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Trails in Kinetic Theory

Foundational Aspects and Numerical Methods

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Preface

In the last decades, kinetic theory has emerged as one of the most prominent fields of modern mathematics. It was originally developed as a field of Mathematical Physics to investigate interacting particle systems and their corresponding continuum descriptions. Yet, in the recent years, there has been an explosion of applications of kinetic theory to other areas of research such as biology and social sciences. Kinetic-type equations currently represent a common ground for the cross-fertilization between a heterogeneity of communities that include both pure and applied disciplines.

The source for the broad applicability of kinetic theory lies on the omnipresence of emergent phenomena in real-life applications. Emergent phenomena correspond to the appearance of large-scale (observable) structures from the underlying microscopic, discrete dynamics. Kinetic theory provides precisely the mathematical framework to link these discrete dynamics with their corresponding continuum equations at the macro-scale. Nowadays, countless applications, ranging from plasma physics to socio-economic and soft matter, have roots in kinetic theory. At the same time, the investigation of emerging phenomena in these new fields of applications presents many mathematical challenges at the level of modelling, mathematical analysis and numerics.

All these aspects have been presented in the School *Trails in Kinetic Theory: Foundational Aspects and Numerical Methods* organized by the editors of the present book during the Trimester Program on Kinetic Theory at the Hausdorff Institute for Mathematics of Bonn in June 2019. During this event, four eminent lecturers and eleven invited speakers of the highest profile presented research advances and cutting-edge results to broadest scientific community. In the following, we briefly describe the main topics of this event, which will constitute the backbone for the book.

Theoretical Aspects

The problem of deriving macroscopic evolution equations from the microscopic description based on the fundamental laws of mechanics, through suitable scaling limits, is a central problem of non-equilibrium statistical mechanics. The resulting

kinetic equations are essential to describe the relevant physical properties of the system and their time evolution. Classical examples of kinetic equations are the Boltzmann equation, the Landau equation and the Vlasov equation. During the last couple of decades, several important results have been achieved, both in the derivation and in the analysis of the kinetic equations, which have increased the current understanding of kinetic theory. Most of the rigorous results available in this direction have been obtained for dilute gases or for systems with weak interactions. The rigorous analysis of the mean-field limit and the Vlasov equation is a well-understood subject in the case of smooth potentials, but the interesting case of the Coulomb interaction is still open. Much more subtle are the limiting physical situations leading to the Boltzmann equation and the Landau equation. After the fundamental Lanford's result on the rigorous derivation of the Boltzmann equation from a gas of hard spheres even though only for a short time interval, the case of smooth short-range interaction potentials has been studied, but the validity (or non-validity) of the Boltzmann equation in the case of long-range potentials is still open and challenging, as well as the short-time validity limitation. Concerning the derivation of the Landau equation, a rigorous proof is still missing, even for short time. Recently, some partial results on the extension to long-range interactions have been proposed in the simplified case of the Lorentz gas, which consists of a single particle moving through infinitely heavy, randomly distributed scatterers. Significant progress has been made in the derivation of kinetic equations for quantum particles, despite that the understanding of interacting particle systems is much more partial than the classical ones.

Another well-established research direction in kinetic theory is the qualitative behaviour of the solutions of kinetic equations, and the analysis of their long-time asymptotics, closely related to the problem of hydrodynamics. It is worth to mention that also in this direction several hard problems still need further investigation, as, for instance, the proof of global well-posedness of classical solutions of Boltzmann equation, which is still elusive. Kinetic equations are also used in various other settings. One example is, for instance, the Smoluchowski equation arising from problems of polymerization, particle aggregation in aerosols or drop formation in rain. The analysis and validation from particle systems of the Smoluchowski equation is much less developed, but there has recently been progress in understanding the properties of self-similar solutions and a first step in understanding the derivation of coagulation equations from mechanical particle systems.

Applications in Socio-Economic and Life Sciences

As mentioned before, emergent phenomena are ubiquitous in nature. Being able to link phenomena at the different scales is crucial to giving answers to questions in the experimental sciences: How do we explain the self-organization of a tissue from its underlying constituents? How crowds of pedestrian self-organize into lanes? How does opinion dynamics evolve over time from local and macroscopic interactions?

Kinetic theory has found in the recent years multiple applications in biology and social sciences, especially in the fields of collective motion and opinion dynamics.

This poses many new challenges: first, at the level of the modelling, we need to consider simple models that are tractable enough but that, at the same time, capture the phenomena under investigation; second, new coarse-graining tools need to be developed to obtain macroscopic equations since the classical tools cannot be applied (this can be due, for example, to the lack of conserved quantities or to the appearance of phase transitions); third, new numerical methods need to be developed (for example, for hyperbolic non-conservative equations); finally, some systems exhibit violation of the propagation of chaos, making it impossible to derive kinetic equations with classical methods. In conclusion, the applications of kinetic theory to emergent phenomena in biology and social sciences open new fascinating questions that will push further the borders of our mathematical understanding and methodologies.

Numerical Methods and Uncertainty Quantification

The development of numerical methods for kinetic equations has been the subject of extremely active investigations in the past decades, especially in relation to efficient approximations of equilibrium states and in connection to the multi-scale limits of collisional equations. Significant progress has been obtained in a variety of problems in kinetic theory, e.g. granular gases, kinetic methods for soft matter physics, optimal control of kinetic equations.

