Solid Mechanics and Its Applications

Chyanbin Hwu

Anisotropic Elasticity with Matlab

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Preface

Because the analytical solutions are usually expressed in the closed-form of mathematical expressions, they often offer the advantages of transparency and efficiency, and can serve as the benchmarks for the numerical solutions. However, unlike the numerical techniques that can provide a general-purpose engineering analysis, such as finite element method and boundary element method, due to the limit of simple models very few computer programs were developed for the analytical solutions. In my previous book, Anisotropic Elastic Plates (Hwu, 2010), more than one hundred analytical solutions are presented for the problems of anisotropic elasticity. Most of the solutions are applicable to all kinds of linear anisotropic elastic materials, and can be extended to piezoelectric, piezomagnetic, magneto-electro-elastic, and viscoelastic materials. So it would be useful if these analytical solutions can be computed numerically through a well-designed computer program. With this consideration, in this book I plan to provide the computer program for the analytical solutions and boundary element analysis presented in my previous book.

Since the analytical solutions presented in (Hwu, 2010) are expressed in complex variable matrix form, the programming language *Matlab*, which is good for array operation and matrix operation, is selected for our computer program. The program is named as AEPH (Anisotropic Elastic Plates - Hwu). To handle different kinds of problems, several Matlab functions are created in this program. These functions are modularly structured so that the readers can readily extend or adapt them to suit their own needs. To help the readers understand the codes, appropriate comments as well as the associated equation number are added in the related lines of the function to enhance its readability. If special numerical techniques are used in the programming, some remarks will be stated in the related sections. Representative examples are shown in all chapters (except Chaps. 1 and 2) to illustrate how to prepare the input files to get the analytical solutions and boundary element solutions. All the solutions of the representative examples together with many other unpublished examples calculated by AEPH have been verified internally by different approaches and/or externally by the commercial finite element software ANSYS. Even though every attempt has been made to verify all the functions created in AEPH, due to the vast range it may cover still no guarantee can be given for their performance in practice.

For the convenience of the readers' reference, most of the chapters and the sections are purposely arranged according to my previous book (Hwu, 2010). Also, a control parameter named by the section number is used to direct the function flow. To avoid duplication with the previous book and to provide complete information for the computer program, all the analytical solutions are presented in this book without providing the detailed derivation. The basic theory of anisotropic elasticity and complex variable formalism are summarized in Chaps. 1 and 2 without going into details. Chapter 3 is a totally new chapter stating program structures, the main program, and the related functions. The computational procedures as well as the nomenclature of control parameters, global variables, input, and output used in AEPH are described in this chapter.

Chapters 4–9 provide the analytical solutions and their associated functions for the problems of wedges, interface corners, holes, cracks, inclusions, and contact. Chapter 10 deals with thermoelastic problems. Chapters 11 and 12 extend all the previous solutions to the cases of piezoelectric, magneto-electro-elastic and viscoelastic materials. Chapter 13 presents the solutions for plate bending analysis which are all implemented as special cases of coupled stretching-bending analysis stated in Chap. 14.

Chapter 15 starts with an overview of the boundary element method (BEM) that includes the brief introduction of the boundary integral equation, fundamental solution, interpolation function, boundary element formulation, boundary-based finite element, computational procedure, and program structure. All the problems of boundary element analysis are managed by two main functions, *BEMbankB* and *BEMbankIN,* except those for three-dimensional (3D) analysis which instead are managed by *BEM3DelasticB* and *BEM3DelasticIN*. The function whose name ends with the letter *B* is responsible for the solutions of boundary nodes, whereas the one that ends with *IN* is for the solutions of internal points. The boundary elements collected in *BEMbankB* and *BEMbankIN* are further categorized by (1) linear or quadratic, (2) static, dynamic or contact, (3) two-dimensional or coupled stretching-bending, (4) anisotropic, piezoelectric, magneto-electro-elastic, viscoelastic or thermoelastic, and (5) problem with a traction-free line, an interface, a hole, a crack, a rigid inclusion, an elastic inclusion, or multiple holes/cracks/inclusions. To implement the boundary element provided in AEPH, additional input files are required to input mesh information, boundary conditions, and other necessary data for different kinds of boundary element methods.

To compute the analytical solutions numerically, several standard numerical algorithms such as those for numerical integration, solving systems of linear equations, and finding zeros of functions have been employed in our programs. For the convenience of the readers' reference, these algorithms are briefly introduced in Appendix A. To provide a way of improving the efficiency of our computer program, part of 3D-BEM was coded through 'for loop' vectorization, whose details are presented in Appendix B. Since totally 204 functions are provided in our computer program - AEPH, a lot of global variables are used and different input files are required for different problems. To have a clear picture of them, lists of functions, global variables, and input files are provided in Appendix C, D, and E. The source codes of all these functions are then provided in Appendix F. To provide a friendly interface for the users to prepare their input files via messages from the operating system, a Windows program written using Microsoft Visual C++ is also designed for running the program AEPH, but not included in this book.

Just like all the other computational products, even if the readers have difficulties to understand the computer codes of AEPH, they still can use this program to get the solutions for the problems presented in this book. Since this program can deal with the two-dimensional, coupled stretching-bending, and three-dimensional problems with linear anisotropic elastic, piezoelectric, magneto-electro-elastic, and viscoelastic materials, I believe it is helpful for engineers and scientists who want to have the analytical or boundary element solutions for the general anisotropic elastic plates or solids. The topics such as wedges, interfaces, cracks, holes, inclusions, contact, and thermal stresses are all included. The solutions obtained here, especially the analytical solutions, can then serve as the benchmarks or play as the alternatives to the numerical solutions commonly calculated by commercial finite element software.

This book is appropriate to be a university textbook for the courses related to *anisotropic elasticity* and *computational mechanics*, which is generally offered for the graduate students majoring in aerospace, mechanical, civil, and naval engineering, applied mechanics, and engineering science. It is difficult to cover all the contents in a one-semester course. By properly selecting the related book chapters, it may be a good reference book for the standard courses such as *elasticity*, *mechanics of composite materials*, *fracture mechanics*, *plates and shells*, *boundary element method*, *smart materials and structures*, etc. and for the advanced courses such as *advanced elasticity, viscoelasticity, themoelasticity, advanced mechanics of composite materials, contact mechanics*, *micromechanics*, etc. It will be helpful for engineers and scientists who want to have an advanced knowledge of the theory of elasticity and mechanics of composite materials and structures. Moreover, students, engineers, researchers, and scientists can all use the results run by the codes provided in this book as their benchmark for the solutions from other methods.

