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Factorization of Matrix and Operator Functions: The State Space Method

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Dedicated to the memory of

Moshe Livsic

the founding father of the
characteristic function
in operator theory

Contents

Preface

The present book deals with various types of factorization problems for matrix and operator functions. The problems appear in different areas of mathematics and its applications. A unified approach to treat them is developed. The main theorems yield explicit necessary and sufficient conditions for the factorizations to exist and explicit formulas for the corresponding factors. Stability of the factors relative to a small perturbation of the original function is also studied in this book.

The unifying theory developed in the book is based on a geometric approach which has its origins in different fields. A number of initial steps can be found in:

(1) the theory of non-selfadjoint operators, where the study of invariant subspaces of an operator is related to factorization of the characteristic matrix or operator function of the operator involved,

(2) mathematical systems theory and electrical network theory, where a cascade decomposition of an input-output system or a network is related to a factorization of the associated transfer function, and

(3) the factorization theory of matrix polynomials in terms of invariant subspaces of a corresponding linearization.

In all three cases a state space representation of the function to be factored is used, and the factors are expressed in state space form too. We call this approach the *state space method*. It has a large number of applications. For instance, besides the areas referred to above, Wiener-Hopf factorizations of some classes of symbols can also be treated by the state space method.

The present book is the second book which is devoted to the state space factorization theory. The first was published in 1979 as the monograph by H. Bart, I. Gohberg and M.A. Kaashoek, "Minimal factorization of matrix and operator functions," Operator Theory: Advances and Applications **1**, Birkhäuser Verlag. This 1979 book appeared very soon after the first main results were obtained. In fact, some of these results where published in the 1979 book for the first time.

This second book, which is written by the authors of the first book jointly with A.C.M. Ran, consists of four parts. Parts I, II and IV contain a substantial selection from the first book, in a reorganized and updated form. Part III, which covers more than a quarter of the book, is entirely new. This third part is devoted to the theory of factorization into degree one factors and its connection to the combinatorial problem of job scheduling in operations research. It also contains Maple procedures to calculate degree one factorizations. In contrast to the other parts, this third part is completely finite-dimensional and can be considered as a new advanced chapter of Linear Algebra and its Applications. Almost each chapter in this book offers new elements and in many cases new sections, taking into account a number of new results in state space factorization theory and its applications that have appeared in the period of 25 years after publication of the first book. On the other hand in the present book there is less emphasis on Wiener-Hopf integral equation and its applications than in the first book. However these topics are not entirely absent but, for instance, the applications to transport do not appear in this book. The text is largely self-contained, and will be of interest to experts and students in Mathematics, Sciences and Engineering.

The authors are in the process of writing another book, also devoted to the state space approach to factorization. There the emphasis will be on canonical factorization and symmetric factorization with applications to different classes of convolution equations. For the latter we have in mind the transport equation, singular integral equations, equations with symbols analytic in a strip, and equations involving factorization of non-proper rational matrix functions. Furthermore, a large part of this third book will deal with factorization of matrix functions satisfying various symmetries. A main theme will be the effect on factorization of these symmetries and how the symmetries can be used in effective way to get state space formulas for the factors. Applications to H-infinity control theory, which have been developed in the eighties and nineties, will also be included.

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In conclusion, we would like to express our gratitude to Johan Kaashoek who wrote for this book two new sections with Maple procedures for computing certain degree one factorizations. Without his help these sections would not have been. He also read Part III of the book in detail and provided us with several useful comments. We thank our friends and colleagues who made comments on earlier drafts of this book. In particular, we would like to mention Sanne ter Horst for his corrections to the first part of the book, Leonia Lerer for his comments on the first two parts, and Rob Zuidwijk for his remarks about the third part.

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Chapter 0

Introduction

This monograph is devoted to theory and applications of various types of factorizations for matrix and operator functions belonging to different classes. The types of factorizations described in the book appear in several branches of algebra, analysis and applications. Let us mention a few examples.

In the theory of non-selfadjoint operators [30, 108] there exists the notion of regular factorization of the characteristic matrix or operator function of a given operator. This type of factorization leads to the description of an invariant subspace of the operator involved and, what is more important, to triangular representation of this operator [61]. In systems theory and electrical network theory [27, 84] one encounters the notion of minimal factorization of the transfer function of a system or a network. Such a factorization allows one to represent a system or a network as a cascade connection of systems or networks with simpler synthesis. Sometimes, the situation allows for so-called complete factorizations. These are minimal factorizations where the factors are of the simplest possible type, namely of (McMillan) degree one. Dropping the minimality requirement, factorizations into degree one factors are always possible. Those that have the least possible number of factors are called quasicomplete. Via this notion a connection is made with the two machine flow shop problem from the theory of combinatorial job scheduling. Another type of factorization that we shall consider is that of canonical Wiener-Hopf factorization [45, 57] of some classes of symbols. This factorization, when it exists, allows one to invert Wiener-Hopf, Toeplitz and singular integral operators, and when the factors are known one can also build explicitly the inverses of these operators. Factorization of matrix or operator polynomials into polynomials of lower degree [69, 101] is also a type of factorization we shall discuss.

The matrix and operator functions that are considered have in common that they appear in a natural way as functions of the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B \tag{1}$$

or (after some transformations) can be represented in this form. In the above formula λ is a complex variable, and A, B, C, and D are matrices or (bounded) linear operators acting between appropriate Banach or Hilbert spaces. When A, B, C, and D are matrices or the underlying spaces are all finite-dimensional, the function W is a rational matrix function which is analytic at infinity. From mathematical systems theory it is known that conversely any rational matrix function which is analytic at infinity admits a representation of the above form. In systems theory the right-hand side of (1) is called a *state space realization* of the function W, and one refers to the space in which A is acting as the *state space*. For this reason we call the method that we are using in this book the *state space method*.

The state space approach has been used very successfully in mathematical systems theory and network theory. In this book we use the method to deal with various classes of factorization problems. The method has also been used in other branches of analysis, for instance, in interpolation theory [8, 43].

Realizations allow us to deal with factorization from a geometric point of view. Special attention is paid to various types of factorizations, for example, to canonical factorization, minimal and non-minimal factorizations, pseudo-canonical factorization, degree one factorizations and others. The problem of numerical computations of the factors of a given matrix or operator function leads in a natural way to questions of stability of divisors under small perturbations. It turns out that in general the factors are unstable. In this book the stable cases are described and estimates are given for the measures of stability.