Furthermore, in recent years, significant efforts have been dedicated to incorporate possible deviations from the systems' prescribed deterministic behaviour. Our analytical understanding of the structural randomness seems to be crucial to provide reliable descriptions of real-world models. A step towards realistic modelling demands a quantification of the possible deviations of a model-driven approach measuring errors and uncertainties. In a kinetic setting, the general strategy to take into consideration the realistic lack in information due, for example, to empirical assumptions or incomplete knowledge of boundary terms or initial data relies on an increased dimensionality of the particles' distribution. In the recent literature, authors deal with this challenge through *uncertainty quantification* (UQ) methods whose approach provides accurate algorithms for the a priori estimation of the impact of uncertainties in terms of statistical moments. All these methods seem to be particularly appropriate also in connection to socio-economic and life sciences phenomena since most of the models are generally not derived from first principles.

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Recent Development in Kinetic Theory of Granular Materials: Analysis and Numerical Methods

José Antonio Carrillo, Jingwei Hu, Zheng Ma, and Thomas Rey

Abstract Over the past decades, kinetic description of granular materials has received a lot of attention in mathematical community and applied fields such as physics and engineering. This article aims to review recent mathematical results in kinetic granular materials, especially for those which arose since the last review Villani (J Stat Phys 124(2):781–822, 2006) by Villani on the same subject. We will discuss both theoretical and numerical developments. We will finally showcase some important open problems and conjectures by means of numerical experiments based on spectral methods.

Keywords Granular gases equation · Inelastic Boltzmann equation · Fast spectral method · Asymptotic behavior · Review · Hydrodynamic equations · Granular flows · GPU

1 The Boltzmann Equation for Granular Gases

Granular gases have been initially introduced to describe the nonequilibrium behavior of materials composed of a large number of unnecessarily microscopic particles, such as grains or sand. These particles form a gas, interacting *via* energy dissipating inelastic collisions. Statistical mechanics description of particle systems through inelastic collisions faces basic derivation problems such as the inelastic collapse [72], i.e. infinite many collisions in finite time. Nevertheless, the kinetic

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description of rapid granular flows [57, 65, 66] has been able to compute transport coefficients for hydrodynamic descriptions successfully used in situations that are a long way from their supposed limits of validity, to describe, for instance, shock waves in granular gases [28, 82], clustering [32, 37, 61], and the Faraday instability for vibrating thin granular layers [27, 28, 36, 46, 73, 88–90]. A large amount of practical systems can be described as a granular gas, such as for example spaceship reentry in a dusty atmosphere (Mars for instance), planetary rings [7, 70] and sorting behavior in vibrating layers of mixtures. A lot of other examples can be found in the thesis manuscript [40], and in the seminal book of Brilliantov and Pöschel [31].

Usually, a granular gas is composed of 10^6–10^{16} particles. The study of such a system will then be impossible with a direct approach, and we shall adopt a kinetic point of view, studying the behavior of a one-particle distribution function f, depending on time $t \geq 0$, space $x \in \Omega \subset \mathbb{R}^{d_x}$ and velocity $v \in \mathbb{R}^d$, for $d_x \leq d \in \{1, 2, 3\}$. The statistical mechanics description of the system has been then admitted in the physical community as the tool to connect the microscopic description to macroscopic system of balance laws in rapid granular flows [30, 57, 59, 65, 66] as in the classical rarefied gases [39]. In this first section, we shall review some basics on the inelastic Boltzmann equation, and present the mathematical state of the art since the previous review paper on the subject [91].

Microscopic Dynamics The microscopic dynamics can be summarized with the following hypotheses:

1. The particles interact via *binary* collisions. More precisely, the gas is rarefied enough so that collisions between 3 or more particles can be neglected.
2. These binary collisions are localized in space and time. In particular, all the particles are considered as point particles, even if they describe macroscopic objects.
3. Collisions preserve mass and momentum, but dissipate a fraction $1 - e$ of the kinetic energy in the impact direction, where the inelasticity parameter $e \in [0, 1]$ is called *restitution coefficient*:

$$\begin{cases} v' + v'_* = v + v_*, \\ |v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\dfrac{1 - e^2}{2} |(v - v_*) \cdot \omega|^2 \leq 0, \end{cases} \tag{1}$$

with $\omega \in \mathbb{S}^{d-1}$ being the impact direction. Using these conservation, one has the following two possible parametrizations (see also Fig. 2) of the post-collisional velocities, as a function of the pre-collisional ones:

- The ω-representation or reflection map, given for $\omega \in \mathbb{S}^{d-1}$ by

$$\begin{aligned} v' &= v - \frac{1 + e}{2} ((v - v_*) \cdot \omega) \, \omega, \\ v'_* &= v_* + \frac{1 + e}{2} ((v - v_*) \cdot \omega) \, \omega. \end{aligned} \tag{2}$$

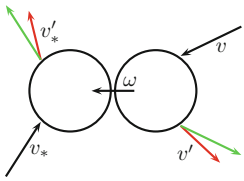

Fig. 1 Geometry of the inelastic collision in the physical space (green is elastic, red is inelastic)

- The σ-representation or swapping map, given for $\sigma \in \mathbb{S}^{d-1}$ by

$$v' = \frac{v + v_*}{2} + \frac{1-e}{4}(v - v_*) + \frac{1+e}{4}|v - v_*|\sigma,$$
$$v'_* = \frac{v + v_*}{2} - \frac{1-e}{4}(v - v_*) - \frac{1+e}{4}|v - v_*|\sigma. \tag{3}$$

Remark 1 Taking $e = 1$ in both (2) and (3) yields the classical energy-conservative elastic collision dynamics, as illustrated in Fig. 1.

