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Singularly Perturbed Boundary-Value Problems

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$$U_\varepsilon = U_0(x,t) + \varepsilon U_1(x,t) + V_0(x,\tau) + \varepsilon V_1(x,\tau) + R_\varepsilon(x,t),$$
$$(x,t) \in D_T,\ \tau = t/\varepsilon\,,$$

$$\|R_{1\varepsilon}\|_{C(\overline{D}_T)} = \mathcal{O}(\varepsilon^{9/8})\,, \quad \|R_{2\varepsilon}\|_{C(\overline{D}_T)} = \mathcal{O}(\varepsilon^{11/8})$$

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To George Soros,
a generous supporter
of Mathematics

Contents

Preface

It is well known that many phenomena in biology, chemistry, engineering, physics can be described by boundary value problems associated with various types of partial differential equations or systems. When we associate a mathematical model with a phenomenon, we generally try to capture what is essential, retaining the important quantities and omitting the negligible ones which involve small parameters. The model that would be obtained by maintaining the small parameters is called the perturbed model, whereas the simplified model (the one that does not include the small parameters) is called unperturbed (or reduced model). Of course, the unperturbed model is to be preferred, because it is simpler. What matters is that it should describe faithfully enough the respective phenomenon, which means that its solution must be "close enough" to the solution of the corresponding perturbed model. This fact holds in the case of regular perturbations (which are defined later). On the other hand, in the case of singular perturbations, things get more complicated. If we refer to an initial-boundary value problem, the solution of the unperturbed problem does not satisfy in general all the original boundary conditions and/or initial conditions (because some of the derivatives may disappear by neglecting the small parameters). Thus, some discrepancy may appear between the solution of the perturbed model and that of the corresponding reduced model. Therefore, to fill in this gap, in the asymptotic expansion of the solution of the perturbed problem with respect to the small parameter (considering, for the sake of simplicity, that we have a single parameter), we must introduce corrections (or boundary layer functions).

More than half a century ago, A.N. Tikhonov [43]–[45] began to systematically study singular perturbations, although there had been some previous attempts in this direction. In 1957, in a fundamental paper [50], M.I. Vishik and L.A. Lyusternik studied linear partial differential equations with singular perturbations, introducing the famous method which is today called the Vishik-Lyusternik method. From that moment on, an entire literature has been devoted to this subject.

> This book offers a detailed asymptotic analysis of some important classes of singularly perturbed boundary value problems which are mathematical models for various phenomena in biology, chemistry, engineering.

> We are particularly interested in nonlinear problems, which have hardly been examined so far in the literature dedicated to singular perturbations. This book proposes to fill in this gap, since most applications are described by nonlinear models. Their asymptotic analysis is very interesting, but requires special methods and tools. Our treatment combines some of the most successful results from different parts of mathematics, including functional analysis, singular perturbation theory, partial differential equations, evolution equations. So we are able to offer the reader a complete justification for the replacement of various perturbed models with corresponding reduced models, which are simpler but in general have a different character. From a mathematical point of view, a change of character modifies dramatically the model, so a deep analysis is required.
>
> Although we address specific applications, our methods are applicable to other mathematical models.

We continue with a few words about the structure of the book. The material is divided into four parts. Each part is divided into chapters, which, in turn, are subdivided into sections (see the Contents). The main definitions, theorems, propositions, lemmas, corollaries, remarks are labelled by three digits: the first digit indicates the chapter, the second the corresponding section, and the third the respective item in the chapter.

Now, let us briefly describe the material covered by the book.

The first part, titled *Preliminaries*, has an introductory character. In Chapter 1 we recall the definitions of the regular and singular perturbations and present the Vishik-Lyusternik method. In Chapter 2, some results concerning existence, uniqueness and regularity of the solutions for evolution equations in Hilbert spaces are brought to attention.

In Part II, some nonlinear boundary value problems associated with the telegraph system are investigated. In Chapter 3 (which is the first chapter of Part II) we present the classes of problems we intend to study and indicate the main fields of their applications. In Chapters 4 and 5 we discuss in detail the case of algebraic boundary conditions and that of dynamic boundary conditions, respectively. We determine formally some asymptotic expansions of the solutions of the problems under discussion and find out the corresponding boundary layer functions. Also, we establish results of existence, uniqueness and high regularity for the other terms of our asymptotic expansions. Moreover, we establish estimates for the components of the remainders in the asymptotic expansions previously deducted in a formal way, with respect to the uniform convergence topology, or with respect to some weaker topologies. Thus, the asymptotic expansions are validated.

