deg\ \Theta_1\Theta_2 = \deg\ \Theta_1 + \deg\ \Theta_2.$$

Thus the product $\Theta_1\Theta_2$ is minimal. Finally from the equality

$$K_{\Theta_1\Theta_2}(z,w) = K_{\Theta_1}(z,w) + \Theta_1(z) K_{\Theta_2}(z,w) \Theta_1(w)^*$$

we see that

$$\mathcal{H}(\Theta_1\Theta_2) = \mathcal{H}(\Theta_1) + \Theta_1\mathcal{H}(\Theta_2).$$

The sum is direct and orthogonal since the product $\Theta_1\Theta_2$ is minimal, and this proves (vi). $\square$

In the next theorem we precise the structure of $\mathcal{H}(\Theta)$ spaces.

Theorem 2.2. *Let $\Theta \in P(J)$. The space $\mathcal{H}(\Theta)$ has a basis which consists of $k \le p$ chains of the form*

$$\begin{aligned} f_1(z) &= u_1, \\ f_2(z) &= zu_1 + u_2, \\ &\vdots \\ f_m(z) &= z^m u_1 + z^{m-1}u_2 + \cdots + u_m, \end{aligned} \tag{2.2}$$

where $u_1, \ldots, u_m \in \mathbb{C}^{2p}$.

Proof. The elements of $\mathcal{H}(\Theta)$ are polynomials (see (iv) of Theorem 2.1) and therefore the only eigenvalue of R_0 is 0, and the corresponding eigenvectors are vectors in $\mathbb{C}^{2p}$. Let $f_1, \ldots, f_k$ be the linear independent elements of $\mathbb{C}^{2p}$ in $\mathcal{H}(\Theta)$. The space spanned by the f_j is a strictly positive subspace of $\mathbf{H}_{2,J}$. On constant vectors the inner product of $\mathbf{H}_{2,J}$ coincides with the inner product of $\mathbb{C}_J$ (see Definition (1.14)) and so $k \le p$. To conclude we note that each Jordan chain corresponding to an eigenvector is of the form (2.2). $\square$

In general we can only state that $m \le \deg \Theta$. Here we are in a more special situation. The $\Psi_n(z)$ defined by (1.11) have moreover the following property, which is important here: $\deg \Psi_n = np$ *and the entries of $\Psi_n(z)$ are scalar polynomials of degree less or equal to n*. Therefore, by Theorem 2.1 the components of the elements of $\mathcal{H}(\Psi_n)$ are polynomials of degree less or equal to $n-1$ and the following theorem shows that the space $\mathcal{H}(\Psi_n)$ is spanned by p chains of length n.

Theorem 2.3. *There exist matrices $S_0, S_1, \ldots, S_{n-1}$ such that a basis of $\mathcal{H}(\Psi_n)$ is given by the columns of $F_0(z), \ldots, F_{n-1}(z)$ where*

$$\begin{aligned} F_0(z) &= \begin{pmatrix} I_p \\ S_0^* \end{pmatrix}, \\ F_1(z) &= z\begin{pmatrix} I_p \\ S_0^* \end{pmatrix} + \begin{pmatrix} 0 \\ S_1^* \end{pmatrix}, \\ &\vdots \\ F_{n-1}(z) &= z^{n-1}\begin{pmatrix} I_p \\ S_0^* \end{pmatrix} + z^{n-2}\begin{pmatrix} 0 \\ S_1^* \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ S_{n-1}^* \end{pmatrix}. \end{aligned} \tag{2.3}$$

Proof. By Theorem 2.2, a basis of $\mathcal{H}(\Psi_n)$ is made of $k \le p$ chains of the form (2.2). Since the components of the elements of $\mathcal{H}(\Psi_n)$ are polynomials of degree less or equal to $n-1$, these chains generate a space of dimension less or equal to kn. On the other hand,

$$\deg \Psi_n = np = \dim \mathcal{H}(\Psi_n).$$

Therefore, $k = p$ and each chain has length n. The space $\mathcal{H}(\Psi_n)$ contains therefore p linearly independent vectors $f_1, f_2, \ldots, f_p \in \mathbb{C}^{2p}$. Set

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_p \end{pmatrix} \stackrel{\text{def.}}{=} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

where X_1 and X_2 are in $\mathbb{C}^{p\times p}$. Since the f_j span a strictly positive subspace of $\mathbf{H}_{2,J}$ we have $X_1^*X_1 > X_2^*X_2$. Thus X_1 is invertible, and we can chose:

$$F_0(z) = \begin{pmatrix} I_p \\ X_2X_1^{-1} \end{pmatrix} \in \mathcal{H}(\Psi_n).$$

We set $S_0^* = X_2X_1^{-1}$. The next p elements in a basis of $\mathcal{H}(\Theta)$ form the columns of a matrix-function of the form

$$zF_0(z) + V = z\begin{pmatrix} I_p \\ S_0^* \end{pmatrix} + V, \quad V \in \mathbb{C}^{2p\times p}.$$

By subtracting a multiple of $F_0(z)$ to this function we obtain $F_1(z)$. The rest of the argument is proved by induction in the same way: if we know at rank ℓ that $F_\ell(z)$ is of the asserted form, then the next p elements in a basis of $\mathcal{H}(\Theta)$ form a matrix-function of the form $zF_\ell(z) + V$. Removing a multiple of $F_0(z)$ from this function we obtain $F_{\ell+1}(z)$. □

The following uniqueness theorem will be used in the solution of the inverse spectral problem; see the proof of Theorem 4.1:

Theorem 2.4. *Let (α_n, β_n) and (α'_n, β'_n) be two admissible sequences with associated sequences of diagonal matrices Δ_n and Δ'_n respectively, normalized by $\Delta_0 = \Delta'_0 = I_{2p}$. Let C_n be given by (1.10) and let C'_n be defined in a similar way, with (α'_n, β'_n) and Δ'_n. Assume that*

$$C_1^*\begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix}\cdots C_m^*\begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix}U = (C'_1)^*\begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix}\cdots (C'_m)^*\begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix}U'$$
$$\stackrel{\text{def.}}{=} \Theta(z),$$

where U and U' are J-unitary constants. Then $U = U'$ and $C_\ell = C'_\ell$ for $\ell = 1, \ldots m$.