The geometry of collisions is more complex than the classical elastic one. Indeed, fixing $v, v_* \in \mathbb{R}^d$, denote by

$$\Omega_\pm := \frac{v + v_*}{2} \pm \frac{1-e}{4}(v_* - v), \qquad O := \frac{v + v_*}{2} = \frac{v' + v'_*}{2}.$$

Then if $u := v - v_*$ is the *relative velocity*, one has

$$|\Omega_+ - v'| = |\Omega_- - v'_*| = \frac{1+e}{4}|u|,$$

namely $v' \in \mathcal{S}\left(\Omega_+, |u|(1+e)/4\right)$ and $v'_* \in \mathcal{S}\left(\Omega_-, |u|(1+e)/4\right)$, where $\mathcal{S}(x, r)$ is the sphere centered in x and of radius r (see also Fig. 2).

Restitution Coefficient The physics literature is quite divided on the question of whether the restitution coefficient e should be a constant or not [31]. Although most of the early mathematical results on the topic consider a constant e [91], it seems that this case is only realistic in dimension 1 of velocity (the so-called "collisional cannon" described in the chapter 4 of [31] is a famous counter-example). The true realistic case considers that e depends on the relative velocity $|v-v_*|$ of the colliding particles. Even more precisely, it must be close to the elastic case 1 for small relative velocities (namely no dissipation, elastic case), and decay towards 0 when this relative velocity is large. The first mathematical result on this direction can be found in [85], where

$$e(|v - v_*|) = \frac{1}{1 + c\,|v - v_*|^\gamma}, \tag{4}$$

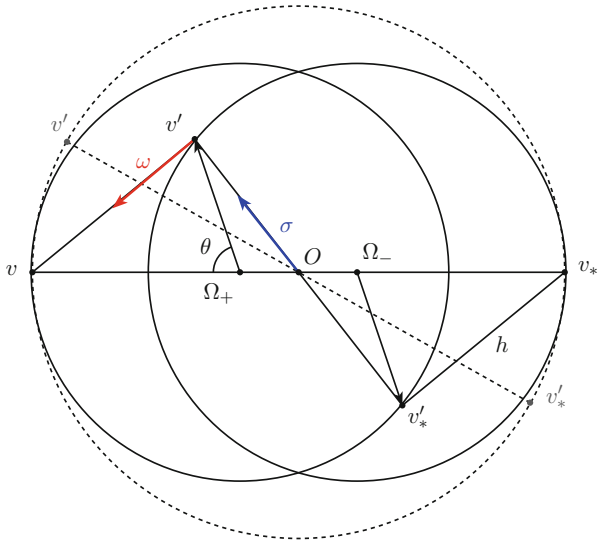

Fig. 2 Geometry of the inelastic collision in the phase space (dashed lines represent the elastic case)

for a nonnegative constant c characterizing the inelasticity strength ($c = 0$ being elastic), and $\gamma \in \mathbb{R}$.

Another important case is the so-called *viscoelastic* hard spheres one, thoroughly studied mathematically in a series of papers [4–6, 11], where e is given by the implicit relation

$$e(|v - v_*|) + a|v - v_*|^{1/5} e(|v - v_*|)^{3/5} = 1, \tag{5}$$

for $a > 0$. More details on the derivation of this expression can be found in [31, Chapter 3].

Another quite rigorous study has been made in [81], with a threshold-dependent restitution coefficient:

$$e(r) = \begin{cases} 1 & \text{if } r < r_*, \\ \bar{e} & \text{if } r \geq r_*, \end{cases}$$

$\bar{e} < 1$ and $r_* > 0$ being fixed.

Finally, the case $e = 0$ describes *sticky collisions*: the normal component of the kinetic energy being completely dissipated during impact, the particles stick and travel together in the tangent direction after impact. A derivation of the model from the microscopic dynamics on the line can be found in [29, 42].

Remark 2 This model is meaningful even in dimension 1, which is not the case for elastic collisions. Indeed, such monodimensional collisions are only

$$\{v', v'_*\} = \{v, v_*\},$$

meaning that the particle velocities are either swapped or preserved. The particles being indistinguishable, nothing happens.[1] In the 1d inelastic case, the collisions are given using (3) by

$$\{v', v'_*\} = \{v, v_*\} \quad \text{or} \quad \left\{\frac{v + v_*}{2} \pm \frac{e}{2}(v - v_*)\right\}$$

depending on the value of $\sigma \in \{\pm 1\}$.