Part III, titled *Singularly perturbed coupled problems*, is concerned with the coupling of some boundary value problems, considered in two subdomains of a given domain, with transmission conditions at the interface.

In the first chapter of Part III (Chapter 6) we introduce the problems we are going to investigate in the next chapters of this part. They are mathematical models for diffusion-convection-reaction processes in which a small parameter is present. We consider both the stationary case (see Chapter 7) and the evolutionary one (see Chapter 8). We develop an asymptotic analysis which in particular allows us to determine appropriate transmission conditions for the reduced models.

What we do in Part III may also be considered as a first step towards the study of more complex coupled problems in Fluid Mechanics.

While in Parts II and III the possibility to replace singular perturbation problems with the corresponding reduced models is discussed, in Part IV we aim at reversing the process in the sense that we replace given parabolic problems with singularly perturbed, higher order (with respect to t) problems, admitting solutions which are more regular and approximate the solutions of the original problems. More precisely, we consider the classical heat equation with homogeneous Dirichlet boundary conditions and initial conditions. We add to the heat equation the term $\pm\varepsilon u_{tt}$, thus obtaining either an elliptic equation or a hyperbolic one. If we associate with each of the resulting equations the original boundary and initial conditions we obtain new problems, which are incomplete, since the new equations are of a higher order with respect to t. For each problem we need to add one additional condition to get a complete problem. We prefer to add a condition at $t = T$ for the elliptic equation, either for u or for u_t, and an initial condition at $t = 0$ for u_t for the hyperbolic equation. So, depending on the case, we obtain an elliptic or hyperbolic regularization of the original problem. In fact, we have to do with singularly perturbed problems, which can be treated in an abstract setting. In the final chapter of the book (Chapter 11), elliptic and hyperbolic regularizations associated with the nonlinear heat equation are investigated.

Note that, with the exception of Part I, the book includes original material mainly due to the authors, as considerably revised or expanded versions of previous works, including in particular the 2000 authors' Romanian book [6].

The present book is designed for researchers and graduate students and can be used as a two-semester text.

The authors December 2006

Part I

Preliminaries

Chapter 1

Regular and Singular Perturbations

In this chapter we recall and discuss some general concepts of singular perturbation theory which will be needed later. Our presentation is mainly concerned with singular perturbation problems of the boundary layer type, which are particularly relevant for applications.

In order to start our discussion, we are going to set up an adequate framework. Let $D \subset \mathbb{R}^n$ be a nonempty open bounded set with a smooth boundary S. Denote its closure by $\overline{D}$. Consider the following equation, denoted E_ε,

$$L_\varepsilon u = f(x, \varepsilon), \quad x \in D,$$

where ε is a small parameter, $0 < \varepsilon \ll 1$, L_ε is a differential operator, and f is a given real-valued smooth function. If we associate with E_ε some condition(s) for the unknown u on the boundary S, we obtain a boundary value problem P_ε. We assume that, for each ε, P_ε has a unique smooth solution $u = u_\varepsilon(x)$. Our goal is to construct approximations of u_ε for small values of ε. The usual norm we are going to use for approximations is the sup norm (or max norm), i.e.,

$$\|g\|_{C(\overline{D})} = \sup\{|g(x)|;\ x \in \overline{D}\},$$

for every continuous function $g : \overline{D} \longrightarrow \mathbb{R}$ (in other words, $g \in C(\overline{D})$). We will also use the weaker L^p-norm

$$\|g\|_{L^p(D)} = \left(\int_D |g|^p \, dx\right)^{1/p},$$

where $1 \leq p < \infty$. For information about L^p-spaces, see the next chapter.

In many applications, operator L_ε is of the form

$$L_\varepsilon = L_0 + \varepsilon L_1,$$

where L_0 and L_1 are differential operators which do not depend on ε. If L_0 does not include some of the highest order derivatives of L_ε, then we should associate with L_0 fewer boundary conditions. So, P_ε becomes

$$L_0 u + \varepsilon L_1 u = f(x, \varepsilon), \quad x \in D,$$

with the corresponding boundary conditions. Let us also consider the equation, denoted E_0,

$$L_0 u = f_0, \quad x \in D,$$

where $f_0(x) := f(x, 0)$, with some boundary conditions, which usually come from the original problem P_ε. Let us denote this problem by P_0. Some of the original boundary conditions are no longer necessary for P_0. Problem P_ε is said to be a *perturbed problem* (*perturbed model*), while problem P_0 is called *unperturbed* (or *reduced model*).