Not only motivations but also applications play an important role in the book. We shall deal with applications to problems in mathematical systems theory and control, to problems in the theory of algebraic Riccati equations, and to inversion problems for convolution operators. Another special feature is the connection between (generally non-minimal) factorizations into elementary factors and a problem of job scheduling from combinatorial operations research. Applications to the theory of matrix and operator polynomials and rational matrix functions are included too.

Our intention was to make this monograph accessible for readers working in different areas of mathematics. We have in mind Linear Algebra, Linear Operator Theory, Integral Equations, Mathematical Systems Theory and Applied Mathematics. This forced us to make the exposition reasonably self-contained. In particular, we included some known material about characteristic operator functions, angular operators, minimal factorizations of rational matrix functions, the gap between subspaces et cetera.

We shall now give a short description of the contents of the book. Not counting the present introduction, the book consists of four parts.

Part I. The first part has a preparatory character. The motivating problems are described, and the underlying concepts are developed. In this part also the notions of nodes and characteristic function, and of systems and transfer functions are introduced. The main operations on nodes and systems are studied, and the effect of these operations on the characteristic or transfer functions are described. The basic factorization principle used throughout the book already appears in this part, including its version in terms of angular subspaces and Riccati equations. The problem of realization is also addressed, and the connection with linearization of operator functions is clarified.

Part II. The second part deals with the notions of minimality of realizations and minimality of factorizations. For finite-dimensional systems minimality is equivalent to controllability and observability. For rational matrix functions minimal realizations are constructed in terms of the pole-zero structure of the given function, and minimal factorizations are described in terms of pole-zero cancellation. This part contains also a study of the notion of minimality for various classes of finite- and infinite-dimensional systems. Using the notion of local minimality, the concept of a pseudo-canonical factorization relative to a curve is introduced and analyzed for rational matrix functions with singularities on the given curve.

Part III. The third part is devoted to the problem of factorization into elementary functions, that is, into factors that have a minimal realization with a state space of dimension one, the so-called degree one factors. A new feature is the connection to a job scheduling issue, namely to the two machine flow shop problem from operations research. The latter involves quasicomplete factorizations, that is, generally non-minimal factorizations into degree one factors with the smallest number of factors. Maple procedures are provided to calculate degree one factorizations, complete as well as quasicomplete, of companion based 2×2 rational matrix functions. This part is completely finite-dimensional and can be considered as a new advanced chapter of Linear Algebra and its Applications.

Part IV. The fourth part deals with the behavior of the factors in a factorization under small perturbations of the original function. Canonical factorization is stable in the sense that a rational matrix function which has a canonical factorization keeps this property under small perturbation. In this part we analyze the dependence of the factors on the perturbations using state space realizations. For minimal factorization the situation is different. It can happen that a rational matrix function admits a non-trivial minimal factorization while after a small perturbation the perturbed function has no such factorization. Using the realization theory the minimal factorizations that do not have this kind of instability are identified. The notions of stable, Lipschitz stable and isolated invariant subspaces turn out to play an important role in the analysis. Applications to Riccati equations are included. The case of factorization of real matrix functions is also treated in this part and yields results that differ from the case when the underlying field is complex.

Part I

Motivating Problems, Systems and Realizations

An important notion in this book is that of a time-invariant linear, discrete or continuous, input-output system. This notion is taken from mathematical systems theory. A related notion is that of an operator node. The latter originates from the theory of non-selfadjoint operators. Nodes can be considered as finite- or infinite-dimensional systems with some additional restrictions on the system coefficients. In the two theories different terminologies have been developed for objects that are essentially the same. For instance, the transfer function of a system is the same as the characteristic function of a node. On the other hand the type of problems considered in the two theories are quite different.

This first part, which consists of six chapters, is of a preparatory character. It presents in a unified way various aspects of the two theories. In Chapters 1 and 2 systems and nodes are introduced. The notions of transfer function and characteristic function are defined and discussed. The main operations on systems and nodes – inversion, product, factorization – are introduced and studied in detail. The effects of these operations on the transfer function or characteristic function are analyzed. The main principle of state space factorization theory, used throughout this book, already appears in Chapter 2 (see Section 2.4). Chapter 1 contains also a number of motivating problems. These problems and their variations reappear in different parts of the book. Chapter 3 contains the classification of systems and nodes. Chapter 4 is dedicated to the problem of linearization of analytic operator functions and of transfer functions of systems. In Chapter 5 the state space factorization theorem from Chapter 2 is reformulated in terms of angular subspaces and solutions of algebraic Riccati equations. The final chapter (Chapter 6) presents a first analysis of canonical factorization in terms of the state space method. Included are also applications to convolution equations.

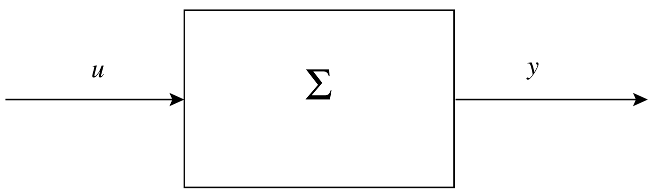
Chapter 1

Motivating Problems

This chapter has an introductory character. It presents a number of problems involving factorization of matrix- and operator-valued functions of different types. The functions considered appear as transfer functions of input output systems (Section 1.1), as characteristic functions of Hilbert space operators (Sections 1.2 and 1.3), as monic matrix polynomials (Section 1.4) or as symbols of Wiener-Hopf and singular integral equations of various types (Sections 1.5 and 1.6). For each of these classes the corresponding factorization is described. This chapter also motivates the state space setting for solving factorization problems.

1.1 Linear time invariant systems and cascade connection

A system Σ can be considered as a physical object which produces an output in response to an input. Schematically,



where u denotes the input and y denotes the output. Mathematically, the input u and the output y are vector-valued functions of a parameter t. The input can be chosen freely (at least in principle), but the output is uniquely determined by the choice of the input. Hence the map $u \mapsto y$ is a well-defined transformation, which is called the *input output operator* of the system.

The way in which the output is generated by the input can be quite complicated. In this section we consider the simplest model and assume that the relation between input and output is described by a system of differential equations of the following type:

$$\Sigma \begin{cases} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \qquad t \geq 0, \\ x(0) &= 0. \end{cases} \tag{1.1}$$

Here the coefficients are linear operators acting between Euclidean spaces,

$$A : \mathbb{C}^m \to \mathbb{C}^m, \quad B : \mathbb{C}^p \to \mathbb{C}^m, \quad C : \mathbb{C}^m \to \mathbb{C}^q, \quad D : \mathbb{C}^p \to \mathbb{C}^q.$$

Whenever convenient, we identify these operators with the corresponding matrices (using the standard bases in the Euclidean spaces).