Proof. We denote by the superscript $'$ all the quantities related to the C_n' and we set $\Delta_n = \text{diag}\,(d_{1,n}, d_{2,n})$. Equation (1.2) can be rewritten as:

$$\begin{aligned} d_{1,n} - \alpha_n d_{2,n}\alpha_n^* &= d_{1,n-1}, & (2.4)\\ d_{1,n}\beta_n^* &= \alpha_n d_{2,n} & (2.5)\\ d_{2,n} - \beta_n d_{1,n}\beta_n^* &= d_{2,n-1}. & (2.6)\end{aligned}$$

We set

$$\theta_n(z) = C_n^* \begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix}, \tag{2.7}$$

so that $\Theta(z) = \theta_1(z)\cdots\theta_m(z)$.

By Theorem 2.1 (item (vi)) we have:

$$\begin{aligned}\mathcal{H}(\Theta) &= \mathcal{H}(\theta_1) \oplus \theta_1\mathcal{H}(\theta_2) \oplus \theta_1\theta_2\mathcal{H}(\theta_3) \oplus \cdots \\ &= \mathcal{H}(\theta_1') \oplus \theta_1'\mathcal{H}(\theta_2') \oplus \theta_1'\theta_2'\mathcal{H}(\theta_3') \oplus \cdots .\end{aligned}$$

By Theorem 2.3, the constant functions of $\mathcal{H}(\Theta)$ span both the spaces $\mathcal{H}(\theta_1)$ and $\mathcal{H}(\theta_1')$. Thus,

$$\mathcal{H}(\theta_1) = \mathcal{H}(\theta_1').$$

These two spaces have the same reproducing kernel and we get

$$K_{\theta_1}(z,w) = K_{\theta_1'}(z,w).$$

Since

$$\begin{aligned} K_{\theta_1}(z,w) &= \frac{J - C_1^*\begin{pmatrix} zw^*I_p & 0 \\ 0 & -I_p\end{pmatrix} C_1}{1 - zw^*} \\ &= \frac{J - C_1^*\begin{pmatrix} (zw^*-1+1)I_p & 0 \\ 0 & -I_p\end{pmatrix} C_1}{1 - zw^*} \\ &= \frac{J - C_1^*JC_1}{1-zw^*} + C_1^*\begin{pmatrix} I_p & 0 \\ 0 & 0\end{pmatrix} C_1 \\ &= \begin{pmatrix} I_p \\ \beta_1 \end{pmatrix} d_{1,1} \begin{pmatrix} I_p & \beta_1^* \end{pmatrix}, \end{aligned} \tag{2.8}$$

we get

$$\begin{pmatrix} I_p \\ \beta_1 \end{pmatrix} d_{1,1} \begin{pmatrix} I_p & \beta_1^* \end{pmatrix} = \begin{pmatrix} I_p \\ \beta_1' \end{pmatrix} d_{1,1}' \begin{pmatrix} I_p & (\beta_1')^* \end{pmatrix}.$$

It follows that $d_{1,1} = d_{1,1}'$ and $\beta_1 = \beta_1'$. From the normalization $\Delta_0 = \Delta_0' = I_{2p}$ and equations (2.4)–(2.6) it follows that $d_{2,1} = d_{2,1}'$ and $\alpha_1 = \alpha_1'$.

By induction we see that

$$\mathcal{H}(\theta_n) = \mathcal{H}(\theta_n'), \quad n = 2, 3, \ldots.$$

But, in a way similar to (2.8),

$$K_{\theta_n}(z,w) = \frac{J - C_n^* \begin{pmatrix} zw^* I_p & 0 \\ 0 & -I_p \end{pmatrix} C_n}{1 - zw^*} = \Delta_{n-1}^{-1/2} \begin{pmatrix} I_p \\ \beta_n \end{pmatrix} d_{1,n} \begin{pmatrix} I_p & \beta_n^* \end{pmatrix} \Delta_{n-1}^{-1/2},$$

and it follows from $\Delta_{n-1} = \Delta'_{n-1}$ (induction hypothesis at rank $n-1$) that $\beta_n = \beta'_n$ and $d_{n,1} = d'_{n,1}$. Equations (2.4)–(2.6) imply then that $\alpha_n = \alpha'_n$ and $d_{n,2} = d'_{n,2}$, and finally that $U = U'$. □

Theorem 2.5. *Let $X(z)$ be analytic and contractive in the open unit disk and let $R(z) = \lim_{n\to\infty} T_{\Psi_n(z)}(X(z))$. Let $R(z) = R_0 + R_1 z + \cdots$ be the Taylor expansion of $R(z)$ at the origin. Then, the space $\mathcal{H}(\Psi_n)$ is spanned by the functions* (2.3) *with the coefficients $R_0, R_1, \ldots, R_{n-1}$.*

Proof. Let $A_0, A_1, \ldots$ be matrices such that $\mathcal{H}(\Psi_n)$ is spanned by the columns of the functions

$$F_0(z) = \begin{pmatrix} I_p \\ A_0^* \end{pmatrix},$$

$$F_1(z) = z \begin{pmatrix} I_p \\ A_0^* \end{pmatrix} + \begin{pmatrix} 0 \\ A_1^* \end{pmatrix},$$

$$\vdots$$

$$F_{n-1}(z) = z^{n-1} \begin{pmatrix} I_p \\ A_0^* \end{pmatrix} + z^{n-2} \begin{pmatrix} 0 \\ A_1^* \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ A_{n-1}^* \end{pmatrix}.$$

Since $\mathcal{H}(\Psi_n) = \mathbf{H}_{2,J} \ominus \Psi_n \mathbf{H}_{2,J}$ (see Theorem 2.1) we have that

$$\begin{aligned} \begin{pmatrix} I_p & -A_0 \end{pmatrix} \Psi_n(0) &= 0 \\ \begin{pmatrix} I_p & -A_0 \end{pmatrix} \Psi'_n(0) + \begin{pmatrix} 0 & -A_1 \end{pmatrix} \Psi_n(0) &= 0 \\ &\vdots \end{aligned} \tag{2.9}$$

The first equation leads to $T_{\Psi_n(z)}(0) = A_0$. Letting $n \to \infty$ we have

$$A_0 = R(0) = R_0.$$

The second equation will lead in a similar way to $R'(0) = A_1$. More generally, equations (2.9) lead to

$$\begin{pmatrix} I_p & -(A_0 + A_1 + \cdots + A_{n-1} z^{n-1}) \end{pmatrix} \Psi_n(z) = O(z^n). \tag{2.10}$$

Set

$$\Psi_n(z) = \begin{pmatrix} \alpha_n(z) & \beta_n(z) \\ \gamma_n(z) & \delta_n(z) \end{pmatrix}.$$