I wish to express my gratitude to my Ph.D. thesis adviser, Prof. T. C. T. Ting, and my mentors, Prof. C. S. Yeh of National Taiwan University and Prof. W. H. Chen of National Tsing-Hua University, for their guidance during my studies. Since the publication of my previous book (Hwu, 2010), some of my friends encouraged me to write down the computer program for the analytical solutions of anisotropic elasticity. Here, I would like to thank my friends, Profs. K. Kishimoto (Tokyo Institute of Technology), M. Omiya (Keio University), N. Miyazaki (Kyoto University), T. Ikeda (Kagoshima University), Y. W. Mai (Sydney University), T. Aoki (Tokyo University), T. Yokozeki (Tokyo University), and C. Zhang (Siegen University), who have helped me during my visit to their departments. I would also like to thank my friends, V. Mantic of University of Seville, C. K. Chao of National Taiwan University of Science and Technology, T. Chen, Y. C. Shiah of National Cheng Kung University, and C. C. Ma, C. K. Wu, and T. T. Wu of National Taiwan University for their helpful discussions during my research on anisotropic elasticity. Special thanks also to my former assistant H. Shen, who helped me in drawing parts of figures presented in this book, and my students C. W. Hsu, V. T. Nguyen, C. L. Hsu, H. W. Chang, H. Y. Huang, Y. C. Chen, J. H. Shu, R. F. Liang, S. T. Huang, W. R. Chen, Z. H. Zhou, B. H. Shu, C. C. Li, S. H. Ho, T. H. Lo, J. C. Yu, H. B. Ko, C. Y. Chen, D. W. Huang, S. Y. Duan, H. S. Huang, J. Y. Hu, etc., who debugged and verified the computer codes, and Y. J. Yang, C. F. Chou, R. J. Wang, K. H. Weng, P. M. Chan, W. S. Huang, P. F. Ho, J. G. Jang, etc., who designed the Windows program, and many other former students who have ever contributed to part of AEPH. I acknowledge the Ministry of Science and Technology of Taiwan for the support of my research in the area of anisotropic elasticity.

Finally, I would like to dedicate this book to my wife, Wenling, and my daughters, Frannie and Vevey, with thanks for their constant support and encouragement in everything.

Tainan, Taiwan
October 2020

Chyanbin Hwu

Reference

Hwu, C., 2010, *Anisotropic Elastic Plates*, Springer, New York.

The original version of the book backmatter was revised: The MATLAB codes was added in Appendix F.

Contents

1 Anisotropic Elasticity

To study the behavior of an elastic continuous medium, the theory of elasticity is a generally accepted model. A simple idealized linear stress-strain relationship gives a good description of the mechanical properties of many elastic materials around us. By this relation, we need 21 elastic constants to describe a linear anisotropic elastic material if the materials do not possess any symmetry properties. Consideration of the material symmetry may reduce the number of elastic constants. If the two-dimensional deformation is considered, the number of elastic constants used in the theory of elasticity can be further reduced. If the materials are under thermal environment, additional thermal properties are needed to express the temperature effects on the stress-strain relation. If the materials exhibit the piezoelectric effects, the stress-strain relation should be further expanded to include the electric displacements and the electric fields. If not only the inplane deformation but also the out-of-plane deflection are considered for the laminates made by laying up various unidirectional fiber-reinforced composites, the elastic constants will generally be reorganized into the extensional, coupling and bending stiffnesses to suit for the classical lamination theory. Since the computer program developed in this book covers all these kinds of materials, their constitutive relations are now described in this Chapter. Further extensions to magneto-electro-elastic and viscoelastic materials will then be described in Chaps. 11 and 12.

1.1 Theory of Elasticity

In the study of elasticity, the body is generally considered to be a deformable continuous medium. Usually the objectives of analysis are the determination of stresses and strains induced by the external loads. The state of stress at a given point of a continuous body, which is either in equilibrium or in motion as a result of external forces, is known to be represented by the stress components σ_{ij}. The first subscript of σ indicates the direction of normal to the plane, while the second indicates the direction of the stress component. The components σ_{11}, σ_{22} and σ_{33} which are normal to the element surfaces are called *normal stresses*. The remaining components that are parallel to the surfaces are called *shear stresses.* The stress components are symmetric if the body moment is neglected. Knowing the stress components on the planes normal to the coordinate axes, we can always determine the stress vector **t** on any surface with unit outer normal vector **n**, which passes through the given point. This stress vector is determined by *Cauchy's formula* as

$$t_i = \sigma_{ij} n_j, \tag{1.1}$$

in which repeated indices imply summation through 1–3. This rule of summation convention will be applied in the whole text unless stated otherwise. The stress components in a continuous body that is in equilibrium must satisfy the *equilibrium equations*, which in Cartesian coordinates are

$$\sigma_{ij,j} + f_i = 0, \tag{1.2}$$

where $f_i, i = 1, 2, 3$, designate the body forces referred to a unit volume in directions x_1, x_2 and x_3, and a subscript comma stands for differentiation.

Forces applied to solids cause deformation. When the relative position of points in a continuous body is altered, the body is *strained*. The change in the relative position of points is a *deformation*. All material bodies are to some extent deformable. If there exists an ideal body which is nondeformable such that the distance between every pair of its points remains invariant

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throughout the history of the body, we call it a *rigid body*. The motion of a rigid body is usually described by translation and rotation. A deformable solid will experience an additional change in shape, i.e. deformation. Let the variables, a_1, a_2, a_3, refer to any particle in the original configuration of the body, and let x_1, x_2, x_3 be the coordinates of that particle when the body is deformed. The deformation of the body is known if x_1, x_2, x_3 are known functions of a_1, a_2, a_3. The displacement vector $\mathbf{u}$ is defined by its components

$$u_i = x_i - a_i. \tag{1.3}$$

Since the displacements defined in (1.3) may include the rigid body motion and deformation in which the former induces no stress. Thus, the displacements themselves are not directly related to the stresses. To relate deformation with stress, we must consider the stretching and distortion of the body, which is related to the changes in distance and angle among points of the body. For this purpose, when the deformation is infinitesimal the *Cauchy's infinitesimal strain tensor* is defined as

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}). \tag{1.4}$$

Like the stresses, the components of the strains that reflect the stretching or shortening of the body, i.e., $\varepsilon_{11}, \varepsilon_{22}$ and ε_{33} are called *normal strains*. The remaining components related to the distortion of the body are called *shear strains.* Sometimes $\gamma_{ij} = 2\varepsilon_{ij},\ i \neq j$ are used to represent *engineering shear strains*.