The space $\mathbb{C}^m$ is called the *state space*, and its elements (vectors) are called *states*. The spaces $\mathbb{C}^p$ and $\mathbb{C}^q$ will be referred to as the *input space* and *output space*, respectively. The operator A is the so-called *state operator* or *main operator* of (1.1), B is the *input operator*, C is the *output operator*, and D is the *external operator*, which is also called the *feed through coefficient*. In what follows we shall call (1.1) a *finite-dimensional linear time-invariant system* or just a *system*. The qualification "finite-dimensional" refers to the finite dimensionality of the underlying spaces, and the word "time-invariant" is reflected by the fact that the coefficients A, B, C and D do not depend on the variable t.

We shall assume that the inputs u of (1.1) are taken from the space $PCE(\mathbb{C}^p)$ which consists of all piecewise continuous $\mathbb{C}^p$-valued functions on $[0, \infty)$ that are exponentially bounded. The latter means that for each $u \in PCE(\mathbb{C}^p)$ there exists real constants M and γ (depending on u), $M \geq 0$, such that $\|u(t)\| \leq Me^{\gamma t},\ t \geq 0$. Then the output y belongs to the space $PCE(\mathbb{C}^q)$ which consists of all piecewise continuous exponentially bounded $\mathbb{C}^q$-valued functions. In fact, the input output operator of (1.1) is the operator $T : PCE(\mathbb{C}^p) \to PCE(\mathbb{C}^q)$ given by

$$y(t) = (Tu)(t) = Du(t) + \int_0^t Ce^{(t-s)A}Bu(s)\, ds, \qquad t \geq 0, \tag{1.2}$$

To see this, note that

$$x(t) = \int_0^t e^{(t-s)A}Bu(s)\, ds, \qquad t \geq 0, \tag{1.3}$$

is the unique solution of the first equation in (1.1) satisfying the initial condition $x(0) = 0$. Inserting (1.3) into the second equation in (1.1) yields formula (1.2) for the input output operator.

From (1.2) it follows that the input output operator is linear. This explains the use of the term "linear" in connection with (1.1). Furthermore, one sees that (1.1) is *causal*. This means that future inputs do not affect past outputs, i.e., for each $\tau > 0$ the output $y(t)$ on $[0, \tau]$ does not depend on the input $u(t),\ t > \tau$.

Taking Laplace transforms in (1.1) we arrive at the equivalent form in *frequency domain*:

$$\begin{cases} \lambda\widehat{x}(s) &= A\widehat{x}(\lambda) + B\widehat{u}(\lambda), \\ \widehat{y}(\lambda) &= C\widehat{x}(\lambda) + D\widehat{u}(\lambda). \end{cases} \tag{1.4}$$

Here, for any exponentially bounded vector-valued function v the symbol $\widehat{v}$ denotes its Laplace transform

$$\widehat{v}(\lambda) = \int_0^\infty e^{-\lambda t} v(t)\, dt, \qquad \Re\lambda \geq c,$$

where c is some constant depending on v. From (1.4) one can solve $\widehat{y}(\lambda)$ in terms of $\widehat{u}(\lambda)$. Indeed, on some open open right half-plane of $\mathbb{C}$ we have

$$\widehat{y}(\lambda) = \big(D + C(\lambda I_m - A)^{-1}B\big)\widehat{u}(\lambda),$$

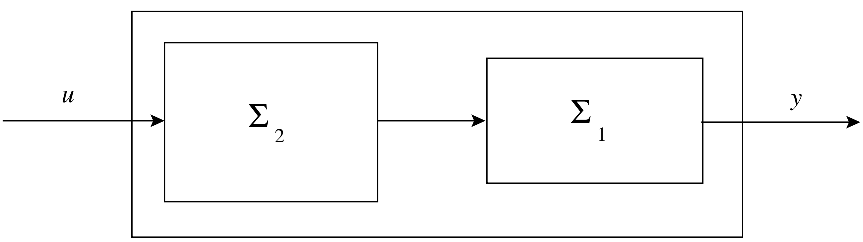
where I_m is the $m \times m$ identity matrix (or, if one prefers, the identity operator on $\mathbb{C}^m$). So in the frequency domain the input output behavior of (1.1) is determined by the function

$$W(\lambda) = D + C(\lambda I_m - A)^{-1}B, \tag{1.5}$$

which is called the *transfer function* of the system (1.1). Since the system is finite-dimensional, the transfer function is a $q \times p$ matrix function all of whose entries are rational functions. Such a function will be referred to as a *rational matrix function*. Notice that the rational matrix function W in (1.5) is analytic at infinity. A rational matrix function with this additional property is said to be *proper*.

We shall see later (in Chapter 4) that any proper rational matrix function is the transfer function of a finite-dimensional time-invariant linear system. That is, given a proper rational matrix function W, one can find matrices A, B, C, D such that (1.5) holds. In this case we call the right-hand side of (1.5) or the corresponding system (1.1) a *realization* of W. This connection allows one to study problems involving a rational matrix function in terms of the four matrices appearing in its realization. We refer to this approach as the *state space method*. In particular, we shall use the state space method to solve factorization problems.