The Granular Gases Operator: Weak Form Using the microscopic hypotheses (1–1–1), one can derive the granular gases collision operator $\mathcal{Q}_{\mathcal{I}}$, by following the usual elastic procedure (see [91] for more details). Its *weak form* in the σ-representation is given by

$$\int_{\mathbb{R}^d} \mathcal{Q}_{\mathcal{I}}(f, f)(v)\, \psi(v)\, dv = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_* f \left(\psi' + \psi'_* - \psi - \psi_*\right) \cdot B(|v - v_*|, \cos\theta, E(f))\, d\sigma\, dv\, dv_*, \tag{6}$$

where the collision kernel is typically of the form $B(|u|, \cos\theta, E(f)) = \Phi(|u|)b(\cos\theta, E(f))$, and $E(f)$ is the kinetic energy of f, namely its second moment in velocity, the postcollisional velocities are computed by (3), and θ is the angle between σ and u. We shall assume in all the following of this section that the collision kernel is of *generalized hard sphere* type, namely

$$B(|u|, \cos\theta, E)) = \Phi(|u|)\widehat{b}(\cos\theta, E) = |u|^{\lambda}\, b(\cos\theta)\, E^{\gamma}, \tag{7}$$

where $\lambda \in [0, 1]$ ($\lambda = 0$ being the simplified *Maxwellian pseudo-molecules case* and $\lambda = 1$ the more relevant *hard sphere* case), $\gamma \in \mathbb{R}$ and the angular cross section b verifies

$$0 < \beta_1 \leq b(x) \leq \beta_2 < \infty, \quad \forall x \in [-1, 1]. \tag{8}$$

Remark 3 Note that we assumed that the collision kernel B in (6) depends on the relative velocity, the angle of collision, and on $E(f)$. These former dependencies are quite classical, but the latter is not. Nevertheless, it makes a lot of sense physically speaking, as one can see in [83].

[1] Because of that, the elastic collision operator is simply equal to 0 for a one-dimensional velocity space, the Boltzmann equation reducing only to the free transport equation.

The weak form in the ω-representation can be written analogously as

$$\int_{\mathbb{R}^d} \mathcal{Q}_{\mathcal{I}}(f,f)(v)\,\psi(v)\,dv = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{S}^{d-1}} f_*\,f\,\left(\psi' + \psi'_* - \psi - \psi_*\right) \cdot \widetilde{B}(|u|,\cos\theta,E(f))\,d\omega\,dv\,dv_*, \tag{9}$$

where the postcollisional velocities are computed by (2), θ is the angle between ω and u, and

$$\widetilde{B}(|u|,\cos\theta,E) = |u|^{\lambda}\widetilde{b}(\cos\theta)E^{\gamma}$$

with $\widetilde{b}(t) = 3|t|b(1-2t^2)$ for $-1 \le t \le 1$ by the change of variables between the σ- and the ω-representation, see [34] for details.

The Granular Gases Operator: Strong Form Deriving a strong form of $\mathcal{Q}_{\mathcal{I}}$ with the reflection map in the ω-representation is a matter of a change of variables. However, deriving a strong form of $\mathcal{Q}_{\mathcal{I}}$ is not as easy as in the elastic case in the σ-representation since the collisional transform $(v, v_*, \sigma) \to (v', v'_*, \sigma)$ is not an involution and we have to go through the ω-representation, see [34] for details.

More precisely, given the restitution coefficient $e = e(|u|)$ depending on the relative velocity of the particles $u = v - v_*$, we assume the collisional transform's Jacobian for (2) is $J(|u|,\cos\theta) \neq 0$ for all z. Notice $J = e$ in the constant restitution case. It is in general a complicated expression of the relative speed $r = |u|$ and $s = \cos\theta$ involving e and its derivative. Then, the precollisional velocities read as

$$\begin{aligned} {}'v &= v - \frac{1+e}{2e}\left((v-v_*)\cdot\omega\right)\omega, \\ {}'v_* &= v_* + \frac{1+e}{2e}\left((v-v_*)\cdot\omega\right)\omega. \end{aligned} \tag{10}$$

The final strong from of the operator is $\mathcal{Q}_{\mathcal{I}}(f,f)(v) = \mathcal{Q}^+_{\mathcal{I}}(f,f)(v) - f(v)\,L(f)(v)$ with the loss part of the operator given by

$$L(f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widetilde{B}(|v-v_*|,\cos\theta,E(f))\,f_*\,d\omega\,dv_*$$

and the gain part of the operator in strong form written as

$$\mathcal{Q}^+_{\mathcal{I}}(f,f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \widetilde{\Phi}^+_e(|u|,\cos\theta)\widetilde{b}^+_e(\cos\theta)\frac{E^{\gamma}}{J(|u|,\cos\theta)}\,{}'f\,{}'f_*\,d\omega\,dv_*, \tag{11}$$

with $\widetilde{\Phi}_e^+(r,s)$ and $\widetilde{b}_e^+(s)$ given by

$$\widetilde{b}_e^+(s) = \widetilde{b}\left(\frac{s}{\sqrt{e^2+(1-e^2)s^2}}\right), \tag{12}$$

and

$$\widetilde{\Phi}_e^+(r,s) = \Phi\left(\frac{r}{e}\sqrt{e^2+(1-e^2)s^2}\right) = \left(\frac{r}{e}\sqrt{e^2+(1-e^2)s^2}\right)^{\lambda}. \tag{13}$$