With reference to the definition given in (1.4), it is clear that a sufficiently well-behaved displacement field u_i will generate an equally well-behaved strain field by differentiation. The converse, however, is not necessarily true. It is not always possible to find a continuous, single-valued displacement field for any set of six well-behaved scalar functions ε_{ij} by integration of (1.4). For this reason we need to have the compatibility conditions for the strain fields to insure the existence of a single-valued, continuous displacement field for simply connected continuous body. The equations of *compatibility* obtained by St. Venant for infinitesimal strains are

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0. \tag{1.5}$$

The system of equations (1.5) consists of $3^4 = 81$ equations, but some of these are identically satisfied, and some are repetitions. Considering the symmetry of the strains, there are only six strain components for three-dimensional problems. Since the six strains are defined in terms of three displacement functions, then only three independent compatibility equations within (1.5) are essential. In the case of two-dimensional problems, only three strain components and two displacement functions are necessary. Thus, only one independent compatibility equation should be satisfied for two-dimensional problems, which can be written as

$$2\varepsilon_{12,12} = \varepsilon_{11,22} + \varepsilon_{22,11}. \tag{1.6}$$

In continuous media the state of stress is completely determined by the stress tensor σ_{ij}, and the state of deformation by the strain tensor ε_{ij}. If a material deforms as it is loaded and will return to its original dimensions during unloading, it is called an *elastic material*. In other words, an elastic material has a one-to-one analytical relation between stresses and strains. If the materials obey a linear relationship between stresses and strains, which is usually called the *generalized Hooke's law*, they are linear elastic materials. When a linear elastic material is maintained at a fixed temperature and the stresses vanish when the strains are all zero, i.e., the initial unstrained state of the solid is unstressed, the generalized Hooke's law can be written as

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}, \tag{1.7}$$

where C_{ijkl} are *elastic constants* which characterize the elastic behavior of the solid.

From the previous description, we know that the basic equations for anisotropic elasticity consist of equilibrium equations for static loading conditions (1.2), strain–displacement relations for small deformations (1.4) as well as stress-strain laws for linear anisotropic elastic solids (1.7). These three equation sets constitute 15 partial differential equations with 15 unknown functions $u_i,\ \varepsilon_{ij},\ \sigma_{ij},\ i,j = 1,2,3,$ in terms of three coordinate variables $x_i, i = 1,\ 2,\ 3.$ All these basic equations and unknown functions are designated to every point of the elastic body. A general solution for these 15 unknown functions satisfying 15 basic equations has been derived using complex variable formulation and will be stated in Chap. 2. Since all the equations stated are designated to points of the elastic body without considering the structure type and size, their associated general solutions can be applied to the studies of micromechanics or macromechanics, etc. If the structures constructed by the anisotropic elastic body are clearly defined and their associated loading and boundary conditions are well described, the undetermined functions of the general solutions and hence the 15 unknown functions will then be uniquely determined through the satisfaction of the boundary conditions. In other words, the state of stress and deformation will be determined by taking into account the boundary conditions. Depending on what is given at the boundary, there are several distinct problems. Generally, they are separated into the following three types: the first fundamental problem (or called *traction-prescribed problem*), the second fundamental problem (or called *displacement-prescribed problem*), and the third fundamental problem (or called *mixed boundary value problem*).

1.2 Linear Anisotropic Elastic Materials

An elastic body is called *isotropic* when its elastic properties are identical in all directions, and *anisotropic* (or called *triclinic*) when its elastic properties are different for different directions. If the anisotropic elastic materials obey a linear relationship between stresses and strains, the constitutive law may be expressed by using the generalized Hooke's law (1.7) in which C_{ijkl} is a fourth rank tensor and has 81 elastic constants. Because of various symmetry conditions in practice there is no need to deal with such many constants. Reduction of the number of elastic constants will then be described below for three-dimensional constitutive relations, two-dimensional constitutive relations, and laminate constitutive relations.

1.2.1 Three-Dimensional Constitutive Relations

Due to the symmetry of stresses and strains, and the existence of an elastic potential function whose derivative with respect to a strain component determines the corresponding stress component, it is required that

$$C_{ijkl} = C_{jikl},\ C_{ijkl} = C_{ijlk},\ C_{ijkl} = C_{klij}. \tag{1.8}$$

Because of the symmetry properties, the number of independent elastic constants can be drastically decreased from 81 to 21. To avoid dealing with double sums, a *contracted notation* has been introduced as

$$\begin{aligned} &\sigma_{11} = \sigma_1,\ \sigma_{22} = \sigma_2,\ \sigma_{33} = \sigma_3,\ \sigma_{23} = \sigma_4,\ \sigma_{31} = \sigma_5,\ \sigma_{12} = \sigma_6, \\ &\varepsilon_{11} = \varepsilon_1,\ \varepsilon_{22} = \varepsilon_2,\ \varepsilon_{33} = \varepsilon_3,\ 2\varepsilon_{23} = \varepsilon_4,\ 2\varepsilon_{31} = \varepsilon_5,\ 2\varepsilon_{12} = \varepsilon_6. \end{aligned} \tag{1.9}$$

And hence, through the symmetry conditions (1.8), the generalized Hooke's law (1.7) can be written as

$$\sigma_p = C_{pq}\varepsilon_q, \quad C_{pq} = C_{qp}, \quad p, q = 1, 2, \ldots, 6, \tag{1.10a}$$

or, in matrix notation,

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}, \quad \mathbf{C} = \mathbf{C}^T, \tag{1.10b}$$

where the superscript T denotes the transpose of a matrix. Note that $\sigma_p, C_{pq}, \varepsilon_q$ are not tensor quantities and therefore cannot be transformed as tensors. C_{pq} is sometimes called *stiffness matrix*. The transformation between C_{ijkl} and C_{pq} is accomplished by the replacement of the subscript according to the following rules for ij (or kl) $\leftrightarrow$ p (or q):

$$11 \leftrightarrow 1,\ 22 \leftrightarrow 2,\ 33 \leftrightarrow 3,\ 23(\text{or } 32) \leftrightarrow 4,\ 31(\text{or } 13) \leftrightarrow 5,\ 12(\text{or } 21) \leftrightarrow 6. \tag{1.11}$$

The relations between stresses and strains written in (1.7) must be reversible, and we can write

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}, \tag{1.12}$$

where S_{ijkl} are the *compliances* which are components of a fourth rank tensor. They also possess the full symmetry conditions like (1.8), i.e.,

$$S_{ijkl} = S_{jikl},\ S_{ijkl} = S_{ijlk},\ S_{ijkl} = S_{klij}. \tag{1.13}$$

Similar to the contracted notation introduced for the elastic tensor C_{ijkl}, the compliance tensor S_{ijkl} can also be contracted according to the rules shown in (1.11) except suitable factors should be added as (Ting 1996)

$$\begin{aligned} S_{ijkl} &= S_{pq}, \text{ if both } p, q \le 3, \\ 2S_{ijkl} &= S_{pq}, \text{ if either } p \text{ or } q \le 3, \\ 4S_{ijkl} &= S_{pq}, \text{ if both } p,q > 3. \end{aligned} \tag{1.14}$$

With (1.13) and (1.14), the stress-strain law (1.12) in contracted notation is

$$\varepsilon_p = S_{pq}\sigma_q, \quad S_{pq} = S_{qp}, \tag{1.15a}$$

or, in matrix notation,

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}, \quad \boldsymbol{S} = \boldsymbol{S}^T. \tag{1.15b}$$

Substitution of (1.15b) into (1.10b) yields

$$\mathbf{C}\boldsymbol{S} = \mathbf{I} = \boldsymbol{S}\mathbf{C}, \tag{1.16}$$

where $\mathbf{I}$ is the 6×6 unit matrix. The relation (1.16) indicates that the stiffness matrix $\mathbf{C}$ and the compliance matrix $\boldsymbol{S}$ are the inverses of each other.