The problem to factorize a rational matrix function into factors of simpler type appears naturally in system theory when one considers cascade connections. By definition the *cascade connection* of two systems is the system which one obtains when the output of the first system is taken to be the input of the second system. Schematically:



Let W_1 and W_2 be the transfer functions of the systems Σ_1 and Σ_2, respectively. Then, given the input u, the output y_2 of Σ_2 is given by $\widehat{y_2}(\lambda) = W_2(\lambda)\widehat{u}(\lambda)$. Since the input of Σ_1 is the output of Σ_2, it follows that the output y is given $\widehat{y}(\lambda) = W_1(\lambda)\widehat{y_2}(\lambda)$. Clearly then, the transfer function W of the cascade connection of these two systems is given by the product $W = W_1W_2$ of W_1 and W_2, that is,

$$\widehat{y}(\lambda) = W_1(\lambda)W_2(\lambda)\widehat{u}(\lambda) = W(\lambda)\widehat{u}(\lambda).$$

Let us analyze this in terms of the operators appearing in the representation (1.1). For $j = 1, 2$, let W_j be the transfer function of the system

$$\Sigma_j \begin{cases} x_j'(t) &= A_jx_j(t) + B_ju_j(t), \\ y_j(t) &= C_jx_j(t) + D_ju_j(t), \qquad t \geq 0, \\ x(0) &= 0. \end{cases}$$

We take $y_2 = u_1$, in other words, we form the cascade connection. Taking

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

as the state vector for the system thus obtained, we have

$$\begin{aligned} x'(t) &= \begin{bmatrix} A_1x_1(t) + B_1u_1(t) \\ A_2x_2(t) + B_2u_2(t) \end{bmatrix} = \begin{bmatrix} A_1x_1(t) + B_1C_2x_2(t) + B_1D_2u_2(t) \\ A_2x_2(t) + B_2u_2(t) \end{bmatrix} \\ &= \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix} x(t) + \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix} u_2(t) \end{aligned}$$

and

$$\begin{aligned} y(t) &= y_1(t) = C_1x_1(t) + D_1u_1(t) \\ &= C_1x_1(t) + D_1C_2x_2(t) + D_1D_2u_2(t) \\ &= \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} x(t) + D_1D_2u_2(t). \end{aligned}$$

Thus the transfer function $W = W_1W_2$ is also given by

$$W(\lambda) = D_1D_2 + \begin{bmatrix} C_1 & D_1C_2 \end{bmatrix} \left(\lambda - \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}\right)^{-1} \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}. \tag{1.6}$$

The fact that the transfer function of the cascade connection is the product of the transfer functions of the corresponding systems is the basis for the state space approach to factorization used in this monograph. We shall develop the state space factorization method for various types of factorization, including canonical factorization (Chapter 6), minimal factorization (Chapter 9), and factorization of rational matrix functions into a product of elementary ones (see Chapter 10 and also Theorems 2.7 and 8.15). Factorization of the latter type corresponds to cascade synthesis (of systems) involving components of simplest possible type (cf., [39] and the references therein)

As can be expected from (1.6) the problem of finding a factorization of W is related to presence of invariant subspaces of the main operator in a realization of W. This relation is one of the leading principles of this monograph. It also turns up in the theory of characteristic operator functions which we shall discuss in the next two sections.

1.2 Characteristic operator functions and invariant subspaces (1)

In the theory of characteristic functions the main object is a bounded linear operator acting on a Hilbert space, and the characteristic function serves as a unitary invariant for the operator. In this section we consider operators close to selfadjoint ones.

Let A be a bounded linear operator acting on a Hilbert space H. The adjoint of A will be denoted by A^*. The imaginary part of A, given by $\frac{1}{2i}(A - A^*)$, is a selfadjoint operator on H, and hence there exists a Hilbert space G and operators $K : G \to H$ and $J : G \to G$ such that

$$KJK^* = \frac{1}{2i}(A - A^*)$$

and J is a *signature operator*. By definition, the latter means that J is invertible and $J^{-1} = J = J^*$. From A, K and J we construct the following operator-valued function:

$$W(\lambda) = I + 2iK^*(\lambda - A)^{-1}KJ, \qquad \lambda \in \rho(A). \tag{1.7}$$

Here $\rho(A)$ is the *resolvent set* of A, that is, the set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is (boundedly) invertible.

The operator-valued function W defined by (1.7) is called the Livsic-Brodskii *characteristic operator function* of A or, more precisely, of the *operator node* $(A, KJ, 2iK^*, I; H, G)$. A Hilbert space operator J satisfying $J = J^* = J^{-1}$ is called a *signature operator*. This function has special symmetry properties. Indeed, using $KJK^* = \frac{1}{2i}(A - A^*)$ one easily checks that

$$W(\lambda)^* JW(\lambda) = J - 2i(\lambda - \bar{\lambda})JK^*(\bar{\lambda} - A^*)^{-1}(\lambda - A)^{-1}KJ.$$

It follows that

$$\begin{aligned} W(\lambda)^* JW(\lambda) &= J, \qquad \lambda \in \rho(A) \cap \mathbb{R}, \\ W(\lambda)^* JW(\lambda) &\leq J, \qquad \lambda \in \rho(A),\ \Im\lambda \leq 0. \end{aligned}$$

These formulas remain true if the positions of $W(\lambda)$ and $W(\lambda)^*$ are interchanged. It is possible, using the above formulas, to give an intrinsic characterization of the class of functions that appear as Livsic-Brodskii characteristic functions (see [30]).

The characteristic operator function can be viewed as the transfer function of the following system

$$\begin{cases} x'(t) &= Ax(t) + KJu(t), \\ y(t) &= 2iK^*x(t) + u(t), \qquad t \geq 0, \\ x(0) &= 0. \end{cases}$$

The above system will be called a *Brodskii J-system*; this term will also be used for the corresponding operator node $(A, KJ, 2iK^*, I; H, G)$.

Suppose that A is unitary equivalent to an operator B, i.e., $A = UBU^*$, where $U : H_1 \to H$ is unitary. Then

$$KJK^* = \frac{1}{2i}(A - A^*) = \frac{1}{2i}(UBU^* - UB^*U^*) = \frac{1}{2i}U(B - B^*)U^*.$$

Taking $L = U^*K$, we see that the system $(B, LJ, 2iL^*, I; H_1, G)$ is also a Brodskii J-system, and that this system has the same transfer function W as the system $(A, KJ, 2iK^*, I; H, G)$. So the characteristic operator function W does not change under unitary equivalence. Under a certain additional minimality condition the converse is also true. Indeed, if two characteristic operator functions W_1 and W_2 given by

$$\begin{aligned} W_1(\lambda) &= I + 2iK_1^*(\lambda - A_1)^{-1}K_1J, \qquad \lambda \in \rho(A_1), \\ W_2(\lambda) &= I + 2iK_2^*(\lambda - A_2)^{-1}K_2J, \qquad \lambda \in \rho(A_2), \end{aligned}$$

coincide in some neighborhood of infinity and the corresponding systems are simple (cf., Subsection 7.5.1) then the operators A_1 and A_2 are unitary equivalent. Actually there exists a unitary operator U such that $UA_1 = A_2U$ and $UK_1 = K_2$ (see [30], Theorem I.3.2). This fact is of particular interest when the imaginary part of A is small. For instance, when A has rank one, then W reduces to a scalar function, and hence the infinite-dimensional operator A is determined up to unitary equivalence by a scalar function.