We can derive now the following *strong form* of the collision operator also in the σ-representation by just changing variable in the operator from ω to σ, see [34], to find the expressions of the loss and the gain terms in the σ-representation:

$$L(f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} B(|v-v_*|,\cos\theta,E(f))\,f_*\,d\sigma\,dv_*$$

and

$$\mathcal{Q}_{\mathcal{I}}^+(f,f)(v) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \Phi_e^+(|u|,\cos\theta)b_e^+(\cos\theta)\frac{E^{\gamma}}{J(|u|,\cos\theta)}\,{}'f\,{}'f_*\,d\sigma\,dv_*, \tag{14}$$

with $\Phi_e^+(r,s)$ and $b_e^+(s)$ given by

$$b_e^+(s) = b\left(\frac{(1+e^2)s-(1-e^2)}{(1+e^2)-(1-e^2)s}\right)\frac{\sqrt{2}}{\sqrt{(1+e^2)-(1-e^2)s}}, \tag{15}$$

and

$$\Phi_e^+(r,s) = \Phi\left(\frac{r}{\sqrt{2}e}\sqrt{(1+e^2)-(1-e^2)s}\right) = \left(\frac{r}{\sqrt{2}e}\sqrt{(1+e^2)-(1-e^2)s}\right)^{\lambda}. \tag{16}$$

In these expressions, the precollisional velocities are given in the σ-representation by

$$\begin{aligned} {}'v &= \frac{v+v_*}{2} + \frac{1-e}{4e}(v-v_*) + \frac{1+e}{4e}|v-v_*|\sigma,\\ {}'v_* &= \frac{v+v_*}{2} - \frac{1-e}{4e}(v-v_*) - \frac{1+e}{4e}|v-v_*|\sigma. \end{aligned} \tag{17}$$

The granular gases collision operator has then the same structure of the elastic Boltzmann operator under Grad's cutoff assumption, namely it can be seen as the difference between the *inelastic* gain term $\mathcal{Q}_{\mathcal{I}}^{+}(f, f)$ and the loss term $f\, L(f)$, which depends only on the chosen collision kernel, but not on the inelasticity.

We shall call the following *granular gases equation*, or the inelastic Boltzmann equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \mathcal{Q}_{\mathcal{I}}(f, f). \tag{18}$$

We shall denote its first fluid moments (resp. density, mean momentum, and kinetic energy) by

$$(\rho(f),\ \rho(f)\boldsymbol{u}(f),\ E(f)) := \int_{\mathbb{R}^d} \left(1, v, |v|^2/2\right) f(v)\, dv.$$

Remark 4 There is another popular approach to describe granular gases, which uses an *Enskog*-type collision operator. It is more relevant physically because it allows to keep the particles' radii δ positive, hence delocalizing the collision.[2] The microscopic hypothesis 1 is in particular not valid anymore. The strong form of the collision operator in the constant restitution coefficient case is given by

$$\mathcal{Q}_{\mathcal{E}}(f, f)(x, v) = \delta^{d-1} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (\widetilde{\Phi}_e^{+}(|u|, \cos\theta)\widetilde{b}_e^{+}(\cos\theta)\frac{G(\rho_+)}{e}\, {f_+}'\, {f_*}' \\ - G(\rho_-)\, f_-\, f_*)\, d\omega\, dv_*, \tag{19}$$

where ρ is the local density of f, $\pm$ denotes for a given function $g = g(x)$ the shorthand notation

$$g_\pm(x) := g(x \pm \delta\, \omega),$$

and G is the *local* collision rate (also known as the *correlation* rate, see [91]). The global existence of renormalized solutions for the full granular gases equation (18) with the collision operator (19) has been established for both elastic and inelastic collisions in [45]. Existence and $L^1(dx\, dv)$ stability of such solutions has been proved in [95], for close to vacuum initial datum.

[2] Note that using a BBGKY approach [50] to derive (11) is not expected to succeed, because among other problems the macroscopic size of the particles composing a granular gas is incompatible with the Boltzmann-Grad scaling assumption.

1.1 Cauchy Theory of the Granular Gases Equation

The Space Homogeneous Setting Most of the rigorous mathematical results concerning the granular gases equation are obtained in the space homogeneous setting, where $f = f(t, v)$ is the solution to

$$\begin{cases} \partial_t f = \dfrac{\mathcal{Q}_{\mathcal{I}}(f, f)}{\varepsilon}, \\ f(0, v) = f_{in}(v), \end{cases} \tag{20}$$

for a given scaling parameter $\varepsilon > 0$.

The first existence results for solution to (20) can be found in [12, 13, 15, 16, 21, 38]. These works deal with the generalized Maxwellian pseudo-molecule kernel (7) $\lambda = 0$, $b \equiv 1$ and $\gamma = 1/2$, with a velocity dependent restitution coefficient $e = e(|v - v_*|)$. Such a model allows to use some Fourier techniques to deal with the collision operator, altogether with the correct large time behavior for the kinetic energy, the so-called Haff's cooling Law (26), and the correct hydrodynamic limit (31). The main result of these works is the global well-posedness of the solutions to (20) in $L^1_2(\mathbb{R}^3)$, the convergence towards equilibrium, and the contraction in different metrics for the equation. The proof relies in the careful study of the self-similar solutions to (20). Some extensions of these results, using the same collision kernel, can be found in the works [17–19], where in particular the uniqueness is obtained.

The physically relevant case of the hard sphere kernel $\lambda = 1$, $\gamma = 0$ was first considered in [85] in the unidimensional case. This work establishes the global existence of measure solutions with finite kinetic energy for this problem. The proof relies on a priori estimates of the solution to (20) in the Monge-Kantorovich-Wasserstein metrics W_2. This work also investigates the quasi-elastic $1 - e^2 \sim \varepsilon \to 0$ limit of the model, a nonlinear McNamara-Young-like friction equation. This friction model was later investigated in [71], where its global wellposedness was shown in the space of measure of finite energy.