Definition 1.0.1. Problem P_ε is called *regularly perturbed* with respect to some norm $\|\cdot\|$ if there exists a solution u_0 of problem P_0 such that

$$\|u_\varepsilon - u_0\| \longrightarrow 0 \;\text{ as } \varepsilon \to 0.$$

Otherwise, P_ε is said to be *singularly perturbed* with respect to the same norm.

In a more general setting, we may consider time t as an additional independent variable for problem P_ε as well as initial conditions at $t = 0$ (sometimes t is the only independent variable). Moreover, we may consider systems of differential equations instead of a single equation. Note also that the small parameter may also occur in the conditions associated with the corresponding system of differential equations. For example, we will discuss later some coupled problems in which the small parameter is also present in transmission conditions. Basically, the definition above also applies to these more general cases.

In order to illustrate this definition we are going to consider some examples. Note that the problem of determining P_0 will be clarified later. Here, we use just heuristic arguments.

Example 1. Consider the following simple Cauchy problem P_ε :

$$\frac{du}{dt} + \varepsilon u = f_0(t), \quad 0 < t < T; \quad u(0) = \theta,$$

where $T \in (0, +\infty)$, $\theta \in \mathbb{R}$, and $f_0 : \mathbb{R} \longrightarrow \mathbb{R}$ is a given smooth function. The solution of P_ε is given by

$$u_\varepsilon(t) = e^{-\varepsilon t}\Big(\theta + \int_0^t e^{\varepsilon s} f_0(s)\, ds\Big), \quad 0 \le t \le T.$$

Obviously, u_ε converges uniformly on $[0, T]$, as ε tends to 0, to the function

$$u_0(t) = \theta + \int_0^t f_0(s)\, ds,$$

which is the solution of the reduced problem

$$\frac{du}{dt} = f_0(t), \quad 0 < x < T; \quad u(0) = \theta.$$

Therefore, P_ε is regularly perturbed with respect to the sup norm.

Example 2. Let P_ε be the boundary value problem

$$\varepsilon \frac{d^2u}{dx^2} + \frac{du}{dx} = 2x, \quad 0 < x < 1; \quad u(0) = 0 = u(1).$$

Its solution is

$$u_\varepsilon(x) = x(x - 2\varepsilon) + \frac{2\varepsilon - 1}{1 - e^{-1/\varepsilon}}\big(1 - e^{-x/\varepsilon}\big).$$

Note that

$$u_\varepsilon(x) = (x^2 - 1) + e^{-x/\varepsilon} + r_\varepsilon(x),$$

where $r_\varepsilon(x)$ converges uniformly to the null function, as ε tends to 0. Therefore, u_ε converges uniformly to the function $u_0(x) = x^2 - 1$ on every interval $[\delta, 1]$, $0 < \delta < 1$, but not on the whole interval $[0, 1]$. Obviously, $u_0(x) = x^2 - 1$ satisfies the reduced problem

$$\frac{du}{dx} = 2x, \quad 0 < x < 1; \quad u(1) = 0,$$

but $\|u_\varepsilon - u_0\|_{C[0,1]}$ does not approach 0. Therefore, P_ε is singularly perturbed with respect to the sup norm. For a small δ, u_0 is an approximation of u_ε in $[\delta, 1]$, but it fails to be an approximation of u_ε in $[0, \delta]$. This small interval $[0, \delta]$ is called a *boundary layer*. Here we notice a fast change of u_ε from its value $u_\varepsilon(0) = 0$ to values close to u_0. This behavior of u_ε is called a boundary layer phenomenon and P_ε is said to be a *singular perturbation problem of the boundary layer type*. In this simple example, we can see that a uniform approximation for $u_\varepsilon(x)$ is given by $u_0(x) + e^{-x/\varepsilon}$. The function $e^{-x/\varepsilon}$ is a so-called *boundary layer function* (*correction*). It fills the gap between u_ε and u_0 in the boundary layer $[0, \delta]$.

Let us remark that P_ε is a regular perturbation problem with respect to the L^p-norm for all $1 \le p < \infty$, since $\|u_\varepsilon - u_0\|_{L^p(0,1)}$ tends to zero. The boundary layer which we have just identified *is not visible in this weaker norm*.