With the foregoing reduction from 81 to 21 independent constants, the stress-strain relations (1.10a, 1.10b) are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{1.17}$$

which is the most general expression within the framework of linear anisotropic elasticity. For most elastic solids, the number of independent elastic constants is far smaller than 21. The reduction is caused by the existence of material symmetry. If there is one plane of material symmetry such as the plane $x_3 = 0$, the stress-strain relations reduce to

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{1.18}$$

Such a material is termed *monoclinic* or *aelotropic*, which has 13 independent elastic constants.

If a material has two orthogonal planes of material symmetry, it can be proved that the symmetry will exist relative to a third mutually orthogonal plane (Ting 1996). Such materials are said to be *orthotropic* (or *rhombic*), whose stress-strain relations in coordinates aligned with principal material directions are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{1.19}$$

which has 9 independent elastic constants.

If at every point of a material there is one plane in which the mechanical properties are equal in all directions, then the material is termed *transversely isotropic*. If, for example, the $x_3 = 0$ plane is the special plane of isotropy, the stress-strain relations of this kind of materials are

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{1.20}$$

which have only five independent constants.

The greatest reduction in the number of elastic constants is obtained when the material is symmetric with respect to any plane and any axis, or say, the elastic properties are identical in all directions. Such materials are called *isotropic* materials, whose stress-strain relations can be expressed by only two independent elastic constants as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & (C_{11}-C_{12})/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}. \tag{1.21}$$

By letting

$$C_{12} = \lambda, \text{ and } (C_{11} - C_{12})/2 = \mu, \tag{1.22}$$

where λ and μ are the *Lame constants*, the generalized Hooke's law for an isotropic material can also be written in the following form

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \ i,j,k = 1,2,3, \tag{1.23}$$

in which δ_{ij} is defined as $\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$, and is called the *Kronecker delta*.

Restrictions on elastic constants are due to the positive definiteness of the strain energy which implies that the stiffness (or the compliances) matrices must be positive-definite. The necessary and sufficient conditions for C_{ij} (or S_{ij}) to be positive definite are that the eigenvalues of C_{ij} (or S_{ij}) are positive, or alternatively all the leading principal minors of the stiffness (or compliance) matrix are positive. The obtained restrictions on elastic constants can then be used to examine experimental data to see if they are physically consistent within the framework of the mathematical elasticity model.

It's known that the components of stresses and strains will be different in a different coordinate system, and will transform according to the law of second rank tensor, whereas the components of elastic constants and elastic compliances will transform like a tensor of rank 4. In other words, they will transform according to

$$\begin{aligned} &\sigma^*_{pq} = \Omega_{pi}\Omega_{qj}\sigma_{ij}, \quad \varepsilon^*_{pq} = \Omega_{pi}\Omega_{qj}\varepsilon_{ij}, \\ &C^*_{pqrs} = \Omega_{pi}\Omega_{qj}\Omega_{rk}\Omega_{sl}C_{ijkl}, \quad S^*_{pqrs} = \Omega_{pi}\Omega_{qj}\Omega_{rk}\Omega_{sl}S_{ijkl}, \end{aligned} \tag{1.24}$$

where Ω_{pi} are the direction cosines between rotated (starred) and original (unstarred) axes. Since in numerical computation it is much more convenient using contracted notation than tensor notation, a matrix form transformation relation has been derived as

$$\boldsymbol{\sigma}^* = \mathbf{K}\boldsymbol{\sigma}, \ \boldsymbol{\varepsilon}^* = (\mathbf{K}^{-1})^T\boldsymbol{\varepsilon}, \ \mathbf{C}^* = \mathbf{KCK}^T, \ S^* = (\mathbf{K}^{-1})^T S\mathbf{K}^{-1}, \tag{1.25}$$

in which $\mathbf{K}$ is a 6×6 matrix whose explicit form for the general three-dimensional coordinate transformation can be found in (Ting 1996). Here, we just show the special case in which the transformation is a rotation about the x_3-axis an angle θ,

$$\mathbf{K}=\begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix},\ c=\cos\theta,\ s=\sin\theta. \tag{1.26}$$

Note that $\mathbf{K}^{-1}$ which is the inverse of the transformation matrix $\mathbf{K}$, can easily be calculated from $\mathbf{K}$ by setting the angle θ to $-\theta$, i.e., $[\mathbf{K}(\theta)]^{-1}=\mathbf{K}(-\theta)$.

Engineering constants (also known as *technical constants*) are generalized Young's moduli, Poisson's ratios and shear moduli as well as some other behavior constants. These constants are measured in simple tests such as uniaxial tension or pure shear tests. Thus, these constants with their obvious physical interpretation have more direct meaning than the components of the relatively abstract compliance and stiffness matrices discussed previously. Most simple tests are performed with a known load or stress. The resulting displacement or strain is then measured. Thus, the components of the compliance matrix are determined more directly than those of the stiffness matrix. For a general anisotropic elastic material, the compliance matrix components in terms of the engineering constants are

$$\boldsymbol{S}=\begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & \frac{\eta_{1,23}}{G_{23}} & \frac{\eta_{1,31}}{G_{31}} & \frac{\eta_{1,12}}{G_{12}} \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & \frac{\eta_{2,23}}{G_{23}} & \frac{\eta_{2,31}}{G_{31}} & \frac{\eta_{2,12}}{G_{12}} \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & \frac{\eta_{3,23}}{G_{23}} & \frac{\eta_{3,31}}{G_{31}} & \frac{\eta_{3,12}}{G_{12}} \\ \frac{\eta_{23,1}}{E_1} & \frac{\eta_{23,2}}{E_2} & \frac{\eta_{23,3}}{E_3} & \frac{1}{G_{23}} & \frac{\mu_{23,31}}{G_{31}} & \frac{\mu_{23,12}}{G_{12}} \\ \frac{\eta_{31,1}}{E_1} & \frac{\eta_{31,2}}{E_2} & \frac{\eta_{31,3}}{E_3} & \frac{\mu_{31,23}}{G_{23}} & \frac{1}{G_{31}} & \frac{\mu_{31,12}}{G_{12}} \\ \frac{\eta_{12,1}}{E_1} & \frac{\eta_{12,2}}{E_2} & \frac{\eta_{12,3}}{E_3} & \frac{\mu_{12,23}}{G_{23}} & \frac{\mu_{12,31}}{G_{31}} & \frac{1}{G_{12}} \end{bmatrix}, \tag{1.27}$$