Equation (2.10) implies that

$$\beta_n(z) - (A_0 + A_1 + \cdots + A_{n-1} z^{n-1}) \delta_n(z) = O(z^n).$$

From the J-innerness of $\Psi_n(z)$ the matrix-function $\delta_n(z)$ is analytic and invertible in $\mathbb{D}$, with $\|\delta_n(z)^{-1}\| \leq 1$; see [13]. Hence,

$$T_{\Psi_n(z)}(0) = (A_0 + A_1 + \cdots + A_{n-1}z^{n-1}) + O(z^n)$$

and hence the result. □

3. Realization theory

As is well known a rational function $W(z)$ analytic at the origin can be written in the form

$$W(z) = D + zC(I - zA)^{-1}B$$

where $D = W(0)$ and where A, B and C are matrices of appropriate sizes. The realization is called minimal when the size of A is minimal; see [10]. Assume moreover that $W(z)$ is analytic on the unit circle. Then A has no spectrum on the unit circle and the entries of $W(z)$ are in the Wiener algebra $\mathcal{W}$; indeed, let P_0 denote the Riesz projection corresponding to the spectrum of A outside the closed unit disk:

$$P_0 = I - \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta I - A)^{-1} d\zeta.$$

Then,

$$\begin{aligned}
W(z) &= D + zC(I - zA)^{-1}B \\
&= D + zCP_0(I - zA)^{-1}P_0B + zC(I - P_0)(I - zA)^{-1}(I - P_0)B \\
&= D - zCP_0A^{-1}z^{-1}(I - z^{-1}A^{-1})^{-1}P_0B \\
&\quad + zC(I - P_0)(I - zA)^{-1}(I - P_0)B \\
&= D - \sum_{k=0}^{\infty} z^{-k}CP_0A^{-k-1}P_0B \\
&\quad + \sum_{k=0}^{\infty} z^{k+1}C(I - P_0)A^k(I - P_0)B.
\end{aligned}$$

and thus the coefficients r_k in the representation $W(z) = \sum_{\mathbb{Z}} z^k r_k$ (with $|z| = 1$) can be written as

$$r_k = \begin{cases} CA^{k-1}(I - P_0)B, & k > 0, \\ D\delta_{k0} - CA^{k-1}P_0B, & k \leq 0, \end{cases} \tag{3.1}$$

so that

$$\sum_{\mathbb{Z}} \|r_k\| < \infty.$$

The hypotheses of analyticity at the origin and at infinity are restrictive. In fact any rational function analytic on the unit circle belongs to the Wiener algebra.

We now review the relevant theory and follow the analysis in [14]. First recall that any rational function $W(z)$ analytic on the unit circle can be represented as

$$W(z) = I + C(zG - A)^{-1}B,$$

where $zG - A$ is invertible on $\mathbb{T}$; see [14, Theorem 3.1 p. 395]. The separating projection is defined by

$$P = \frac{1}{2\pi i}\int_{\mathbb{T}} G(\zeta G - A)^{-1} d\zeta. \tag{3.2}$$

Next the right equivalence operator E and the associated operator Ω are defined by

$$E = \frac{1}{2\pi i}\int_{\mathbb{T}} (1-\zeta^{-1})(\zeta G - A)^{-1} d\zeta \quad \text{and} \quad \Omega = \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta-\zeta^{-1})(\zeta G - A)^{-1}. \tag{3.3}$$

See [14, Equations (2.2)–(2.4) p. 389]. Then, (see [14, p. 398])

$$r_k = \begin{cases} -CE\Omega^k(I-P)B, & k = 1, 2, \ldots, \\ I - CE(I-P)B, & k = 0, \\ CE\Omega^{-k-1}PB, & k = -1, -2, \ldots . \end{cases}$$

The block entries of T_n^{-1} are now given as follows. Let $A^\times = A - BC$ and define $P^\times, E^\times$ and $\Omega^\times$ in a way analog to P, E and Ω, that is:

$$P^\times = \frac{1}{2\pi i}\int_{\mathbb{T}} G(\zeta G - A^\times)^{-1} d\zeta, \tag{3.4}$$

$$E^\times = \frac{1}{2\pi i}\int_{\mathbb{T}} (1-\zeta^{-1})(\zeta G - A^\times)^{-1} d\zeta, \tag{3.5}$$

and

$$\Omega^\times = \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta-\zeta^{-1})(\zeta G - A^\times)^{-1}. \tag{3.6}$$

Define moreover

$$Q = \frac{1}{2\pi i}\int_{\mathbb{T}} (\zeta G - A)^{-1} d\zeta, \tag{3.7}$$

$$\begin{aligned} V_n &= (I-Q)E^\times(I-P^\times) \\ &\quad + (I-Q)E^\times(\Omega^\times)^{n+1}P^\times + QE^\times(\Omega^\times)^{n+1}(I-P^\times) + QE^\times P^\times, \end{aligned} \tag{3.8}$$

and

$$r_k^\times = \begin{cases} CE^\times(\Omega^\times)^k(I-P^\times)B, & k = 1, 2, \ldots, n, \\ I + CE^\times(I-P^\times)B, & k = 0, \\ -CE^\times(\Omega^\times)^{-k}P^\times B, & k = -1, \ldots, -n, \end{cases}$$

and

$$\begin{aligned} k_{kj}^{(n)} &= CE^\times(\Omega^\times)^{k+1}(I-P^\times)V_n^{-1}(I-Q)E^\times(\Omega^\times)^j P^\times B \\ &\quad - CE^\times(\Omega^\times)^{n-k}P^\times V_n^{-1}QE^\times(\Omega^\times)^{n-j}(I-P^\times)B. \end{aligned}$$

Then, $T_n^{-1} = \left(\gamma_{kj}^{(n)}\right)_{k,j=1,\ldots,n}$ with

$$\gamma_{kj}^{(n)} = r_{k-j}^{\times} + k_{kj}^{(n)}. \tag{3.9}$$

See [14, Theorem 8.2 p. 422].

4. Inverse spectral problem

We focus on the rational case and consider three cases:

1. The weight function is general: it is rational and strictly positive on $\mathbb{T}$.
2. We assume that the weight function is analytic at the origin and at infinity. Then we get concrete formulas.
3. We start from a spectral factor.

The uniqueness theorem (Theorem 2.4) is used in the proof of the following theorem.