The product of two Brodskii characteristic operator functions W_1 and W_2 is again a Brodskii characteristic operator function. To see this, write

$$\begin{aligned} W_1(\lambda) &= I + 2iK_1^*(\lambda - A_1)^{-1}K_1J, \qquad \lambda \in \rho(A_1), \\ W_2(\lambda) &= I + 2iK_2^*(\lambda - A_2)^{-1}K_2J, \qquad \lambda \in \rho(A_2), \end{aligned}$$

Here $A_1 : H_1 \to H_1$ and $A_2 : H_2 \to H_2$. As in the previous section it straightforward to check that the function $W = W_1W_2$ is the transfer function of the system

$$\left(\begin{bmatrix} A_1 & 2iK_1JK_2^* \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} K_1 \\ K_2 \end{bmatrix} J,\, 2i \begin{bmatrix} K_1^* & K_2^* \end{bmatrix},\, I;\, H_1 \oplus H_2, G \right).$$

Here $H_1 \oplus H_2$ is the Hilbert space direct sum of H_1 and H_2. Put

$$A = \begin{bmatrix} A_1 & 2iK_1JK_2^* \\ 0 & A_2 \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}.$$

Then $\frac{1}{2i}(A - A^*) = KJK^*$. So the function W is the characteristic operator function of the operator A.

Notice that the operator A constructed in the previous paragraph has the space H_1 as an invariant subspace. This fact contains a hint for constructing factorizations within the class of characteristic operator functions.

To be more precise, let $\Theta = (A, KJ, 2iK^*, I; H, G)$ be a Brodskii system, and assume that H_0 is an invariant subspace of A. Let Π be the orthogonal projection onto H_0. Put

$$A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \qquad K = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix}$$

with respect to the decomposition $H = H_0 \oplus H_0^\perp$. Then

$$\Theta_1 = (A_{11}, K_1J, 2iK_1^*, I; H_0, G), \qquad \Theta_2 = (A_{22}, K_2J, 2iK_2^*, I; H_0^\perp, G)$$

are Brodskii J-systems. Indeed, the imaginary part of A is given by

$$\frac{1}{2i}\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} - \begin{bmatrix} A_{11}^* & 0 \\ A_{12}^* & A_{22}^* \end{bmatrix} \right) = \begin{bmatrix} K_1JK_1^* & K_1JK_2^* \\ K_2JK_1^* & K_2JK_2^* \end{bmatrix},$$

so in particular $\frac{1}{2i}(A_{11} - A_{11}^*) = K_1JK_1^*$ and $\frac{1}{2i}(A_{22} - A_{22}^*) = K_2JK_2^*$. Moreover, $\frac{1}{2i}A_{12} = K_1JK_2^*$. This implies that the product of the characteristic operator function of A_{11} and the characteristic operator function of A_{22} (i.e., the product of the transfer function of the systems Θ_1 and Θ_2) is the characteristic operator function of A.

Under appropriate minimality conditions there is a one-one correspondence between invariant subspaces of A and factorizations of the characteristic operator function W of A as the product of two characteristic operator functions. Thus, in certain cases, the problem of finding invariant subspaces of an operator A may be solved by factorization of its characteristic operator function. For an example of the application of this technique involving the unicellularity of a Volterra operator, see Section XXVIII.11 in [47].

1.3 Characteristic operator functions and invariant subspaces (2)

The Livsic-Brodskii characteristic operator function has been designed to study operators that are not far from being selfadjoint. There are also characteristic operator functions that have been introduced in order to deal with operators that are close to unitary operators. Among them are the characteristic operator function of Sz.-Nagy and Foias and the one of M.G. Kreĭn (see [33] and [108] for references). Here we shall only discuss the characteristic operator function of Kreĭn.

The Kreĭn *characteristic operator function* has the form

$$V(\lambda) = J(K^*)^{-1}\big(J - R^*(I - \lambda A)^{-1}R\big). \tag{1.8}$$

Here $A : H \to H$, $R : G \to H$, $J : G \to G$, $K : G \to G$ are operators, the underlying spaces G and H are complex Hilbert spaces,

$$J = J^* = J^{-1}, \quad I - AA^* = RJR^*, \quad J - R^*R = K^*JK, \tag{1.9}$$

and the operators A and K are invertible. Instead of $(K^*)^{-1}$ we also write K^{-*}. With this (1.8) becomes $V(\lambda) = JK^{-*}\big(J - R^*(I - \lambda A)^{-1}R\big)$.

Obviously, (1.8) does not directly fit into the framework developed in Section 1.1. However, replacing λ by λ^{-1} and using (1.9), one can transform (1.8) into

$$U(\lambda) = K - JK^{-*}R^*A(\lambda - A)^{-1}R.$$

This is the transfer function of the system

$$\begin{cases} x'(t) &=& Ax(t) + Ru(t), \\ y(t) &=& -JK^{-*}R^*Ax(t) + Ku(t), \quad t \geq 0, \\ x(0) &=& 0. \end{cases}$$

The above system will be called a *Kreĭn J-system*; this term will also be used for the corresponding operator node

$$\Theta = (A, R, -JK^{-*}R^*A, K; H, G). \tag{1.10}$$

Observe that the external operator of a Kreĭn J-system is invertible.

The product of two Kreĭn characteristic operator functions is again a Kreĭn characteristic operator function. To see this, suppose

$$U_j(\lambda) = K_i - JK_j^{-*}R_j^*A_i(\lambda - A_j)^{-1}R_j, \qquad j = 1, 2,$$

where

$$I - A_jA_j^* = R_jJR_j^*, \qquad J - R_j^*R_j = K_j^*JK_j. \tag{1.11}$$

Then $U = U_1U_2$ is the transfer function of the operator node

$$\left(\begin{bmatrix} A_1 & -R_1JK_2^{-*}R_2^*A_2 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} R_1K_2 \\ R_2 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} -JK_1^{-*}R_1^*A_1 & -K_1JK_2^{-*}R_2^*A_2 \end{bmatrix}, K_1K_2 \right).$$