where E_1, E_2, E_3 are the *Young's moduli* in x_1, x_2 and x_3 directions, respectively; ν_{ij} is the *Poisson's ratio* for transverse strain in the x_j-direction when stressed in the x_i-direction, that is, $\nu_{ij}=-\varepsilon_j/\varepsilon_i$ for $\sigma_i=\sigma$ and all other stresses are zero; G_{23}, G_{31}, G_{12} are the *shear moduli* in the x_2x_3, x_3x_1 and x_1x_2 planes, respectively; $\eta_{i,ij}$ is the *coefficient of mutual influence of the first kind* which characterizes stretching in the x_i-direction caused by shear in the x_ix_j-plane, that is, $\eta_{i,ij}=\varepsilon_i/\gamma_{ij}$ for $\tau_{ij}=\tau$ and all other stresses are zero; $\eta_{ij,i}$ is the *coefficient of mutual influence of the second kind* which characterizes shearing in the x_ix_j-plane caused by a normal stress in the x_i-direction, that is, $\eta_{ij,i}=\gamma_{ij}/\varepsilon_i$ for $\sigma_i=\sigma$ and all other stresses are zero; $\mu_{ij,kl}$ is the *Chentsov coefficient* which characterizes the shearing strain in the x_ix_j-plane due to shearing stress in the x_kx_l-plane, that is, $\mu_{ij,kl}=\gamma_{ij}/\gamma_{kl}$ for $\tau_{kl}=\tau$ and all other stresses are zero.

Due to the symmetry of the compliance matrix, the Poisson's ratios, the coefficients of mutual influences and the Chentsov coefficients are subject to the following reciprocal relations

$$\frac{\nu_{ij}}{E_i}=\frac{\nu_{ji}}{E_j},\ \frac{\eta_{i,jk}}{G_{jk}}=\frac{\eta_{jk,i}}{E_i},\ \frac{\mu_{ij,kl}}{G_{kl}}=\frac{\mu_{kl,ij}}{G_{ij}}. \tag{1.28}$$

Full matrix shown in (1.27) for the general anisotropic elastic materials also indicates that application of a normal stress leads not only to extension in the direction of the stress and contraction perpendicular to it, but to shearing deformation. Conversely, shearing stress causes extension and contraction in addition to the distorsion of shearing deformation.

From (1.16) we know by inversion of (1.27) the stiffness matrix components C_{ij} in terms of the engineering constants can be obtained. However, since the compliance matrix shown in (1.27) is a 6×6 full symmetric matrix, it is not easy to get a simple analytical expression of its inverse matrix. Following we just list the stiffness matrix components C_{ij} for *orthotropic materials*,

$$\mathbf{C}=\begin{bmatrix} \frac{1-\nu_{23}\nu_{32}}{E_2E_3\Delta} & \frac{\nu_{12}+\nu_{32}\nu_{13}}{E_1E_3\Delta} & \frac{\nu_{13}+\nu_{12}\nu_{23}}{E_1E_2\Delta} & 0 & 0 & 0 \\ & \frac{1-\nu_{13}\nu_{31}}{E_1E_3\Delta} & \frac{\nu_{23}+\nu_{21}\nu_{13}}{E_1E_2\Delta} & 0 & 0 & 0 \\ & & \frac{1-\nu_{12}\nu_{21}}{E_1E_2\Delta} & 0 & 0 & 0 \\ & \text{symm.} & & G_{23} & 0 & 0 \\ & & & & G_{31} & 0 \\ & & & & & G_{12} \end{bmatrix}, \tag{1.29a}$$

where

$$\Delta = \frac{1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{32}\nu_{13}}{E_1 E_2 E_3}, \tag{1.29b}$$

and the symmetry conditions gives us

$$\frac{\nu_{21} + \nu_{31}\nu_{23}}{E_2 E_3 \Delta} = \frac{\nu_{12} + \nu_{32}\nu_{13}}{E_1 E_3 \Delta}, \quad \frac{\nu_{31} + \nu_{21}\nu_{32}}{E_2 E_3 \Delta} = \frac{\nu_{13} + \nu_{12}\nu_{23}}{E_1 E_2 \Delta}, \quad \frac{\nu_{32} + \nu_{12}\nu_{31}}{E_1 E_3 \Delta} = \frac{\nu_{23} + \nu_{21}\nu_{13}}{E_1 E_2 \Delta}. \tag{1.30}$$

1.2.2 Two-Dimensional Constitutive Relations

Two-dimensional problems usually considered in isotropic elasticity fall into two physical distinct types. One of these arises in the study of deformation of large cylindrical bodies acted by the external forces so distributed that the component of deformation in the direction of the axis of the cylinder vanishes and the remaining components do not vary along the length of the cylinder. This is the class of problems in *plane deformation* (or called *plane strain*). Take the cross section of the cylinder be a plane parallel to x_1x_2-plane, the state of plane deformation may be characterized by

$$u_\alpha = u_\alpha(x_1, x_2), \quad u_3 = 0, \quad \alpha = 1, 2, \tag{1.31}$$

which will lead to

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0. \tag{1.32}$$

The other type appears in the study of the deformation of thin plates, the state of stress in which is characterized by the vanishing of the stress components in the direction of the thickness of the plate. These are the problems in *plane stress* and can be characterized by

$$\sigma_{13} = \sigma_{23} = \sigma_{33} = 0. \tag{1.33}$$

In the plane strain problem, all the displacement and stress components are independent of the x_3-coordinate, whereas in the problem of plane stress these components may depend on x_3. Since the variable x_3 may appear as a parameter in the elasticity equations (for example if $\varepsilon_{33} \neq 0$ through integration we get u_3 which will depend on x_3), the problem of plane stress is not truly two-dimensional. However, by dealing with the mean values of the displacements and stresses, and supposing that the faces of the thin plates are free of applied loads and all external surface forces acting on the edge of the plate lie in the plane parallel to the middle plane, a mathematical-truely two-dimensional problem can be obtained for a thin plate (Sokolnikoff 1956), which is called the problems of *generalized plane stress*. It can be proved that the mathematical formulations of plane strain and generalized plane stress are identical. The relevant differential equations and boundary conditions differ only in the Lame constant λ as well as the use of average displacements and stresses in the generalized plane stress problems. For plane strain problems we use $u_\alpha,\ \sigma_{\alpha\beta},\ \lambda$, etc., while for generalized plane stress problems we use $\widetilde{u}_\alpha, \widetilde{\sigma}_{\alpha\beta}, \widetilde{\lambda}$ where