Example 3. Let P_ε be the following Cauchy problem

$$\varepsilon \frac{du}{dt} + ru = f_0(t), \quad 0 < t < T; \quad u(0) = \theta,$$

where r is a positive constant, $\theta \in \mathbb{R}$ and $f_0 : [0,T] \to \mathbb{R}$ is a given Lipschitzian function. The solution of this problem is given by

$$u_\varepsilon(t) = \theta e^{-rt/\varepsilon} + \frac{1}{\varepsilon}\int_0^t f_0(s)e^{-r(t-s)/\varepsilon}ds,\ 0 \le t \le T,$$

which can be written as

$$u_\varepsilon(t) = \frac{1}{r}f_0(t) + \left(\theta - \frac{1}{r}f_0(0)\right)e^{-rt/\varepsilon} + r_\varepsilon(t),\ 0 \le t \le T,$$

where

$$r_\varepsilon(t) = -\frac{1}{r}\int_0^t f_0'(s)e^{-r(t-s)/\varepsilon}ds.$$

We have

$$| r_\varepsilon(t) | \le \frac{L}{r}\int_0^t e^{-r(t-s)/\varepsilon}ds \le \frac{L}{r^2}\varepsilon,$$

where L is the Lipschitz constant of f_0. Therefore, r_ε converges uniformly to zero on $[0,T]$ as ε tends to 0. Thus u_ε converges uniformly to $u_0(t) = (1/r)f_0(t)$ on every interval $[\delta, T]$, $0 < \delta < T$, but not on the whole interval $[0,T]$ if $f_0(0) \ne r\theta$. Note also that u_0 is the solution of the (algebraic) equation

$$ru = f_0(t),\quad 0 < t < T,$$

which represents our reduced problem. Therefore, if $f_0(0) \ne r\theta$, this P_ε is a singular perturbation problem of the boundary layer type with respect to the sup norm. The boundary layer is a small right vicinity of the point $t = 0$. A uniform approximation of $u_\varepsilon(t)$ on $[0,T]$ is the sum $u_0(t) + \left(\theta - \frac{1}{r}f_0(0)\right)e^{-rt/\varepsilon}$. The function $\left(\theta - \frac{1}{r}f_0(0)\right)e^{-rt/\varepsilon}$ is a boundary layer function, which corrects the discrepancy between u_ε and u_0 within the boundary layer.

Example 4. Let P_ε be the following initial-boundary value problem

$$\varepsilon u_t - u_{xx} = t\sin x,\quad 0 < x < \pi,\quad 0 < t < T,$$

$$u(x,0) = \sin x,\quad x \in [0,\pi];\quad u(0,t) = 0 = u(\pi,t),\quad t \in [0,T],$$

where T is a given positive number. The solution of this problem is

$$u_\varepsilon(x,t) = t\sin x + e^{-t/\varepsilon}\sin x + \varepsilon\big(e^{-t/\varepsilon} - 1\big)\sin x,$$

which converges uniformly, as ε tends to zero, to the function $u_0(x,t) = t\sin x$, on every rectangle $R_\delta = \{(x,t) :\ 0 \le x \le \pi,\ \delta \le t \le T\},\quad 0 < \delta < T$. Note that u_0 is the solution of the reduced problem P_0,

$$-u_{xx} = t\sin x,\quad u(0,t) = 0 = u(\pi,t).$$

However, u_0 fails to be a uniform approximation of u_ε in the strip $B_\delta = \{(x,t) : 0 \le x \le \pi, \quad 0 \le t \le \delta\}$. Therefore, P_ε is a singular perturbation problem of the boundary layer type with respect to the sup norm on the rectangle $[0,\pi] \times [0,T]$. The boundary layer is a thin strip B_δ, where δ is a small positive number. Obviously, a boundary layer function (correction) is given by

$$c(x, t/\varepsilon) = e^{-t/\varepsilon} \sin x,$$

which fills the gap between u_ε and u_0. Indeed, $u_0(x,t) + c(x,t/\varepsilon)$ is a uniform approximation of u_ε.

It is interesting to note that P_ε is regularly perturbed with respect to the usual norm of the space $C\big([0,\pi];\, L^p(0,T)\big)$ for all $1 \le p < \infty$. The boundary layer phenomenon is not visible in this space, but it is visible in $C([0,\pi] \times [0,T])$, as noticed above. In fact, we can see that P_ε is singularly perturbed with respect to the weaker norm $\|\cdot\|_{L^1(0,\pi;\, C[0,T])}$.