Moreover, this operator node is a Kreĭn J-system. Indeed, using (1.11) we have

$$I - \begin{bmatrix} A_1 & -R_1JK_2^{-*}R_2^*A_2 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} A_1^* & 0 \\ -A_2^*R_2K_2^{-1}JR_1^* & A_2^* \end{bmatrix}$$

$$= \begin{bmatrix} R_1K_2 \\ R_2 \end{bmatrix} J \begin{bmatrix} K_2^*R_1^* & R_2^* \end{bmatrix},$$

and

$$J - \begin{bmatrix} K_2^*R_1^* & R_2^* \end{bmatrix} \begin{bmatrix} R_1K_2 \\ R_2 \end{bmatrix} = K_2^*K_1^*JK_1K_2,$$

while finally

$$\begin{bmatrix} -JK_1^{-*}R_1^*A_1 & -K_1JK_2^{-*}R_2^*A_2 \end{bmatrix}$$

$$= -JK_1^{-*}K_2^{-*} \begin{bmatrix} K_2^*R_1^* & R_2^* \end{bmatrix} \begin{bmatrix} A_1 & -R_1JK_2^{-*}R_2^*A_2 \\ 0 & A_2 \end{bmatrix}.$$

This proves that $U = U_1U_2$ is the transfer function of a Kreĭn J-system. Notice that the main operator A is given by

$$A = \begin{bmatrix} A_1 & -R_1JK_2^{-*}R_2^*A_2 \\ 0 & A_2 \end{bmatrix},$$

and hence the space on which A_1 acts is an invariant subspace for A.

Let us consider the reverse implication. Our starting point is a Kreĭn J-system as in (1.10), with A acting on the Hilbert space H and A being invertible. Assume that H_0 is an invariant subspace of A. With respect to the decomposition $H_0 \oplus H_0^\perp$ of the state space H, write

$$A = \begin{bmatrix} A_1 & A_{12} \\ 0 & A_2 \end{bmatrix}, \qquad R = \begin{bmatrix} B_1 \\ R_2 \end{bmatrix}.$$

Suppose that A_1 or, equivalently, A_2 is invertible. From $RJR^* = I - AA^*$ one easily deduces that $R_2JR_2^* = I - A_2A_2^*$. Since A_2 is assumed to be invertible, this shows that $I - R_2JR_2^*$ is invertible. But then $I - JR_2^*R_2$ is invertible, and hence the same holds true for $J - R_2^*R_2$. The invertibility of $J - R_2^*R_2$ implies (see [33]) the existence of an invertible operator K_2 such that $J - R_2^*R_2 = K_2^*JK_2$. Put $K_1 = KK_2^{-1}$ and $R_1 = B_1K_2^{-1}$. Then K_1 is also invertible. We claim that $A_{12} = -R_1JK_2^{-*}R_2^*A_2$. In order to prove this, we first note that $A_{12} = -B_1JR_2^*A_2^{-*} = -R_1K_2JR_2^*A_2^{-*}$. Furthermore we have

$$\begin{aligned} K_2JR_2^*A_2^{-*} &= JK_2^{-*}(K_2^*JK_2)JR_2^*A_2^{-*} \\ &= JK_2^{-*}(J - R_2^*R_2)JR_2^*A_2^{-*} \\ &= JK_2^{-*}(R_2^* - R_2^*R_2JR_2^*)A_2^{-*} \\ &= JK_2^{-*}R_2^*(I - R_2JR_2^*)A_2^{-*} = JK_2^{-*}R_2^*A_2. \end{aligned}$$

Thus $A_{12} = -R_1JK_2^{-*}R_2^*A_2$. It follows that one can decompose the function $U(\lambda) = K - JK^{-*}R^*A(\lambda - A)^{-1}R$ as a product of two functions corresponding to Kreĭn J-systems. In fact, $U = U_1U_2$, where

$$U_j(\lambda) = K_j - JK_j^{-*}R_j^*A_j(\lambda - A_j)^{-1}R_j, \qquad j = 1, 2$$

with the coefficients satisfying (1.11).

We conclude this section with an interesting characterization of Kreĭn J-systems. Let G and H be complex Hilbert spaces, and let J be a signature operator on G, that is, $J = J^* = J^{-1}$. A bounded linear operator T on G is called *J-unitary* if T is invertible and $T^{-1} = JT^*J$. By definition, a node $\Theta = (A, B, C, D; H, G)$ is a Kreĭn J-system if A and D are invertible and

$$I - AA^* = BJB^*, \quad J - B^*B = D^*JD, \quad C = -JD^{-*}B^*A.$$

A straightforward calculation shows that these conditions are equivalent to the requirement that the external operator D of Θ is invertible and the operator

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \tag{1.12}$$

on $H \oplus G$ is $\widetilde{J}$-unitary. Here $\widetilde{J} = I \oplus J$. Notice that $\widetilde{J} = \widetilde{J}^* = \widetilde{J}^{-1}$. The class of nodes $\Theta = (A, B, C, D; H, G)$ for which the operator (1.12) is $\widetilde{J}$-unitary (but D not necessarily invertible) is closed under multiplication. Characteristic operator functions of the form $D + \lambda C(I - \lambda A)^{-1}B = D + C(\lambda^{-1} - A)^{-1}B$, where A, B, C and D are such that (1.12) is unitary, have been investigated (cf., [31]; see also Section 3.3 below).

Observe that if (1.12) is $\widetilde{J}$-unitary and $U(\lambda) = D + C(\lambda - A)^{-1}B$, then

$$U(\lambda)^*(-J)U(\lambda) = -J - (1 - |\lambda|^2)B^*(\bar{\lambda} - A^{-*}(\lambda - A)^{-1}B.$$

So we have

$$\begin{aligned} U(\lambda)^*(-J)U(\lambda) &= -J, \qquad |\lambda| = 1 \\ U(\lambda)^*(-J)U(\lambda) &\leq -J, \qquad |\lambda| < 1. \end{aligned}$$

It is possible to give an intrinsic characterization of the class of functions that appear as transfer functions of Kreĭn J-systems (cf., [33]; see also [1] for the case of matrix functions).

1.4 Factorization of monic matrix polynomials

By definition a *monic* $m \times m$ *matrix polynomial* of *degree* ℓ is a function of the form

$$L(\lambda) = \lambda^\ell I_m + \lambda^{\ell-1} A_{\ell-1} + \cdots + \lambda A_1 + A_0,$$

where $A_0, \ldots, A_{\ell-1}$ are $m \times m$ matrices. Given such a function, introduce

$$A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & I_m \\ -A_0 & -A_1 & \cdots & \cdots & -A_{\ell-1} \end{bmatrix}, \tag{1.13}$$

the *first companion operator matrix* associated with L, and

$$C = \begin{bmatrix} 0 & \cdots & 0 & I_m \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}. \tag{1.14}$$

Then the (*pointwise*) *inverse* L^{-1} of L, given by $L^{-1} = L(\lambda)^{-1}$, has the realization

$$L^{-1}(\lambda) = C(\lambda I_m - A)^{-1} B \tag{1.15}$$

(see [66]).