Theorem 4.1. *Let $W(z)$ be a rational function without poles on the unit circle and which takes strictly positive values there, and which is normalized by*

$$\frac{1}{2\pi}\int_0^{2\pi} W(e^{it})dt = I_p. \tag{4.1}$$

Then, $W(z)$ is the spectral function of a uniquely determined first-order discrete system normalized by $\Delta_0 = I_{2p}$. The associated first-order discrete system is computed as follows: let

$$W(z) = I + C(zG - A)^{-1}B$$

be a realization of $W(z)$ which is regular on $\mathbb{T}$. Then,

$$\begin{aligned}
\alpha_n = CE^{\times} &\left\{(\Omega^{\times})^n(I-P^{\times}) + (\Omega^{\times})^{n+1}(I-P^{\times})V_n^{-1}(I-Q)E^{\times}P^{\times}\right.\\
&\quad \left. -P^{\times}V_n^{-1}QE^{\times}(\Omega^{\times})^n(I-P^{\times})\right\}B\\
&\times\left\{I + CE^{\times}(I-P^{\times})B + CE^{\times}\Omega^{\times}(I-P^{\times})V_n^{-1}E^{\times}P^{\times}P^{\times}B\right.\\
&\quad \left. -CE^{\times}(\Omega^{\times})^nP^{\times}V_n^{-1}QE^{\times}(\Omega^{\times})^n(I-P^{\times})B\right\}^{-1},\\
\beta_n = CE^{\times} &\left\{(\Omega^{\times})^{(n-1)}P^{\times} + \Omega^{\times}(I-P^{\times})V_n^{-1}(I-Q)(\Omega^{\times})^nP^{\times}\right.\\
&\quad \left. -P^{\times}V_n^{-1}QE^{\times}(I-P^{\times})\right\}B\\
&\times\left\{I + CE^{\times}(I-P^{\times})B\right.\\
&\quad + CE^{\times}(\Omega^{\times})^{(n+1)}(I-P^{\times})V_n^{-1}(I-Q)(\Omega^{\times})^nP^{\times}B\\
&\quad \left. -CE^{\times}P^{\times}V_n^{-1}QE^{\times}(I-P^{\times})B\right\}^{-1},
\end{aligned} \tag{4.2}$$

with associated sequence of diagonal matrices given by

$$\Delta_n = \begin{pmatrix} d_{1,n} & 0 \\ 0 & d_{2,n}\end{pmatrix} \tag{4.3}$$

where

$$\begin{aligned} d_{1,n} &= I + CE^{\times}(I-P^{\times})B + CE^{\times}(\Omega^{\times})^{(n+1)}(I-P^{\times})V_n^{-1}(I-Q)(\Omega^{\times})^nP^{\times}B \\ &\quad - CE^{\times}P^{\times}V_n^{-1}QE^{\times}(I-P^{\times})B, \\ d_{2,n} &= I + CE^{\times}(I-P^{\times})B + CE^{\times}\Omega^{\times}(I-P^{\times})V_n^{-1}E^{\times}P^{\times}P^{\times}B \\ &\quad - CE^{\times}(\Omega^{\times})^nP^{\times}V_n^{-1}QE^{\times}(\Omega^{\times})^n(I-P^{\times})B \end{aligned}$$

for $n = 1, 2, \ldots$. *In these expressions, the quantities* P, E, Ω *and* Q *are given by* (3.2), (3.3) *and* (3.7) *respectively and* $P^{\times}, E^{\times}, Q^{\times}$ *and* V_n *are given by* (3.4), (3.5), (3.6) *and* (3.8) *respectively.*

Proof. We first prove the uniqueness of the associated first-order discrete system. Fix $n > 0$. For every $q > 0$ we have (recall that Ψ_n is defined by (1.11) and θ_n by (2.7))

$$\Psi_{n+q}(z) = \Psi_n(z)\theta_{n+1}(z)\cdots\theta_{n+q}(z),$$

and in particular

$$R(z) = \lim_{q\to\infty} T_{\Psi_n(z)}(T_{\theta_{n+1}(z)\cdots\theta_{n+q}(z)}(0)).$$

By Montel's theorem, the limit

$$R_n(z) = \lim_{q\to\infty} T_{\theta_{n+1}(z)\cdots\theta_{n+q}(z)}(0)$$

exists (via maybe a subsequence). The limit is analytic and contractive in the open unit disk. Thus

$$R(z) = T_{\Psi_n(z)}(R_n(z)).$$

By Theorem 2.5, the space $\mathcal{H}(\Psi_n)$ is built from the first n coefficients of the Taylor expansion of $R(z)$ at the origin.

Assume that there are two first-order discrete systems (normalized by $\Delta_0 = I_{2p}$) and with same spectral function $W(z)$. By formula (1.13) these two systems have the same reflection coefficient function $R(z)$. Denoting by a superscript $'$ the second one, we get $\mathcal{H}(\Psi_n) = \mathcal{H}(\Psi'_n)$ for every $n \geq 0$. By Theorem 2.4 it follows that the two systems are equal.

We now turn to the existence of such a system. The function $W(z)$ is rational and has no poles on the unit circle. It belongs therefore to the Wiener algebra $\mathcal{W}^{p\times p}$. We set $W(e^{it}) = \sum_{\mathbb{Z}} r_je^{ijt}$ (note that $r_0 = I_p$ in view of the normalization (4.1)). The block matrices T_n are strictly positive and it follows from [12] that the pair

$$\alpha_n = \gamma_{n0}^{(n)}(\gamma_{00}^{(n)})^{-1} \quad \text{and} \quad \beta_n = \gamma_{0n}^{(n)}(\gamma_{nn}^{(n)})^{-1}, \quad n = 1, 2, 3\ldots, \tag{4.4}$$

form an admissible sequence, with associated sequence of diagonal matrices given by

$$\Delta_n = \begin{pmatrix} \gamma_{nn}^{(n)} & 0 \\ 0 & \gamma_{00}^{(n)} \end{pmatrix}, \quad n = 0, 1, 2, \ldots. \tag{4.5}$$

The normalization (4.1) implies that $\Delta_0 = I_{2p}$. We now proceed in a number of steps:

STEP 1: *The limits* $\lim_{n\to\infty}\gamma_{00}^{(n)}$ *and* $\lim_{n\to\infty}\gamma_{nn}^{(n)}$ *exist and are strictly positive.*

Set $\Delta_n = \operatorname{diag}\,(d_{1,n}, d_{2,n})$. We have $\gamma_{00}^{(n)} = d_{1,n}$ and $\gamma_{nn}^{(n)} = d_{2,n}$. Formula (3.9) implies that the limits exist. Formulas (2.4)–(2.6) imply that $\gamma_{00}^{(n)}$ and $\gamma_{nn}^{(n)}$ are non-decreasing sequences of positive matrices, and so their limits are invertible since $\Delta_0 > 0$.