Example 5. Let $D \subset \mathbb{R}^2$ be a bounded domain with smooth boundary ∂D. Let P_ε be the following typical Dirichlet boundary value problem (see, e.g., [48], p. 83):

$$\begin{cases} -\varepsilon \Delta u + u = f(x,y,\varepsilon) \quad \text{in} \;\; D, \\ u = 0 \quad \text{on} \;\; \partial D, \end{cases}$$

where Δ is the Laplace operator, i.e., $\Delta u := u_{xx} + u_{yy}$ and f is a given smooth function defined on $\overline{D} \times [0,\varepsilon_0]$, for some $\varepsilon_0 > 0$, such that $f(x,y,0) \neq 0$ for all $(x,y) \in \partial D$. It is well known that problem P_ε has a unique classical solution $u_\varepsilon(x,y)$. Obviously, P_0 is an algebraic equation, for which the boundary condition is no longer necessary. Its solution is

$$u_0(x,y) = f(x,y,0), \quad (x,y) \in \overline{D}.$$

Clearly, in a neighborhood of ∂D, u_ε and u_0 are not close enough with respect to the sup norm, since $u_{\varepsilon|\partial D} = 0$, whereas u_0 does not satisfy this condition. Therefore, $\|u_\varepsilon - u_0\|_{C(\overline{D})}$ does not converge to 0, as $\varepsilon \to 0$. According to our definition, problem P_ε is singularly perturbed with respect to $\|\cdot\|_{C(\overline{D})}$. Moreover, this problem is of the boundary layer type. In this example, the boundary layer is a vicinity of the whole boundary ∂D. The existence of the boundary layer phenomenon is not as obvious as in the previous examples, since there is no explicit form of u_ε. Following, e.g., [48] we will perform a complete analysis of this issue below. On the other hand, it is worth mentioning that this P_ε is regularly perturbed with respect to $\|\cdot\|_{L^p(D)}$ for all $1 \le p < \infty$, as explained later.

Example 6. In $D_T = \{(x,t);\; 0 < x < 1,\, 0 < t < T\}$ we consider the telegraph system

$$\begin{cases} \varepsilon u_t + v_x + ru = f_1(x,t), \\ \;\; v_t + u_x + gv = f_2(x,t), \end{cases} \qquad (S)_\varepsilon$$

with initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad 0 < x < 1, \qquad (IC)_\varepsilon$$

and boundary conditions of the form

$$\begin{cases} r_0 u(0,t) + v(0,t) = 0, \\ -u(1,t) + f_0(v(1,t)) = 0, \quad 0 < t < T, \end{cases} \qquad (BC)_\varepsilon$$

where $f_1, f_2 : \overline{D}_T \to \mathbb{R}$, $f_0 : \mathbb{R} \to \mathbb{R}$, $u_0, v_0 : [0,1] \to \mathbb{R}$ are given smooth functions, and r_0, r, g are constants, $r_0 > 0,\ r > 0,\ g \geq 0$. If in the model formulated above and denoted by P_ε we take $\varepsilon = 0$, we obtain the following reduced problem P_0:

$$\begin{cases} u = r^{-1}(f_1 - v_x), \\ v_t - r^{-1} v_{xx} + gv = f_2 - r^{-1} f_{1x} \ \text{ in } \ D_T, \end{cases} \qquad (S)_0$$

$$v(x,0) = v_0(x), \quad 0 < x < 1, \qquad (IC)_0$$

$$\begin{cases} rv(0,t) - r_0 v_x(0,t) + r_0 f_1(0,t) = 0, \\ r f_0(v(1,t)) + v_x(1,t) - f_1(1,t) = 0, \quad 0 < t < T. \end{cases} \qquad (BC)_0$$

In this case, the reduced system $(S)_0$ consists of an algebraic equation and a differential equation of the parabolic type, whereas system $(S)_\varepsilon$ is of the hyperbolic type. The initial condition for u is no longer necessary. We will derive P_0 later in a justified manner.