The tail behavior of the equilibrium solution to the granular gases equation with a thermal bath $\Delta_v f$ was investigated in many papers, the main ones being [20, 44, 51]. They all proved the existence of non-Gaussian, overpopulated tails for diffusively excited granular gases, namely:

Theorem 1 (From [20] and [44]) *Let $F(v) \geq 0$ for $v \in \mathbb{R}^d$ be a solution to the stationary equation*

$$\mathcal{Q}_{\mathcal{I}}(F, F) + \Delta_v F = 0$$

with all polynomial moments in velocity. Then,

$$F(v) \sim_{|v|\to\infty} \exp\left(-|v|^{\alpha}\right),$$

with $\alpha = 1$ *in the Maxwellian molecules case and* $\alpha = 3/2$ *in the hard spheres case.*

Indeed, the thermal bath gives an input of kinetic energy, preventing the appearance of trivial Dirac delta equilibria. The propagation of the Sobolev norms of the space dependent version of this equation was then established in [51]. It uses a careful estimate of the inelastic entropy production (23), and a fixed point argument for the existence and uniqueness of solutions.

Finally, the work [77] establishes the global well posedness of the granular gases equation without a thermal bath, for a general case of collision kernel (which contains (7)) and velocity dependent restitution coefficients:

Theorem 2 (Theorem 1.4 of [77]) *Let* $0 \le f_{in} \in L^1_3 \cap \in BV_4$. *Then for any* $T \in (0, T_c)$, *where* $T_c := \sup\{T > 0 : \mathcal{E}(f)(t) > 0,\ \forall\, t < T\}$ *is the so-called* blowup time, *there exists an unique nonnegative solution* $f \in \mathcal{C}(0, T; L^1_2) \cap L^\infty(0, T; L^1_3)$ *of* (20). *It preserves mass and momentum, and converges in the weak-* topology of measures towards a Dirac delta.*

Their proof relies on careful estimates of the collision operator $\mathcal{Q}_{\mathcal{I}}$ in Orlicz space (specially the $L \log L$ space of finite entropy measures).

Remark 5 The related (but still mostly open) problem of the propagation of chaos was considered in [78] for a very simplified inelastic collision operator with a thermal bath.

Cauchy Problem in the Space Dependent Setting The case of the space inhomogeneous setting[3] has been much less investigated.

The first result can be found in [9] for the model introduced in [85] with a restitution coefficient given by (4), in one dimension of space and velocity. This work establishes the existence and uniqueness of mild (perturbative) solutions, first for small $L^1(dx\, dv)$ initial data, and then for compactly supported initial data. Their main argument is reminiscent from a work due to Bony in [23] concerning discrete velocity approximation of the Boltzmann equation in dimension 1.

The global existence of mild solutions in the general $\mathbb{R}^3_x \times \mathbb{R}^3_v$ setting, for a large class of velocity-dependent restitution coefficient, but for initial data close to vacuum, was obtained in [6]. The proof is based on a Kaniel-Shinbrot iteration on a very small functional space. The stability in $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$ under the same assumptions was established in [94]. Finally the existence and convergence to equilibrium in $\mathbb{T}^3_x \times \mathbb{R}^3_v$ for a weakly inhomogeneous granular gas[4] with a thermal bath was proved in [87], using a perturbative approach.

[3] Physically more realistic, in part because of the spontaneous loss of space homogeneity that has been observed in [58].

[4] Namely, the initial condition is chosen with a lot of exponential moments in velocity, and close to a space homogeneous profile.

1.2 Large Time Behavior

Macroscopic Properties of the Granular Gases Operator Modeling-wise, the main microscopic difference between a granular gas and a perfect molecular gas is the dissipation of the kinetic energy. Using the weak form (6) among with the microscopic relations (1) of the inelastic collision operator, this yields

$$\int_{\mathbb{R}^d} \mathcal{Q}_{\mathcal{I}}(f,f)(v) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv \;=\; \begin{pmatrix} 0 \\ 0 \\ -D(f) \end{pmatrix},$$

where $D(f) \geq 0$ is the *energy dissipation* functional, which depends only on the collision kernel:

$$D(f) \;:=\; \int_{\mathbb{R}^d \times \mathbb{R}^d} f\, f_*\, \Delta\left(|v - v_*|, E(f)\right) dv\, dv_*. \tag{21}$$

The quantity $\Delta(|u|, E)$ is the so-called *energy dissipation* rate, given using (1) by

$$\Delta(|u|, E) \;:=\; \frac{1-e^2}{4} \int_{\mathbb{S}^{d-1}} |u \cdot \omega|^2\, B(|u|, \cos\theta, E)\, d\omega \geq 0, \quad \forall e \in [0,1]. \tag{22}$$

This dissipation of kinetic energy has a major consequence on the behavior of the solutions to the granular gases equation. Indeed, combined with the conservation of mass and momentum, it implies (at least formally) an *explosive* behavior, namely convergence in the weak-* topology of solutions to (18) towards Dirac deltas, centered in the mean momentum $\boldsymbol{u}$:

$$f(t,\cdot) \rightharpoonup \delta_{v=\boldsymbol{u}}, \quad t \to \infty.$$

As for the entropy, it is not possible to obtain any entropy dissipation for this equation, in order to precise this large time behavior. Indeed, as noticed in [51], taking $\psi(v) = \log f(v)$ in (6) yields

$$\begin{aligned} \int_{\mathbb{R}^d} \mathcal{Q}_{\mathcal{I}}(f,f)(v)\, \log f(v)\, dv &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_*\, f \log\left(\frac{f'\, f'_*}{f\, f_*}\right) B\, d\sigma\, dv\, dv_* \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} f_*\, f \left[\log\left(\frac{f'\, f'_*}{f\, f_*}\right) - \frac{f'\, f'_*}{f\, f_*} + 1\right] B\, d\sigma\, dv\, dv_* \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \left(f'_*\, f' - f_*\, f\right) B\, d\sigma\, dv\, dv_*. \end{aligned} \tag{23}$$