$$\begin{aligned} \widetilde{u}_\alpha(x_1, x_2) &= \frac{1}{h} \int_{-h/2}^{h/2} u_\alpha(x_1, x_2, x_3)dx_3, \\ \widetilde{\sigma}_{\alpha\beta}(x_1, x_2) &= \frac{1}{h} \int_{-h/2}^{h/2} \sigma_{\alpha\beta}(x_1, x_2, x_3)dx_3, \quad \alpha, \beta = 1, 2, \end{aligned} \tag{1.34a}$$

and

$$\widetilde{\lambda} = \frac{2\lambda\mu}{\lambda + 2\mu}. \tag{1.34b}$$

In (1.34a), h is the thickness of the plate.

In practical applications, to describe the isotropic materials it is common to use the engineering constants E and v instead of the Lame constants λ and μ. Thus, it is useful to know their relations which are shown below,

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}, \quad v = \frac{\lambda}{2(\lambda + \mu)}, \tag{1.35a}$$

or

$$\mu = \frac{E}{2(1+v)}, \quad \lambda = \frac{Ev}{(1+v)(1-2v)}. \tag{1.35b}$$

With these relations, the replacement of μ, λ by $\mu, \tilde{\lambda}$ from plane strain problem to generalized plane stress problem can also be made with the replacement of E, v by $\tilde{E}, \tilde{v}$ where (Hwu 2010)

$$\tilde{E} = \frac{E(1+2v)}{(1+v)^2}, \quad \tilde{v} = \frac{v}{1+v}. \tag{1.36}$$

Due to the mathematical equivalence, the plane strain and generalized plane stress problems are usually referred to be *plane problems* or *two-dimensional problems*. However, in a body with anisotropy of a general form, plane deformation is usually not possible except for some special forms because, assuming $u_3 = 0$, it is generally impossible to satisfy the equations of equilibrium of an elastic body (Lekhnitskii 1963). We can only assert that all the components of stresses and displacements will not depend on x_3. The deformation of such a body (one of infinite length bounded by a cylindrical surface and possessing anisotropy of a general form) which corresponds to plane deformation in an isotropic body is called *generalized plane deformation* or *generalized plane strain*. In other words, a body is said to be in the state of generalized plane deformation (or generalized plane strain), parallel to the x_1x_2-plane, if all the displacement components u_1, u_2, and u_3 are functions of the coordinates x_1 and x_2, but not of x_3. Thus, the state of generalized plane deformation is characterized by.

$$u_i = u_i(x_1, x_2), \quad i = 1, 2, 3, \tag{1.37a}$$

which will lead to

$$\varepsilon_3 = 0. \tag{1.37b}$$

When $\varepsilon_3 = 0$, the stress-strain relation written in (1.10a) becomes

$$\sigma_p = \sum_{q \neq 3} C_{pq}\varepsilon_q, \quad p = 1, 2, \ldots\ldots, 6, \tag{1.38}$$

and its inverse relation (1.15a) becomes

$$\varepsilon_p = \sum_{q \neq 3} \hat{S}_{pq}\sigma_q, \quad p = 1, 2, 4, 5, 6, \tag{1.39a}$$

where $\hat{S}_{pq}$ are the *reduced elastic compliances* which are defined as.

$$\hat{S}_{pq} = S_{pq} - \frac{S_{p3}S_{3q}}{S_{33}} = \hat{S}_{qp}. \tag{1.39b}$$

The relation (1.39a, 1.39b) is derived by employing the requirement $\varepsilon_3 = 0 = S_{3q}\sigma_q$, which will lead to $\sigma_3 = -\sum_{q \neq 3} S_{3q}\sigma_q / S_{33}$. Substituting the result of σ_3 into (1.15a), we obtain the relation (1.39a). Equations (1.38) and (1.39a) can also be written in matrix notation as

$$\boldsymbol{\sigma}^0 = \mathbf{C}^0\boldsymbol{\varepsilon}^0, \quad \boldsymbol{\varepsilon}^0 = \hat{\mathbf{S}}\boldsymbol{\sigma}^0, \tag{1.40a}$$

where

$$\boldsymbol{\sigma}^0 = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}, \quad \boldsymbol{\varepsilon}^0 = \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix}, \tag{1.40b}$$

$$\mathbf{C}^0 = \begin{bmatrix} C_{11} & C_{12} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{24} & C_{25} & C_{26} \\ C_{14} & C_{24} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{46} & C_{56} & C_{66} \end{bmatrix}, \quad \widehat{\boldsymbol{S}} = \begin{bmatrix} \widehat{S}_{11} & \widehat{S}_{12} & \widehat{S}_{14} & \widehat{S}_{15} & \widehat{S}_{16} \\ \widehat{S}_{12} & \widehat{S}_{22} & \widehat{S}_{24} & \widehat{S}_{25} & \widehat{S}_{26} \\ \widehat{S}_{14} & \widehat{S}_{24} & \widehat{S}_{44} & \widehat{S}_{45} & \widehat{S}_{46} \\ \widehat{S}_{15} & \widehat{S}_{25} & \widehat{S}_{45} & \widehat{S}_{55} & \widehat{S}_{56} \\ \widehat{S}_{16} & \widehat{S}_{26} & \widehat{S}_{46} & \widehat{S}_{56} & \widehat{S}_{66} \end{bmatrix}. \tag{1.40c}$$

Similarly, when $\sigma_3 = 0$ the stress-strain relation can be written as

$$\sigma_p = \sum_{q\neq 3} \widehat{C}_{pq}\varepsilon_q, \quad \varepsilon_p = \sum_{q\neq 3} S_{pq}\sigma_q, \quad p = 1,2,4,5,6, \tag{1.41a}$$

or, in matrix notation,

$$\boldsymbol{\sigma}^0 = \widehat{\mathbf{C}}\boldsymbol{\varepsilon}^0, \quad \boldsymbol{\varepsilon}^0 = \boldsymbol{S}^0\boldsymbol{\sigma}^0, \tag{1.41b}$$

where $\boldsymbol{S}^0$ is a 5×5 *compliance matrix* whose components are S_{pq}, p, q = 1, 2, 4, 5, 6, and $\widehat{\mathbf{C}}$ is a 5×5 *reduced stiffness matrix* whose components $\widehat{C}_{pq}$ are the reduced elastic stiffnesses defined by

$$\widehat{C}_{pq} = C_{pq} - \frac{C_{p3}C_{3q}}{C_{33}} = \widehat{C}_{qp}. \tag{1.42}$$