Alternatively, one can prove STEP 1 as follows: That the first limit exits follows from the projection method (see [17]). The invertibility of $\lim_{n\to\infty}\gamma_{00}^{(n)}$ is proved in [16, p. 123]. The second limit is reduced to the first one by considering $W(1/z)$. See the end of the proof of Theorem 1.8 in [3] for more information.

Thus (1.6) is in force. From (1.7) we have

$$\lim_{n\to\infty}\gamma_{00}^{(n)} = \delta_1 \quad \text{and} \lim_{n\to\infty}\gamma_{nn}^{(n)} = \delta_2.$$

STEP 2: *Condition* (1.3) *is in force.*

This follows from the explicit formulas (3.9) for $\gamma_{0n}^{(n)}$ and $\gamma_{nn}^{(n)}$.

As proved in [3] it follows from STEP 2 that the first-order discrete system (1.1) has a unique solution $X_n(z)$ such that (1.5) holds:

$$\lim_{n\to\infty}\begin{pmatrix} z^{-n}I_p & 0 \\ o & I_p\end{pmatrix} X_n(z) = \begin{pmatrix} I_p & 0 \\ 0 & I_p\end{pmatrix}, \quad |z| = 1.$$

We set (see [12, p. 80])

$$\begin{aligned} A_n(z) &= \sum_{\ell=0}^{n} z^\ell \gamma_{\ell 0}^{(n)}, & C_n(z) &= \sum_{\ell=0}^{n} z^\ell \gamma_{\ell n}^{(n)}, \\ A_n^\circ(z) &= 2I_p - \sum_{\ell=0}^{n} p_\ell(z)\gamma_{\ell 0}^{(n)}, & C_n^\circ(z) &= \sum_{\ell=0}^{n} p_\ell(z)\gamma_{\ell n}^{(n)}, \end{aligned}$$

where $p_\ell(z) = z^\ell r_0 + 2\sum_{s=1}^{\ell} z^{\ell-s} r_s^*$.

STEP 3: *It holds that*

$$\begin{aligned} \lim_{n\to\infty} A_n(\overline{z})^* &= \delta_2(Y_{21}(z) + Y_{22}(z)), \\ \lim_{n\to\infty} z^{-n}C_n(\overline{z})^* &= \delta_1(Y_{11}(z) + Y_{12}(z)), \quad |z| = 1. \end{aligned} \tag{4.6}$$

Indeed, set

$$\Theta_n(z) = \begin{pmatrix} zC_n(z) & A_n(z) \\ zC_n^\circ(z) & -A_n^\circ(z)\end{pmatrix}.$$

We have (see [12, Theorem 13.2 p. 127])

$$\Theta_n(z)\Delta_n^{-1} = \Theta_{n-1}(z)\Delta_{n-1}^{-1}\begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p\end{pmatrix}\begin{pmatrix} zI_p & 0 \\ 0 & I_p\end{pmatrix}, \quad n = 1, 2, \ldots.$$

It follows that the matrix-functions

$$X_n(z) = \Delta_n^{-1}\begin{pmatrix} z^{-1}I_p & 0 \\ 0 & I_p \end{pmatrix}\Theta_n(\overline{z})^* = \Delta_n^{-1}\begin{pmatrix} C_n(\overline{z})^* & C_n^\circ(\overline{z})^* \\ A_n(\overline{z})^* & -A_n^\circ(\overline{z})^* \end{pmatrix}$$

satisfy the recursion

$$X_n(z) = \begin{pmatrix} I_p & \alpha_n \\ \beta_n & I_p \end{pmatrix}^* \begin{pmatrix} zI_p & 0 \\ 0 & I_p \end{pmatrix} X_{n-1}(z), \quad n = 1, 2, \ldots.$$

Since, as already noticed, $\Delta_0 = I_{2p}$, we have:

$$\Delta_n^{-1}\begin{pmatrix} C_n(\overline{z})^* & C_n^\circ(\overline{z})^* \\ A_n(\overline{z})^* & -A_n^\circ(\overline{z})^* \end{pmatrix}\begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix}\frac{1}{2} = M_n(z),$$

where we recall that $M_n(z)$ is the solution of (1.1) subject to the initial condition $M_0(z) = I_{2p}$. Hence, with $Y(z)$ defined by (1.4),

$$M_n(z) = X_n(z)Y(z) = \Delta_n^{-1}\begin{pmatrix} C_n(\overline{z})^* & C_n^\circ(\overline{z})^* \\ A_n(\overline{z}) & -A_n^\circ(\overline{z})^* \end{pmatrix}\begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix}\frac{1}{2},$$

where $X_n(z)$ is the solution to (1.1) subject to the asymptotic (1.5). Recalling (1.7) we obtain:

$$\begin{aligned}\lim_{n\to\infty}&\begin{pmatrix} z^{-n}I_p & 0 \\ 0 & I_p \end{pmatrix} X_n(z)Y(z) \\ &= \begin{pmatrix} \delta_1^{-1} & 0 \\ 0 & \delta_2^{-1} \end{pmatrix}\begin{pmatrix} \lim_{n\to\infty} z^{-n}C_n(\overline{z})^* & \lim_{n\to\infty} z^{-n}C_n^\circ(\overline{z})^* \\ \lim_{n\to\infty} A_n(\overline{z}) & -\lim_{n\to\infty} A_n^\circ(\overline{z})^* \end{pmatrix}\begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix}\frac{1}{2}.\end{aligned}$$

Hence,

$$\begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix} Y(z)\begin{pmatrix} I_p & I_p \\ I_p & -I_p \end{pmatrix} = \begin{pmatrix} \lim_{n\to\infty} z^{-n}C_n(\overline{z})^* & \lim_{n\to\infty} z^{-n}C_n^\circ(\overline{z})^* \\ \lim_{n\to\infty} A_n(\overline{z}) & -\lim_{n\to\infty} A_n^\circ(\overline{z})^* \end{pmatrix}.$$

In particular we have (4.6).

STEP 4: *$W(z)$ is the spectral function of the first-order discrete system associated to the pair* (4.7).