Let us remark that if the solution of P_ε, say $U_\varepsilon(x,t) = (u_\varepsilon(x,t), v_\varepsilon(x,t))$, would converge uniformly in $\overline{D}_T$ to the solution of P_0, then necessarily

$$v_0'(x) + r u_0(x) = f_1(x,0), \ \forall x \in [0,1].$$

If this condition is not satisfied then that uniform convergence is not true and, as we will show later, U_ε has a boundary layer behavior in a neighborhood of the segment $\{(x,0);\ 0 \leq x \leq 1\}$. Therefore, this P_ε is a singular perturbation problem of the boundary layer type with respect to the sup norm $\| \cdot \|_{C(\overline{D}_T)^2}$. However, using the form of the boundary layer functions which we are going to determine later, we will see that the boundary layer is not visible in weaker norms, like for instance $\| \cdot \|_{C([0,1];\, L^p(0,T))^2}$, $1 \leq p < \infty$, and P_ε is regularly perturbed in such norms.

Example 7. Let P_ε be the following simple initial value problem

$$\begin{cases} \varepsilon \frac{du_1}{dx} - u_2 = \varepsilon f_1(x), \\ \varepsilon \frac{du_2}{dx} + u_1 = \varepsilon f_2(x), \quad 0 < x < 1, \\ u_1(0) = 1, \quad u_2(0) = 0, \end{cases}$$

where $f_1,\ f_2 \in C[0,1]$ are given functions. It is easily seen that this P_ε is singularly perturbed with respect to the sup norm, but not of the boundary layer type. This conclusion is trivial in the case $f_1 = 0,\ f_2 = 0$, when the solution of P_ε is

$$u_\varepsilon = \big(\cos(x/\varepsilon), -\sin(x/\varepsilon)\big).$$

Definition 1.0.2. Let u_ε be the solution of some perturbed problem P_ε defined in a domain D. Consider a function $U(x,\varepsilon),\ x \in D_1$, where D_1 is a subdomain of D. The function $U(x,\varepsilon)$ is called an asymptotic approximation in D_1 of the solution $u_\varepsilon(x)$ with respect to the sup norm if

$$\sup_{x\in D_1} \|u_\varepsilon(x) - U(x,\varepsilon)\| \to 0 \ \text{ as } \ \varepsilon \to 0.$$

Moreover, if

$$\sup_{x\in D_1} \|u_\varepsilon(x) - U(x,\varepsilon)\| = \mathcal{O}(\varepsilon^k),$$

then we say that $U(x,\varepsilon)$ is an asymptotic approximation of $u_\varepsilon(x)$ in D_1 with an accuracy of the order ε^k. We have similar definitions with respect to other norms. In the above definition we have assumed that U and u_ε take values in $\mathbb{R}^n$, and $\|\cdot\|$ denotes one of the norms of this space.

For a real-valued function $E(\varepsilon)$, the notation $E(\varepsilon) = \mathcal{O}(\varepsilon^k)$ means that $|E(\varepsilon)| \le M\varepsilon^k$ for some positive constant M and for all ε small enough.

In Example 4 above u_0 is an asymptotic approximation of u_ε with respect to the sup norm in the rectangle R_δ, with an accuracy of the order ε. Function u_0 is not an asymptotic approximation of u_ε in $[0,\pi] \times [0,T]$ with respect to the sup norm, but it has this property with respect to the norm of $C\big([0,\pi];\ L^p(0,T)\big)$, with an accuracy of the order $\varepsilon^{1/p}$, for all $1 \le p < \infty$. Note also that the function $t\sin x + e^{-t/\varepsilon}\sin x$ is an asymptotic approximation in $[0,\pi] \times [0,T]$ of u_ε with respect to the sup norm, with an accuracy of the order ε.

In the following we are going to discuss the celebrated **Vishik-Lyusternik method** [50] for the construction of asymptotic approximations for the solutions of singular perturbation problems of the boundary layer type. To explain this method we consider the problem used in Example 5 above, where ε will be replaced by ε^2 for our convenience, i.e.,

$$\begin{cases} -\varepsilon^2\Delta u + u = f(x,y,\varepsilon) \quad \text{in} \ \ D, \\ u = 0 \quad \text{on} \ \ \partial D. \end{cases}$$