Similar to the inversion relation for $\mathbf{C}$ and $\boldsymbol{S}$ proved in (1.16), it can easily be proved that

$$\mathbf{C}^0\widehat{\boldsymbol{S}} = \mathbf{I} = \widehat{\boldsymbol{S}}\mathbf{C}^0, \quad \boldsymbol{S}^0\widehat{\mathbf{C}} = \mathbf{I} = \widehat{\mathbf{C}}\boldsymbol{S}^0. \tag{1.43}$$

By using the relations given in (1.21) and (1.22) for the isotropic materials, we see that the replacement given in (1.34b) is a special case of (1.42). Therefore, like the equivalence of the mathematical formulation for plane strain and generalized plane stress in isotropic elasticity, we can conclude that when the general anisotropic materials are considered the elastic stiffnesses C_{pq} or the reduced elastic compliances $\widehat{S}_{pq}(= C_{pq}^{-1})$ should be employed for the generalized plane strain problems. While for the generalized plane stress problems, we employ the elastic compliances S_{pq} or the reduced elastic stiffnesses $\widehat{C}_{pq}(= S_{pq}^{-1})$.

The elastic constants for the monoclinic materials with symmetry plane $x_3 = 0$ have been given in (1.18). By deleting the third row and third column of the stiffness matrix for the general two-dimensional problems, we see that the in-plane and anti-plane properties are decoupled. The stress-strain relation (1.40a, 1.40b, 1.40c) for the problems of generalized plane deformation can therefore be splitted into two parts. One is the *in-plane relation*, and the other is the *anti-plane relation*. They are,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}. \tag{1.44}$$

For the problems of generalized plane stress, similar relations can be written in terms of S_{ij} or $\widehat{C}_{ij}$.

The elastic constants for the orthotropic materials in coordinates aligned with principal material directions have been given in (1.19). There is no interaction between normal stresses and shearing strains. Also, there is no interaction between shearing stresses and normal strains as well as none between shearing stresses and shearing strains in different planes. Similar to (1.44), the in-plane and anti-plane relations for the problems of generalized plane deformation can then be written as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} C_{44} & 0 \\ 0 & C_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}. \tag{1.45}$$

With (1.29a, 1.29b), the above relation can be written in terms of the engineering constants. For generalized plane stress, we have

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} \widehat{C}_{11} & \widehat{C}_{12} & 0 \\ \widehat{C}_{12} & \widehat{C}_{22} & 0 \\ 0 & 0 & \widehat{C}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{Bmatrix}, \quad \begin{Bmatrix} \sigma_4 \\ \sigma_5 \end{Bmatrix} = \begin{bmatrix} \widehat{C}_{44} & 0 \\ 0 & \widehat{C}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix}. \tag{1.46}$$

By using the engineering constants given in (1.29a, 1.29b) and the definitions for the reduced stiffness constants (1.42), we obtain

$$\begin{aligned} &\widehat{C}_{11} = \frac{E_1}{1 - \nu_{12}\nu_{21}}, \quad \widehat{C}_{22} = \frac{E_2}{1 - \nu_{12}\nu_{21}}, \\ &\widehat{C}_{12} = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}}, \\ &\widehat{C}_{66} = G_{12}, \quad \widehat{C}_{44} = G_{23}, \quad \widehat{C}_{55} = G_{31}. \end{aligned} \tag{1.47}$$

From $(1.43)_2$ we know that the reduced stiffness constants can also be determined by the inversion of the compliance matrix. That is,

$$\begin{bmatrix} \widehat{C}_{11} & \widehat{C}_{12} & 0 \\ \widehat{C}_{12} & \widehat{C}_{22} & 0 \\ 0 & 0 & \widehat{C}_{66} \end{bmatrix}^{-1} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & 0 \\ 0 & 0 & 1/G_{12} \end{bmatrix}, \tag{1.48a}$$

and

$$\begin{bmatrix} \widehat{C}_{44} & 0 \\ 0 & \widehat{C}_{55} \end{bmatrix}^{-1} = \begin{bmatrix} 1/G_{23} & 0 \\ 0 & 1/G_{31} \end{bmatrix}. \tag{1.48b}$$

1.2.3 Laminate Constitutive Relations

In engineering applications, one commonly used orthotropic material is *unidirectional fiber-reinforced composite*. The *laminated composites* are made by laying up various unidirectional fiber-reinforced composites. A single layer of the laminated composites is generally referred to as a *ply* or *lamina*. A single lamina is generally too thin to be directly used in engineering applications. Several laminae are bonded together to form a structure termed a *laminate*. Properties of a lamina may be predicted by knowing the properties of its constituents, i.e., fibers, matrices and their volume fractions. The overall properties of the laminates can be designed by changing the fiber orientation and the stacking sequence of laminae.To describe the overall properties and macromechanical behavior of a laminate, the most popular way is the *classical lamination theory* (Jones 1974). According to the observation of actual mechanical behavior of laminates, *Kirchhoff assumptions* are usually made in this theory: (1) The laminate consists of perfectly bonded laminae and the bonds are infinitesimally thin as well as non-shear-deformable. Thus, the displacements are continuous across lamina boundaries so that no lamina can slip relative to another. (2) A line originally straight and perpendicular to the middle surface of the laminate remains straight and perpendicular to the middle surface of the laminate when the laminate is deformed. In other words, the transverse shear strains are ignored, i.e., $\gamma_{xz} = \gamma_{yz} = 0$. (3) The normals have constant length so that the strain perpendicular to the middle surface is ignored, i.e. $\varepsilon_z = 0$.