By [12, Theorem 10.4 p. 116], we have for $|z| = 1$

$$\begin{aligned} W(z) &= \lim_{n\to\infty} A_n(z)^{-*}\gamma_{00}^{(n)}A_n(z)^{-1} \\ &= \lim_{n\to\infty} C_n(z)^{-*}\gamma_{nn}^{(n)}C_n(z)^{-1}. \end{aligned}$$

and thus, still on the unit circle

$$\begin{aligned} W(1/z) &= \lim_{n\to\infty} A_n(\overline{z})^{-*}\gamma_{00}^{(n)}A_n(\overline{z})^{-1} \\ &= \lim_{n\to\infty} C_n(\overline{z})^{-*}\gamma_{nn}^{(n)}C_n(\overline{z})^{-1}. \end{aligned}$$

Hence, by the preceding two steps,

$$\begin{aligned} W(1/z) &= (Y_{21}(z)+Y_{22}(z))^{-1}\delta_1^{-1}\delta_1\delta_1^{-1}(Y_{21}(z)+Y_{22}(z))^{-*} \\ &= (Y_{21}(z)+Y_{22}(z))^{-1}\delta_1^{-1}(Y_{21}(z)+Y_{22}(z))^{-*} \\ &= (Y_{11}(z)+Y_{12}(z))^{-1}\delta_2^{-1}\delta_2\delta_2^{-1}(Y_{11}(z)+Y_{12}(z))^{-*} \\ &= (Y_{11}(z)+Y_{12}(z))^{-1}\delta_2^{-1}(Y_{11}(z)+Y_{12}(z))^{-*} \end{aligned}$$

and hence the result. □

In [3] we called admissible sequences of the form (4.4)–(4.5) Szegö admissible sequences.

In the next theorem we assume that the weight function is analytic at the origin and at infinity. This allows us to use formulas from [15].

Theorem 4.2. *Let $W(z)$ be a rational function analytic at infinity and at the origin, and without poles on the unit circle. Assume that $W(e^{it}) > 0$ for $t \in [0, 2\pi]$ and that the normalization (4.1) is in force. Then, $W(z)$ is the spectral function of a uniquely determined first-order system. The corresponding associated sequence is obtained as follows: let*

$$W(z) = D + zC(I - zA)^{-1}B$$

be a minimal realization of W. Then α_n and β_n are given by

$$\begin{aligned} \alpha_n &= (D - CA^{-1}B)^{-1}CA^{-1}\left((I-P_0)(A^\times)^{-n}\big|_{\ker P_0}\right)^{-1}(I-P_0)B, \\ \beta_n &= -D^{-1}C\left(P_0(A^\times)^n\big|_{\operatorname{Im} P_0}\right)^{-1}P_0A^{-1}B, \end{aligned} \tag{4.7}$$

and the associated sequence of diagonals is given by $\Delta_n = \operatorname{diag}(d_{1,n}, d_{2,n})$ with

$$\begin{aligned} d_{1,n} &= D^{-1} + D^{-1}C(A^\times)^n W_{n+1}^{-1}P_0A^{-1}BD^{-1}, \\ d_{2,n} &= D^{-1} + D^{-1}CW_{n+1}^{-1}P_0A^{-(n+1)}(A^\times)^nBD^{-1}, \end{aligned}$$

where P_0 denotes the Riesz projection corresponding to the spectrum of A outside the closed unit disk,

$$P_0 = I - \frac{1}{2\pi i}\int_{\mathbb{T}}(zI - A)^{-1}dz, \tag{4.8}$$

and where W_n is given by

$$W_n(I - P_0 + P_0A)^{-n}(I - P_0 + P_0A^{\times n}). \tag{4.9}$$

The proof is a special case of the previous theorem. Formulas (4.7) have been proved in our previous paper [3], and are the discrete analogue of [6, (3.1) p. 9], where the potential associated to a canonical differential expression was computed in terms of a minimal realization of the spectral function.

We now turn to the third case, where we start from a spectral factor.

Theorem 4.3. *Let $g_+(z)$ be a $\mathbb{C}^{p\times p}$-valued rational function analytic and invertible in the closed unit disk, and at infinity. Let*

$$W(z) = g_+(z)g_+(1/z^*)^*,$$

and assume that the normalization (4.1) is in force. Then $W(z)$ is the spectral function of a first-order discrete system of type (1.1). Let $g_+(z) = d + zc(I - za)^{-1}b$ be a minimal realization of $g_+(z)$ and let X and Y be the solutions of the Stein equations

$$X - aXa^* = bb^* \tag{4.10}$$

and

$$Y - a^{\times *}Ya^{\times} = (d^{-1}c)^*(d^{-1}c). \tag{4.11}$$

Assume that a is invertible (that is, $W(z)$ is analytic at the origin and at infinity). Then the following formulas hold:

$$\begin{aligned}\alpha_n &= (d - ca^{-1}b)d^*ca^{-1}(a^{\times})^n(I + X(Y - (a^{\times})^{*n}Y(a^{\times})^n))^{-1}(bd^* + aXc^*),\\ \beta_n &= (d(d^* - b^*a^{-*}c^*))^{-1}(cX + db^*a^{-*})\left(I + (Y - (a^{\times *})^nY(a^{\times})^n)X\right)^{-1}(a^{\times *})^n c^*.\end{aligned} \tag{4.12}$$

The associated sequence of diagonals is given by $\Delta_n = \mathrm{diag}\,(d_{1,n}, d_{2,n})$ where

$$\begin{aligned}d_{1,n} &= (d(d^* - c^*a^{-*}b^*))^{-1}\\ &\quad\times \Big(I + (-c(a^{\times})^n(I + X(Y - (a^{\times})^{*n}Y(a^{\times})^n))^{-1}X(a^{\times})^{*(n+1)}\\ &\qquad -d^*a^{-*}c^*(a^{\times})^*(I + (Y - (a^{\times})^{*n}Y(a^{\times})^n)X)\\ &\quad\times(I + (Y - (a^{\times})^{*(n+1)}Y(a^{\times})^{(n+1)})X)^{-1}(a^{\times})^{(n+1)}a^{-*}c^*(d(d^* - b^*a^{-*}c^*)^{-1}\Big),\\ d_{2,n} &= (d(d^* - c^*a^{-*}b^*))^{-1}\\ &\quad\times \left(I - (cX + d^*b^*a^{-*})\right.\\ &\quad\times(I + (Y - (a^{\times})^{*(n+1)}Y(a^{\times})^{(n+1)})X)^{-1}(a^{\times})^*\\ &\quad\times \left(b(d^* - b^*a^{-*}c^*) + ((a^{\times})^{*n}Y(a^{\times})^n - Y)a^{-*}c^*\right)\left.(d(d^* - c^*a^{-*}b^*))^{-1}\right).\end{aligned}$$

Proof. The fact that $W(z)$ is the spectral function of a system (1.1) stems from Theorem 4.1. We now prove formulas (4.12). In the arguments to obtain a formula for the Schur coefficients α_n and β_n in terms of a minimal realization of $g_+(z)$ we make much use of computations from our previous paper [5].