We will seek the solution of P_ε in the form

$$u_\varepsilon = u + c, \tag{1.1}$$

where u and c are two series: $u = \sum_{j=0}^{\infty} \varepsilon^j u_j(x,y)$ is the so-called *regular series* and does not in general satisfy the boundary condition; the discrepancy in the boundary condition is removed by the so-called *boundary layer series* c, which will be introduced in the following. Let the equations of the boundary ∂D have the following parametric form:

$$x = \varphi(p), \quad y = \psi(p), \quad 0 \le p \le p_0.$$

More precisely, when p increases from 0 to p_0, the point $(\varphi(p), \psi(p))$ moves on ∂D in such a way that D remains to the left. Consider an internal δ-vicinity of ∂D, $\delta > 0$ small, which turns out to be our boundary layer. Any point (x, y) of the boundary layer is uniquely determined by a pair $(\rho, p) \in [0, \delta] \times [0, p_0]$. Indeed, let $p \in [0, p_0]$ be the value of the parameter for which the normal at $(\varphi(p), \psi(p))$ to ∂D contains the point (x, y). Then ρ is defined as the distance from (x, y) to $(\varphi(p), \psi(p))$. It is obvious that (x, y) and (ρ, p) are connected by the following equations

$$x = \varphi(p) - \rho\psi'(p)/\left(\varphi'(p)^2 + \psi'(p)^2\right)^{1/2},$$

$$y = \psi(p) + \rho\varphi'(p)/\left(\varphi'(p)^2 + \psi'(p)^2\right)^{1/2}.$$

We have the following expression for the operator $L_\varepsilon u = -\varepsilon^2 \Delta u + u$ with respect to the new coordinates (ρ, p)

$$L_\varepsilon u = -\varepsilon^2 \Big(u_{\rho\rho} + (p_x^2 + p_y^2) u_{pp} + (\rho_{xx} + \rho_{yy}) u_\rho + (p_{xx} + p_{yy}) u_p \Big) + u.$$

We stretch the variable ρ by the transformation $\tau = \rho/\varepsilon$. The new variable τ, called *fast*indexfast variable or *rapid variable*, helps us to describe the behavior of the solution u_ε inside the boundary layer. The construction of the fast variable depends on the problem P_ε under investigation (see, e.g., [18] and [29]). It turns out that for the present problem $\tau = \rho/\varepsilon$ is the right fast variable. If we expand the coefficients of L_ε in power series in ε, we get the following expression for L_ε with respect to (τ, p)

$$L_\varepsilon u = \big(- u_{\tau\tau} + u \big) + \sum_{j=1}^{\infty} \varepsilon^j L_j u,$$

where L_j are differential operators containing the partial derivatives u_τ, u_p and u_{pp}. We will seek the solution of problem P_ε in the form of the following expansion, which is called *asymptotic expansion*,

$$u_\varepsilon(x, y) = u + c = \sum_{j=0}^{\infty} \varepsilon^j \Big(u_j(x, y) + c_j(\tau, p) \Big). \tag{1.2}$$

Now, expanding $f(x, y, \varepsilon)$ into a power series in ε and substituting (1.2) in P_ε, we get

$$\sum_{j=0}^{\infty} \varepsilon^j \Big(-\varepsilon^2 \Delta u_j(x,y) + u_j(x,y) \Big) + \sum_{j=0}^{\infty} \varepsilon^j \Big(-c_{j\tau\tau}(\tau,p) + c_j(\tau,p) \Big) \\ + \sum_{j=0}^{\infty} \varepsilon^j \Big(\sum_{i=1}^{\infty} \varepsilon^i L_i c_j(\tau,p) \Big) = \sum_{j=0}^{\infty} \varepsilon^j f_j(x,y), \tag{1.3}$$

$$\sum_{j=0}^{\infty} \varepsilon^j \Big(u_j(\varphi(p),\psi(p)) + c_j(0,p) \Big) = 0. \tag{1.4}$$

We are going to equate coefficients of the like powers of ε in the above equations, separately for terms depending on (x,y) and (τ,p). This distinction can be explained as follows: the boundary layer part is sizeable within the boundary layer and negligible outside this layer, so in the interior of the domain we have to take into account only regular terms, thus deriving the equations satisfied by $u_j(x,y)$; then we continue with boundary layer terms. For our present example we obtain

$$u_j(x,y) = f_j(x,y), \quad j = 0,\, 1,$$
$$u_j(x,y) = f_j(x,y) + \Delta u_{j-2}(x,y), \quad j = 2, 3, \ldots.$$