The first term in (23), the elastic contribution, is nonpositive because $\log\lambda - \lambda + 1 \leq 0$ (this is Boltzmann's celebrated *H Theorem*). Nevertheless, the second term has no sign *a priori*: it is 0 only in the elastic case (because of the involutive collisional transformation $(v, v_*, \sigma) \to (v', v'_*, \sigma)$). Boltzmann's entropy

$$\mathcal{H}(f) := \int_{\mathbb{R}^{d_v}} f(v) \log f(v)\, dv$$

is then not dissipated by the solution of the granular gases equation if $e < 1$. Some work has been done on that direction in the numerical side. Indeed, adding a drift term or a thermal bath in velocity can yield numerical entropy dissipation, as noticed in [53].

Kinetic Energy Dissipation and the Haff's Cooling Law Let us assume in this subsection that the granular gas considered is space homogeneous, namely f is solution to (20). Having no known entropy, one has then to use other macroscopic quantities to study the large time behavior of solutions to (18). Because of its explicit dissipation functional, kinetic energy is a good candidate for this. Moreover, being related to the variance, it allows to measure the concentration in velocity of the solution.

In order to have an explicit bound for the energy dissipation, let us assume that the collision kernel is of the general type (7). Using polar coordinates, it is straightforward to compute the dissipation rate (22):

$$\Delta\left(|u|, E\right) = b_1 \frac{1-e^2}{4} |u|^{\lambda+2} E^{\gamma}, \tag{24}$$

where thanks to (8)

$$b_1 = \left|\mathbb{S}^{d-2}\right| \int_0^{\pi} \cos^2(\theta) \sin^{d-3}(\theta)\, b(\cos(\theta))\, d\theta < \infty.$$

Using the conservation of mass and momentum, one can always assume that the initial condition is of unit mass and zero momentum. Plugging (24) into (21) then yields using Hölder and Jensen inequalities

$$\begin{aligned}
\frac{d}{dt} E(f)(t) &= -b_1 \frac{1-e^2}{4} E(f)^{\gamma}(t) \int_{\mathbb{R}^d \times \mathbb{R}^d} f\, f_*\, |v - v_*|^{\lambda+2}\, dv\, dv_* \\
&\leq -b_1 \frac{1-e^2}{4} E(f)^{\gamma}(t) \int_{\mathbb{R}^d} f(v)\, |v|^{\lambda+2}\, dv \\
&\leq -b_1 \frac{1-e^2}{4} E(f)^{1+\gamma+\lambda/2}(t).
\end{aligned} \tag{25}$$

In particular, one will have the following large time behaviors: Setting $C_e = b_1\, \rho\, (1-e^2)/4$ and $\alpha := \gamma + 1/2$,

- Maxwellian pseudo-molecules ($\lambda = \gamma = 0$) decays exponentially fast towards the Dirac delta :

$$E(f)(t) = E\left(f_{in}\right) e^{-C_e t};$$

  Notice that the inequality in (25) is an identity for this case.
- Hard spheres ($\lambda = 1, \gamma = 0$) exhibits the seminal quadratic *Haff's cooling Law* [60]:

$$E(f)(t) \leq \left(E\left(f_{in}\right)^{-1/2} + C_e\, t/2\right)^{-2}. \tag{26}$$

- Anomalous gases ($\gamma \neq 0$) exhibits more general behaviors:

$$E(f)(t) \leq \begin{cases} \left(E\left(f_{in}\right)^{\alpha} + C_e \alpha\, t\right)^{-\frac{1}{\alpha}} & \text{if } \gamma > -1/2 \ (\alpha > 0, \quad \text{finite time extinction}); \\ E\left(f_{in}\right) e^{-C_e t} & \text{if } \gamma = -1/2; \\ \left(E\left(f_{in}\right)^{\alpha} - C_e \alpha\, t\right)^{-\frac{1}{\alpha}} & \text{if } \gamma < -1/2 \ (\alpha < 0). \end{cases}$$

All of these formal results have been proven to be rigorous and sharp, with explicit lower bounds, in [13, 15] for the Maxwellian and hard sphere cases [74], and in [83] for the anomalous cases. Extension to the viscoelastic case can be found in [5, 6], where the energy is shown to behave as

$$E(f)(t) \sim_{t\to\infty} C\,(1+t)^{-5/3}.$$

All these papers share a common approach of proof, using the fact that the space homogeneous granular gases equation admits a self-similar behavior. Hence, introducing some well chosen time-dependent scaling function ω and τ, the distribution f is written as

$$f(t, v) = \omega(t)^d\, g\left(\tau(t), \omega(t)\, v\right),$$

to take into account the concentration in the velocity variables.[5] The rescaled function g is then solution to the granular gases equation, with an *anti-drift* term in velocity:

$$\partial_t g + \nabla_v \cdot (v\, g) = \mathcal{Q}_{\mathcal{I}}(g, g).$$

[5] One can see the velocity scaling function ω as the inverse of the variance of the distribution f. This scaling is then a continuous "zoom" on the blowup, and can be used to develop numerical methods for solving the full granular gases equation, see [49].