Let $g_+(z) = d + zc(I - za)^{-1}b$ be a minimal realization of $g_+(z)$. By hypothesis the matrix a is invertible. Hence, a minimal realization of $g_+(1/z^*)^*$ is given by

$$\begin{aligned}g_+(1/z^*)^* &= d^* + b^*(zI - a^*)^{-1}c^*\\ &= d^* - b^*a^{-*}c^* + b^*\left((zI - a^*)^{-1} + a^{-*}\right)c^*\\ &= d^* - b^*a^{-*}c^* - zb^*(I - a^{-*})^{-1}c^*\\ &= d^* - b^*a^{-*}c^* - zb^*a^{-*}(I - za^{-*})^{-1}a^{-*}c^*,\end{aligned}$$

and hence the matrices

$$A = \begin{pmatrix} a & -bb^*a^{-*} \\ 0 & a^{-*} \end{pmatrix}, \quad B = \begin{pmatrix} b(d^* - b^*a^{-*}c^*) \\ a^{-*}c^* \end{pmatrix}, \quad C = \begin{pmatrix} c & -db^*a^{-*} \end{pmatrix}, \tag{4.13}$$

and

$$D = d(d^* - b^*a^{-*}c^*) \tag{4.14}$$

define a minimal realization $W(z) = D + zC(I - zA)^{-1}B$ of $W(z)$. See [5, Theorem 3.3 p. 155]). Furthermore, the Riesz projection (4.8) is given by

$$P_0 = \begin{pmatrix} 0 & -X \\ 0 & I \end{pmatrix},$$

where X is the solution of the Stein equation (4.10). We have (see [5, Equation (3.21) p. 156])

$$(A^\times)^n = \begin{pmatrix} (a^\times)^n & 0 \\ Y(a^\times)^n - (a^{\times *})^{-n}Y & (a^{\times *})^{-n} \end{pmatrix}, \tag{4.15}$$

where Y is the solution to the Stein equation (4.11). Therefore

$$P_0(A^\times)^n P_0 = \begin{pmatrix} 0 & X(Ya^{\times n} - (a^{\times *})^{-n}Y)X - X(a^{\times *})^{-n} \\ 0 & -(Ya^{\times n} - (a^{\times *})^{-n}Y)X + (a^{\times *})^{-n} \end{pmatrix},$$

and hence

$$(P_0(A^\times)^n\big|_{\operatorname{Im} P_0})^{-1} = (a^{\times *})^n \left(I + (Y - (a^{\times *})^n Y a^{\times n})X\right)^{-1}.$$

We remark that the matrix $I + (Y - (a^{\times *})^n Y a^{\times n})X$ is indeed invertible since $X > 0$ and since, for every $n \geq 0$,

$$Y - (a^{\times *})^n Y a^{\times n} \geq 0.$$

The formula for β_n follows.

To prove the formula for α_n we first note that (using (4.15))

$$(I - P_0)(A^\times)^{-n}(I - P_0) = \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} (a^\times)^{-n} & 0 \\ Y(a^\times)^{-n} - a^{\times n}Y & (a^\times)^{*n} \end{pmatrix} \begin{pmatrix} I & X \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} (I + X(Y - (a^\times)^{*n}Y(a^\times)^n)(a^\times)^{-n} & (I + X(Y - (a^\times)^{*n}Y(a^\times)^n)(a^\times)^{-n}X \\ 0 & 0 \end{pmatrix}.$$

Moreover,

$$D - CA^{-1}B = W(\infty) = (d - ca^{-1}b)d^*,$$

$$CA^{-1}(I - P_0) = \begin{pmatrix} ca^{-1} & ca^{-1}X \end{pmatrix},$$

and (using the Stein equation (4.10))

$$(I - P_0)B = \begin{pmatrix} b(d^* - b^*a^{-*}c^*) + Xa^{-*}c^* \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} bd^* + aXc^* \\ 0 \end{pmatrix}.$$

The formula for α_n follows.

We now compute $d_{1,n} = \gamma_{nn}^{(n)}$. Using [15, p. p. 36] we have

$$\gamma_{nn}^{(n)} = D^{-1}(I + C(A^\times)^n W_{n+1}^{-1} P_0 A^{-(n+1)} BD^{-1}),$$

where A, B, C and D are given by (4.13)–(4.14) and where W_n is defined by (4.9). In [5, (4.8) p. 164] we proved that

$$W_{n+1} P_0 A^{-(n+1)} = \begin{pmatrix} 0 & a_{n+1} \\ 0 & b_{n+1} \end{pmatrix} \tag{4.16}$$

where

$$\begin{aligned} a_{n+1} &= -X(I + (Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)})X)^{-1} (a^\times)^{*(n+1)}, \\ b_{n+1} &= (I + (Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)})X)^{-1} (a^\times)^{*(n+1)}. \end{aligned}$$

Using (4.15) we have

$$(A^\times)^n W_{n+1} P_0 A^{-(n+1)} = \begin{pmatrix} 0 & (a^\times)^n a_{n+1} \\ 0 & h_n \end{pmatrix}$$

where

$$\begin{aligned} h_n &= (Y(a^\times)^n - (a^\times)^{-*n} Y) a_{n+1} + (a^\times)^{-*n} b_{n+1} \\ &= (a^\times)^{-*n} \Big\{ -((a^\times)^{*n} Y (a^\times)^n - Y)(I + X(Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)}))^{-1} \\ &\qquad \times X (a^\times)^{*(n+1)} \\ &\qquad\qquad + (I + (Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)} X)^{-1} (a^\times)^{*(n+1)} \Big\} \\ &= (a^\times)^{-*n} \Big\{ (Y - (a^\times)^{*n} Y (a^\times)^n)(I + X(Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)}))^{-1} X \\ &\qquad + (I + (Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)})X)^{-1} \Big\} (a^\times)^{*(n+1)} \\ &= (a^\times)^{-*} \\ &\times (I + (Y - (a^\times)^{*n} Y (a^\times)^n)X)(I + (Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)})X)^{-1} (a^\times)^{*(n+1)}. \end{aligned}$$

Since

$$C(A^\times)^n W_{n+1} P_0 A^{-(n+1)} B = (c(a^\times)^n a_{n+1} - d^* a^{-*} c^* h_n a^{-*} c^*,$$

we get the formula for $d_{1,n}$.