For the boundary layer functions we obtain the following ordinary differential equations in τ

$$c_{j\tau\tau}(\tau,p) - c_j(\tau,p) = g_j(\tau,p), \quad \tau \geq 0, \tag{1.5}$$

where $g_0(\tau,p) = 0$, $g_j(\tau,p) = \sum_{i=1}^{j} L_i c_{j-i}(\tau,p)$ for all $j = 1, 2, \ldots$, together with the conditions

$$c_j(0,p) = -u_j(\varphi(p),\psi(p)). \tag{1.6}$$

In addition, having in mind that the boundary layer functions should be negligible outside the boundary layer, we require that

$$\lim_{\tau\to\infty} c_j(\tau,p) = 0, \quad \forall p \in [0,p_0]. \tag{1.7}$$

We can solve successively the above problems and find

$$c_0(\tau,p) = -u_0(\varphi(p),\psi(p))e^{-\tau},$$

while the other c_j's are products of some polynomials (in τ) and $e^{-\tau}$. Therefore,

$$|c_j(\tau,p)| \leq K_j e^{-\tau/2}, \quad j = 0, 1, \ldots,$$

where K_j are some positive constants. In fact, these corrections c_j should act only inside the boundary layer, i.e., for $0 \leq \tau \leq \delta/\varepsilon$. Let $\alpha(\rho)$ be an infinitely differentiable function, which equals 0 for $\rho \geq 2\delta/3$, equals 1 for $\rho \leq \delta/3$, and $0 \leq \alpha(\rho) \leq 1$ for $\delta/3 < \rho < 2\delta/3$. So, we can consider the functions $\alpha(\varepsilon\tau)c_j(\tau,p)$ as our new boundary layer functions, which are defined in the whole $\overline{D}$ and still satisfy the estimates above. This smooth continuation procedure will be used whenever we need it, without any special mention.

So, we have constructed an asymptotic expansion for u_ε. It is easily seen (see also [48], p. 86) that the partial sum

$$U_n(x,y,\varepsilon) = \sum_{j=0}^{n} \varepsilon^j \Big(u_j(x,y) + c_j(\tau,p)\Big)$$

is an asymptotic approximation in D of u_ε with respect to the sup norm, with an accuracy of the order of ε^{n+1}. Indeed, for a given n, $w_\epsilon = u_\varepsilon - U_n(\cdot,\cdot,\varepsilon)$ satisfies an equation of the form

$$-\varepsilon^2 \Delta w_\varepsilon(x,y) + w_\varepsilon(x,y) = h_\varepsilon(x,y),$$

with a homogeneous Dirichlet boundary condition, where $h_\varepsilon = \mathcal{O}(\varepsilon^{n+1})$. Now, the assertion follows from the fact that $\Delta w_\varepsilon \leq 0$ (≥ 0) at any maximum (respectively, minimum) point of w_ε.

On the other hand, since

$$\| \, c_0 \, \|_{L^p(D)} = \mathcal{O}(\varepsilon^{1/p}) \; \forall \; 1 \leq p < \infty,$$

we infer that u_0 is an asymptotic approximation in D of u_ε with respect to the norm $\| \cdot \|_{L^p(D)}$, with an accuracy of the order of $\varepsilon^{1/p}$, $\forall \; 1 \leq p < \infty$. In fact, c_0 is not important if we use this weaker norm.

We may ask ourselves what would happen if the data of a given P_ε were not very regular. For example, let us consider the same Dirichlet P_ε problem above, in a domain D with a smooth boundary ∂D, but in which $f = f(x,y,\varepsilon)$ is no longer a series expansion with respect to ε. To be more specific, we consider the case in which f admits a finite expansion of the form

$$f(x,y,\varepsilon) = \sum_{j=0}^{n} \varepsilon^j f_j(x,y) + \varepsilon^{n+1} g_\varepsilon(x,y),$$

for some given $n \in \mathbb{N}$, where $f_j, \; g_\varepsilon(\cdot,\cdot)$ are smooth functions defined on $\overline{D}$, and $\|g_\varepsilon(\cdot,\cdot)\|_{C(\overline{D})} \leq M$, for some constant M. In this case, we seek the solution of P_ε in the form

$$u_\varepsilon(x,y) = \sum_{j=0}^{n} \varepsilon^j \Big(u_j(x,y) + c_j(\tau,p)\Big) + r_\varepsilon(x,y),$$

where u_j and c_j are defined as before, and r_ε is given by

$$r_\varepsilon(x,y) = u_\varepsilon(x,y) - \sum_{j=0}^{n} \varepsilon^j \Big(u_j(x,y) + c_j(\tau,p)\Big),$$

and is called *remainder of the order* n. Using exactly the same argument as before, one can prove that

$$\|r_\varepsilon\|_{C(\overline{D})} = \mathcal{O}(\varepsilon^{n+1}).$$