Using some regularity estimates of the gain term of $\mathcal{Q}_{\mathcal{I}}$ "à la" Lions/Bouchut-Desvillettes [24] and some new Povzner-like estimates [3, 74] then obtains a lower bound for the energy of g, yielding the generalized Haff's law by coming back to f.

Remark 6 In the viscoelastic case, note that the rescaling in velocity induces a time dependency on the restitution coefficient, complicating the proof of the Haff's cooling law [4]. It is also the case in the anomalous setting, where the rescaling function depends nonlinearly on the solution f.

The question of the uniqueness, stability and exponential return to an universal equilibrium profile (*hypocoercivity*, see [92]) of the self-similar solutions has then been fully addressed in the series of work [75, 76], for a constant restitution coefficient, with and without a thermal bath. Extension of these results to the viscoelastic case has been done in [5].

Remark 7 These results are reminiscent of the works [8, 12, 13, 21, 22, 35], where the exponential convergence to equilibrium of the solution to the nonlocal granular media equation or the Maxwellian case has been shown in the W_2 or Fourier metrics.

1.3 Compressible Hydrodynamic Limits

Let us consider in this subsection the following hyperbolic scaling of the granular gases equation:

$$\frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon} \mathcal{Q}_{\mathcal{I}}(f_\varepsilon, f_\varepsilon). \tag{27}$$

Determining the precise hyperbolic limit $\varepsilon \to 0$ of Eq. (27) is a fundamental, yet very difficult question.

Indeed, for the elastic case $e = 1$, one simply has to use the fact that the equilibria of the collision operator are at the thermodynamical equilibrium (gaussian distributions) and the conservation of mass, momentum and kinetic energy to obtain the classical compressible Euler–Fourier system [39]. Because of the trivial Dirac equilibria, this question is more intricate for the true inelastic case.

Pressureless Euler Dynamics Adopting the same approach as in the elastic case, one can formally plug the "equilibrium" Dirac deltas in the pressure to obtain the following pressureless Euler system:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \\ \dfrac{\partial (\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = \mathbf{0}. \end{cases} \tag{28}$$

This system can describe various interesting physical situations, such as galactic clusters, but is notoriously difficult to study mathematically. Its solution are in general ill-posed, as classical solutions cannot exists for large times and weak solutions are not unique.

In the unidimensional case, it is however possible to recover a well posed theory by imposing a semi-Lipschitz condition on u. This theory was introduced in [25], and later extended in [26] and [63]. We cite below the main result of [63], where $M^1(\mathbb{R})$ denotes the space of Radon measures on $\mathbb{R}$ and $L^2(\rho)$ for $\rho \geq 0$ in $M^1(\mathbb{R})$ denotes the space of functions which are square integrable against the density ρ.

Theorem 3 (From [63]) *For any $\rho^0 \geq 0$ in $M^1(\mathbb{R})$ and any $u^0 \in L^2(\rho^0)$, there exists $\rho \in L^\infty(\mathbb{R}_+, M^1(\mathbb{R}))$ and $u \in L^\infty(\mathbb{R}_+, L^2(\rho))$ solution to (28) in the sense of distribution and satisfying the semi-Lipschitz Oleinik-type bound*

$$u(t,x) - u(t,y) \leq \frac{x-y}{t}, \quad \text{for a.e. } x > y. \tag{29}$$

Moreover the solution is unique if u^0 is semi-Lipschitz or if the kinetic energy is continuous at $t = 0$

$$\int_{\mathbb{R}} \rho(t,dx)\,|u(t,x)|^2 \longrightarrow \int_{\mathbb{R}} \rho^0(dx)\,|u^0(x)|^2, \qquad \text{as } t \to 0.$$

The proof of Theorem 3 is quite delicate, relying on duality solutions. For this reason, we only explain the rational behind the bound (29), which can be seen very simply from the discrete *sticky particles* dynamics. We refer in particular to [29] for the limit of this sticky particles dynamics as $N \to \infty$.

Consider N particles on the real line. We describe the ith particle at time $t > 0$ by its position $x_i(t)$ and its velocity $v_i(t)$. Since we are dealing with a one dimensional dynamics, we can always assume the particles to be initially ordered

$$x_1^{in} < x_2^{in} < \ldots < x_N^{in}.$$

The dynamics is characterized by the following properties

1. The particle i moves with velocity $v_i(t)$: $\frac{d}{dt}x_i(t) = v_i(t)$.
2. The velocity of the ith particle is constant, as long as it does not collide with another particle: $v_i(t)$ is constant as long as $x_i(t) \neq x_j(t)$ for all $i \neq j$.
3. The velocity jumps when a collision occurs: if at time t_0 there exists $j \in \{1,\ldots,N\}$ such that $x_j(t_0) = x_i(t_0)$ and $x_j(t) \neq x_i(t)$ for any $t < t_0$, then all the particles with the same position take as new velocity the average of all the velocities

$$v_i(t_0+) = \frac{1}{|j|x_j(t_0) = x_i(t_0)|} \sum_{j|x_j(t_0)=x_i(t_0)} v_j(t_0-).$$