Finally, we compute the formula for $d_{2,n} = \gamma_{00}^{(n)}$. By the formula in [15, p. 36] we now have

$$\gamma_{00}^{(n)} = D^{-1} \Big\{ I + C W_{n+1} P_0 A^{-(n+1)} (A^\times)^n BD^{-1} \Big\}.$$

By (4.16) and [15, p. 36] we have

$$\begin{aligned} &C W_{n+1} P_0 A^{-(n+1)} \\ &= \begin{pmatrix} 0 & -(cX + db^* a^{-*})(I + (Y - (a^\times)^{*(n+1)} Y (a^\times)^{(n+1)})X)^{-1} (a^\times)^{*(n+1)} \end{pmatrix}. \end{aligned}$$

Hence, using (4.15) we obtain

$$\begin{aligned}CW_{n+1}P_0A^{-(n+1)}(A^\times)B = -(cX + d^*b^*a^{-*})\\ \times (I + (Y - (a^\times)^{*(n+1)}Y(a^\times)^{(n+1)})X)^{-1}(a^\times)^*\\ \times \big(b(d^* - b^*a^{-*}c^*) + ((a^\times)^{*n}Y(a^\times)^n - Y)a^{-*}c^*\big)\end{aligned}$$

and the formula for $\gamma_{00}^{(n)}$ follows. □

These formulas are the discrete analogs of the formula given in [6, Theorem 3.5 p.9], where we computed the potential associated to a canonical differential expression in terms of a minimal realization of a spectral factor of the spectral function. Connections with the formulas for Nehari admissible sequences given in [3, Section 1.3] will be explored in a separate publication.

5. Connection with the scattering function

The connection between the scattering function and the spectral function allows to reconstruct the discrete system from the scattering function by building first the associated spectral function. We are given two strictly positive matrices δ_1 and δ_2 in $\mathbb{C}^{p\times p}$, and consider a $\mathbb{C}^{p\times p}$-valued rational function $S(z)$ which admits a spectral factorization $S(z) = S_-(z)S_+(z)$ and satisfies the following two conditions:

$$S(z)^*\delta_1 S(z) = \delta_2, \quad |z| = 1, \tag{5.1}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} S_-(e^{it})\delta_1^{-1}S_-(e^{it})^*dt = I_p. \tag{5.2}$$

We also assume that the factors $S_+(z)$ and $S_-(z)$ are normalized by $S_+(0) = S_-(\infty) = I_p$. Note that for a given pair (δ_1, δ_2) there need not exist associated functions $S(z)$ with the required properties. For instance, in the scalar case we necessarily have $\delta_1 = \delta_2$ (see [3]) and then $S(z)$ is unitary on the unit circle.

Using (5.1) we define

$$S_-(1/z)\delta_1^{-1}S_-(1/z)^{-*} = S_+(1/z)\delta_2^{-1}S_+(1/z)^* \overset{\text{def.}}{=} W(z). \tag{5.3}$$

By Theorem (4.1) the function $W(z)$ is the spectral function of a uniquely defined first-order discrete system of the form (1.1) with Szegö admissible sequence defined by (4.4)–(4.5). We know from the proof of Step 1 of Theorem 4.1 that the limits

$$\lim_{n\to\infty}\gamma_{00}^{(n)} \quad \text{and} \quad \lim_{n\to\infty}\gamma_{nn}^{(n)}$$

exist and are strictly positive. For the moment being we denote these limits by k_1 and k_2. Let $Y(z)$ be defined by (1.4). Then (see Section 1)

$$\begin{aligned}W(z) &= (Y_{21} + Y_{22})(1/z)k_2^{-1}(Y_{21} + Y_{22}(1/z))^{-*}\\ &= (Y_{11} + Y_{12})(1/z)k_1^{-1}(Y_{11} + Y_{12}(1/z))^{-*}.\end{aligned} \tag{5.4}$$

By uniqueness of the spectral factorizations and comparing (5.3) and (5.4) we have

$$\begin{aligned}\delta_1 &= k_1,\\ \delta_2 &= k_2,\\ S_-(1/z) &= (Y_{11}+Y_{12})(1/z),\\ S_+(1/z) &= ((Y_{21}+Y_{22})(1/z))^{-1}.\end{aligned}$$

Hence, the associated scattering function is equal to

$$(Y_{11}(z)+Y_{12}(z))(Y_{21}(z)+Y_{22}(z))^{-1} = S_-(z)S_+(z) = S(z).$$

This way, we can reconstruct the system associated to the scattering function using the spectral function.

Theorem 5.1. *Let $S(z)$ be a rational matrix-function which admits a spectral factorization and satisfies conditions* (5.1) *and* (5.2) *for some pair of strictly positive matrices δ_1 and δ_2. Then $S(z)$ is the scattering function of the first-order discrete system with spectral function*

$$S_-(1/z)\delta_1^{-1}S_-(1/z)^{-*} = S_+(1/z)\delta_2^{-1}S_+(1/z)^*.$$

6. Connection with the reflection coefficient function

Let $R \in \mathcal{W}_+^{p\times p}$ be a rational function which is strictly contractive in the closed unit disk. The function

$$W(z) = (I_p - zR(z))^{-1}(I_p - R(z)R(z)^*)(I_p - zR(z))^{-*}, \quad |z| = 1, \tag{6.1}$$

is strictly positive on the unit circle and is the restriction there of the rational function

$$W(z) = \frac{1}{2i}(N(z) - N(1/z^*)^*) \quad \text{with} \quad N(z) = i(I_p - zR(z))(I_p + zR(z))^{-1}.$$

Hence $W(z)$ is the spectral function of a first-order discrete system. Since $R(z)$ defined uniquely $W(z)$ we have:

Theorem 6.1. *Let $R \in \mathcal{W}_+^{p\times p}$ be a rational function which is strictly contractive in the closed unit disk. Then it is the reflection coefficient function of the first-order canonical discrete system* (1.1) *with associated spectral function* (6.1).

Indeed, by Theorem 4.1 the function

$$W(z) = \frac{1}{2i}(N(z) - N(1/z^*)^*), \quad |z| = 1,$$

is the spectral function of a uniquely defined first-order discrete system and $R(z)$ is uniquely determined by $W(z)$.

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