To prove this identity, consider the set of differential equations in $\mathbb{C}^n$-valued vector functions y given by

$$\begin{cases} y^{(\ell)}(t) + A_{\ell-1} y^{(\ell-1)}(t) + \cdots + A_1 y'(t) + A_0 y(t) = u(t), \\ y(0) = 0,\ y'(0) = 0\ ,\ \ldots,\ y^{(\ell-1)}(0) = 0. \end{cases} \tag{1.16}$$

Let us transform this to a higher-dimensional first-order system in the usual way, by introducing

$$x(t) = \left[\, y(t)^\top \; y'(t)^\top \; \dots \; y^{(\ell-1)}(t)^\top \,\right]^\top.$$

Then, because of (1.16), we have

$$\begin{cases} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \qquad\qquad t \geq 0, \\ x(0) &= 0, \end{cases} \tag{1.17}$$

where A, B, and C are defined by (1.13) and (1.14). Taking Laplace transform in (1.17) and eliminating $\widehat{x}(s)$ we obtain that

$$\widehat{y}(s) = C(s-A)^{-1}B\widehat{u}(s).$$

On the other hand, taking Laplace transform in (1.16) we get $L(s)\widehat{y}(s) = \widehat{u}(s)$. Thus (1.15) has been established.

We shall consider the problem of finding and describing factorizations of $L(\lambda)$ of the form $L(\lambda) = L_2(\lambda)L_1(\lambda)$, where L_1 and L_2 are again monic matrix polynomials. Certain invariant subspaces of the operator A play an important role in solving this problem (see Section 3.4).

1.5 Wiener-Hopf integral operators and factorization

In this section we outline the factorization method of [59] to solve systems of Wiener-Hopf integral equations. Such a system may be written as a single *Wiener-Hopf equation*

$$\phi(t) - \int_0^\infty k(t-s)\phi(s)\,ds = f(t), \qquad 0 \leq t < \infty, \tag{1.18}$$

where ϕ and f are m-dimensional vector functions and $k \in L_1^{m\times m}(-\infty,\infty)$, that is, the kernel function k is an $m \times m$ matrix function whose entries are in $L_1(-\infty,\infty)$. We assume that the given vector function f has its component functions in $L_p[0,\infty)$, and we express this property by writing $f \in L_p^m[0,\infty)$. Throughout this section p will be fixed and $1 \leq p < \infty$. The problem we shall consider is to find a solution ϕ of equation (1.18) that also belongs to the space $L_p^m[0,\infty)$.

Equation (1.18) has a unique solution $\phi \in L_p^m[0,\infty)$ for any right-hand side $f \in L_p^m[0,\infty)$ if and only if the *Wiener-Hopf integral operator* $I - \mathbf{K} : L_p^m[0,\infty) \to L_p^m[0,\infty)$ is invertible, where

$$\big(\mathbf{K}\phi\big)(t) = \int_0^\infty k(t-s)\phi(s)\,ds, \qquad t \geq 0.$$

The usual method (see [59]) to solve equation (1.18) is as follows. First assume that (1.18) has a solution ϕ in $L_p^m[0,\infty)$. Extend ϕ and f to the full real line by putting

$$\phi(t) = 0, \qquad f(t) = -\int_0^\infty k(t-s)\phi(s)\,ds, \quad t < 0.$$

Then $\phi, f \in L_p^m(-\infty,\infty)$ and the full line convolution equation

$$\phi(t) - \int_{-\infty}^\infty k(t-s)\phi(s)\,ds = f(t), \qquad -\infty < t < \infty$$

is satisfied. By applying the Fourier transformation and leaving the part of f that is given in the right-hand side, one gets

$$\big(I_m - K(\lambda)\big)\Phi_+(\lambda) - F_-(\lambda) = F_+(\lambda), \qquad \lambda \in \mathbb{R}, \tag{1.19}$$

where

$$K(\lambda) = \int_{-\infty}^\infty e^{i\lambda t}k(t)\,dt, \qquad F_+(\lambda) = \int_0^\infty e^{i\lambda t} f(t)\,dt,$$

$$\Phi_+(\lambda) = \int_0^\infty e^{i\lambda t}\phi(t)\,dt, \qquad F_-(\lambda) = \int_{-\infty}^0 e^{i\lambda t} f(t)\,dt.$$

Note that the functions K and F_+ are given, but the functions Φ_+ and F_- have to be found. In fact in this way the problem to solve (1.18) is reduced to that of finding two functions Φ_+ and F_- such that (1.19) holds, while furthermore Φ_+ and F_- must be as above with $\phi \in L_p^m[0,\infty)$ and $f \in L_p^m(-\infty,0]$.

To find Φ_+ and F_- of the desired form such that (1.19) holds, one factorizes the $m \times m$ matrix function $I_m - K(\lambda)$. This function is called the *symbol* of the integral equation (1.18). Assume that the symbol admits a factorization of the form

$$I_m - K(\lambda) = \big(I_m + G_-(\lambda)\big)\big(I_m + G_+(\lambda)\big), \qquad \lambda \in \mathbb{R}, \tag{1.20}$$

where

$$G_+(\lambda) = \int_0^\infty e^{i\lambda t} g_+(t)\,dt, \qquad G_-(\lambda) = \int_{-\infty}^0 e^{i\lambda t} g_-(t)\,dt,$$

with $g_+ \in L_1^{m\times m}[0,\infty)$ and $g_- \in L_1^{m\times m}(-\infty,0]$ while, in addition, the determinants $\det\big(I_m + G_+(\lambda)\big)$ and $\det\big(I_m + G_-(\lambda)\big)$ do not vanish in the closed upper and lower half-plane, respectively. We shall refer to the factorization (1.20) as a *right canonical factorization of* $I_m - K(\lambda)$ *with respect to the real line*. Under the conditions stated above the functions $\big(I_m + G_+(\lambda)\big)^{-1}$ and $\big(I_m + G_-(\lambda)\big)^{-1}$ admit representations as Fourier transforms:

$$\big(I_m + G_+(\lambda)\big)^{-1} = I_m + \int_0^\infty e^{i\lambda t}\gamma_+(t)\,dt, \tag{1.21}$$

$$\big(I_m + G_-(\lambda)\big)^{-1} = I_m + \int_{-\infty}^0 e^{i\lambda t}\gamma_-(t)\,dt, \tag{1.22}$$

with $\gamma_+ \in L_1^{m\times m}[0,\infty)$ and $\gamma_- \in L_1^{m\times m}(-\infty,0]$. Using the factorization (1.20) and suppressing the variable λ, equation (1.19) can be rewritten as

$$(I_m + G_+)\Phi_+ - (I_m + G_-)^{-1}F_- = (I_m + G_-)^{-1}F_+. \tag{1.23}$$

Let $\mathcal{P}$ be the projection acting on the Fourier transforms of $L_p^m(-\infty,\infty)$-functions according to the following rule

$$\mathcal{P}\left(\int_{-\infty}^{\infty} e^{i\lambda t}h(t)\,dt\right) = \int_0^{\infty} e^{i\lambda t}h(t)\,dt.$$

Applying $\mathcal{P}$ to (1.23), one gets

$$(I_m + G_+)\Phi_+ = \mathcal{P}\big((I_m + G_-)^{-1}F_+\big),$$

and hence

$$\Phi_+ = (I_m + G_+)^{-1}\mathcal{P}\big((I_m + G_-)^{-1}F_+\big),$$

which is the formula for the solution of equation (1.19). To obtain the solution ϕ of the original equation (1.18), i.e., to obtain the inverse Fourier transform of Φ_+, one can employ the formulas (1.21) and (1.22). In fact,

$$\phi(t) = f(t) + \int_0^{\infty} \gamma(t,s)f(s)\,ds, \qquad t \geq 0,$$

where the kernel $\gamma(t,s)$ is given by

$$\gamma(t,s) = \begin{cases} \gamma_+(t-s) + \displaystyle\int_0^s \gamma_+(t-r)\gamma_-(r-s)\,dr, & 0 \leq s < t, \\ \gamma_-(t-s) + \displaystyle\int_0^t \gamma_+(t-r)\gamma_-(r-s)\,dr, & 0 \leq t < s. \end{cases} \tag{1.24}$$

We conclude the description of this factorization method by mentioning that the equation (1.18) has a unique solution in $L_p^m[0,\infty)$ for each f in $L_p^m[0,\infty)$ if and only if its symbol admits a factorization as in (1.20). For details, see [45], [59].

To illustrate the method, let us consider a special choice for the right-hand side f (cf., [59]). Take

$$f(t) = e^{-iqt}x_0, \tag{1.25}$$

where x_0 is a fixed vector in $\mathbb{C}^m$ and q is a complex number with $\Im q < 0$. Then

$$F_+(\lambda) = \int_0^{\infty} e^{i(\lambda-q)t}x_0\,dt = \frac{i}{\lambda - q}x_0, \qquad \Im\lambda \geq 0.$$

Now observe that

$$\frac{i}{\lambda - q}\Big(\big(I_m + G_-(\lambda)\big)^{-1} - \big(I_m + G_-(q)\big)^{-1}\Big)x_0$$

is the Fourier transform of an $L_p^m(-\infty, 0]$-function and hence it vanishes when applying the projection $\mathcal{P}$. It follows that in this case the formula for Φ_+ may be written as

$$\Phi_+(\lambda) = \frac{i}{\lambda - q}\big(I_m + G_+(\lambda)\big)^{-1}\big(I_m + G_-(q)\big)^{-1}x_0.$$

Recall that the solution ϕ is the inverse Fourier transform of Φ_+. So we have

$$\phi(t) = e^{-iqt}\left(I_m + \int_0^t e^{iqs}\gamma_+(s)\,ds\right)\big(I_m + G_-(q)\big)^{-1}x_0. \tag{1.26}$$

1.6 Block Toeplitz equations and factorization

In this section we consider the discrete analogue of a Wiener-Hopf integral equation, that is, a block *Toeplitz equation*. So we consider an equation of the type

$$\sum_{k=0}^{\infty} a_{j-k}\xi_k = \eta_j, \qquad j = 0, 1, 2, \ldots . \tag{1.27}$$

Throughout we assume that the coefficients a_j are given complex $m \times m$ matrices satisfying

$$\sum_{j=-\infty}^{\infty} \|a_j\| < \infty,$$

and $\eta = (\eta_j)_{j=0}^{\infty}$ is a given vector from $\ell_p^m = \ell_p(\mathbb{C}^m)$. The problem is to find $\xi = (\xi_k)_{k=0}^{\infty} \in \ell_p^m$ such that (1.27) is satisfied. We shall restrict ourselves to the case when $1 \le p \le 2$; the final results however are valid for $2 < p \le \infty$ as well.

Assume $\xi \in \ell_p^m$ is a solution of (1.27). Then one can write (1.27) in the form

$$\sum_{k=-\infty}^{\infty} a_{j-k}\xi_k = \eta_j, \qquad j = 0, \pm 1, \pm 2, \ldots , \tag{1.28}$$

where $\xi_k = 0$ for $k < 0$ and η_j is defined by (1.28) for $j < 0$. Multiplying both sides of (1.28) by λ^j with $|\lambda| = 1$ and summing over j, one gets

$$a(\lambda)\xi_+(\lambda) - \eta_-(\lambda) = \eta_+(\lambda), \qquad |\lambda| = 1, \tag{1.29}$$

where the functions a, η_+, η_-, ξ_+ and ξ_+ are given by

$$a(\lambda) = \textstyle\sum_{j=-\infty}^{\infty} \lambda^j a_j, \qquad \eta_+(\lambda) = \sum_{j=0}^{\infty} \lambda^j \eta_j,$$

$$\xi_+(\lambda) = \textstyle\sum_{j=0}^{\infty} \lambda^j \xi_j, \qquad \eta_-(\lambda) = \sum_{j=-\infty}^{-1} \lambda^j \eta_j.$$

In this way the problem to solve (1.27) is reduced to that of finding two functions ξ_+ and η_- such that (1.29) holds, while moreover, ξ_+ and η_- must be as above with $(\xi_j)_{j=0}^{\infty}$ and $(\eta_{-j-1})_{j=0}^{\infty}$ from ℓ_p^m.