Based upon the above assumptions, the laminate displacements u, v and w in the x, y and z directions can be expressed as

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w(x,y)}{\partial x}, \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w(x,y)}{\partial y}, \\ w(x,y,z) &= w_0(x,y), \end{aligned} \tag{1.49}$$

where u_0, v_0 and w_0 are the middle surface displacements. If small deformations are considered, the laminate strains can be written in terms of the middle surface displacements as follows.

$$\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w}{\partial x^2} = \varepsilon_x^0 + z\kappa_x, \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w}{\partial y^2} = \varepsilon_y^0 + z\kappa_y, \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z\frac{\partial^2 w}{\partial x \partial y} = \gamma_{xy}^0 + z\kappa_{xy}, \end{aligned} \tag{1.50}$$

in which $\varepsilon_x^0, \varepsilon_y^0, \gamma_{xy}^0$ and $\kappa_x, \kappa_y, \kappa_{xy}$, denote, respectively, the midsurface strain and curvature. Substituting (1.50) into the stress-strain relation for each lamina, the stresses in the kth lamina can be written as

$$\boldsymbol{\sigma}_k = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11}^* & Q_{12}^* & Q_{16}^* \\ Q_{12}^* & Q_{22}^* & Q_{26}^* \\ Q_{16}^* & Q_{26}^* & Q_{66}^* \end{bmatrix}_k \left(\begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} \right) = \boldsymbol{Q}_k^*(\boldsymbol{\varepsilon}_0 + z\boldsymbol{\kappa}), \tag{1.51a}$$

where

$$\begin{aligned} Q_{11}^* &= Q_{11}\cos^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\sin^4\theta, \\ Q_{12}^* &= (Q_{11} + Q_{22} - 4Q_{66})\sin^2\theta\cos^2\theta + Q_{12}(\sin^4\theta + \cos^4\theta), \\ Q_{22}^* &= Q_{11}\sin^4\theta + 2(Q_{12} + 2Q_{66})\sin^2\theta\cos^2\theta + Q_{22}\cos^4\theta, \\ Q_{16}^* &= (Q_{11} - Q_{12} - 2Q_{66})\sin\theta\cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin^3\theta\cos\theta, \\ Q_{26}^* &= (Q_{11} - Q_{12} - 2Q_{66})\sin^3\theta\cos\theta + (Q_{12} - Q_{22} + 2Q_{66})\sin\theta\cos^3\theta, \\ Q_{66}^* &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})\sin^2\theta\cos^2\theta + Q_{66}(\sin^4\theta + \cos^4\theta). \end{aligned} \tag{1.51b}$$

θ is the angle oriented from the reference coordinate axes to the *principal material axes* of the kth lamina, and the principal material axes means the axes parallel and perpendicular to the fibers of the lamina. When the reference coordinate axes coincide with the principal material directions, the lamina made by the unidirectional fiber-reinforced composites will behave as the orthotropic materials, and hence the stiffness constants $Q_{ij}, i,j = 1,2,6$ can be obtained from (1.47) as

$$\begin{aligned} Q_{11} &= \frac{E_L}{1 - \nu_{LT}\nu_{TL}}, \quad Q_{22} = \frac{E_T}{1 - \nu_{LT}\nu_{TL}}, \\ Q_{12} &= \frac{\nu_{LT}E_T}{1 - \nu_{LT}\nu_{TL}} = \frac{\nu_{TL}E_L}{1 - \nu_{LT}\nu_{TL}}, \quad Q_{66} = G_{LT}, \end{aligned} \tag{1.52}$$

where the subscripts L and T denote, respectively, the directions parallel and perpendicular to fibers, which are generally called the *longitudinal* and *transverse directions.*

Like the classical plate theory, the thickness of the laminate is considered to be small compared to its other dimensions. Therefore, instead of dealing the stress distribution across the laminate thickness, *an integral equivalent system of forces and moments* acting on the laminate cross section is used in the classical lamination theory. By integration of the stresses in each lamina through the laminate thickness, the resultant forces $\boldsymbol{N}$ and moments $\boldsymbol{M}$ acting on a laminate cross section are defined as follows,

$$\boldsymbol{N} = \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \boldsymbol{\sigma}_k dz$$
$$\boldsymbol{M} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} z dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k z dz = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \boldsymbol{\sigma}_k z dz \quad (1.53)$$

where h_k and h_{k-1} are defined in Fig. 1.1. Substituting (1.51a) into (1.53), the resultant forces $\boldsymbol{N}$ and moments $\boldsymbol{M}$ can be written in terms of the laminate middle surface strains $\boldsymbol{\varepsilon}_0$ and curvatures $\boldsymbol{\kappa}$ as

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\kappa} \end{Bmatrix}, \quad (1.54a)$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ are called the *extensional*, *coupling* and *bending stiffness matrices*, respectively, and are determined by

$$\boldsymbol{A} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \boldsymbol{Q}_k^* dz = \sum_{k=1}^{n} \boldsymbol{Q}_k^* (h_k - h_{k-1}),$$
$$\boldsymbol{B} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \boldsymbol{Q}_k^* z dz = \frac{1}{2} \sum_{k=1}^{n} \boldsymbol{Q}_k^* (h_k^2 - h_{k-1}^2), \quad (1.54b)$$
$$\boldsymbol{D} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \boldsymbol{Q}_k^* z^2 dz = \frac{1}{3} \sum_{k=1}^{n} \boldsymbol{Q}_k^* (h_k^3 - h_{k-1}^3).$$

From (1.54a), we see that if the coupling matrix $\boldsymbol{B}$ is a zero matrix, the resultant forces $\boldsymbol{N}$ will induce only the midsurface strains while the resultant moments $\boldsymbol{M}$ will induce only the plate curvatures. The presence of the matrix $\boldsymbol{B}$ implies coupling between bending and extension of a laminate. Thus, when a laminate is subjected to an extensional force or a bending moment, it may suffer extensional as well as bending and/or twisting deformations at the same time. By $(1.54\text{b})_2$, we also know that the presence of a nonzero coupling matrix is not only attributable to the orthotropy or anisotropy of the layers but also to the nonsymmetric stacking of laminae.

The aim of the analysis of laminated composites is to determine the stresses and strains in each of the laminae forming the laminate. These stresses and strains can be used to predict the load at which failure initiates. If the resultant forces $\boldsymbol{N}$ and moments $\boldsymbol{M}$ are known at a particular cross section through the structural analysis, the midsurface strains and curvatures at this cross section may then be determined by the inversion of (1.54a), i.e.,

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix}^{-1} \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A}^* & \boldsymbol{B}^* \\ \boldsymbol{B}^{*T} & \boldsymbol{D}^* \end{bmatrix} \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{Bmatrix}, \quad (1.55a)$$

where

$$\boldsymbol{A}^* = \boldsymbol{A}^{-1} + \boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{D}^*\boldsymbol{B}\boldsymbol{A}^{-1}, \quad \boldsymbol{B}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{D}^*, \quad \boldsymbol{D}^* = (\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}. \quad (1.55b)$$

The stresses and strains in each lamina can therefore be determined from (1.51a, 1.51b) and (1.50). In addition to (1.54a, 1.54b) and (1.55a, 1.55b), two more equivalent expressions of laminate constitutive relations are

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{Bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{A}} & \widetilde{\boldsymbol{B}} \\ -\widetilde{\boldsymbol{B}}^T & \widetilde{\boldsymbol{D}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{M} \end{Bmatrix}, \quad \begin{Bmatrix} \boldsymbol{\varepsilon}_0 \\ \boldsymbol{M} \end{Bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{A}}^* & \widetilde{\boldsymbol{B}}^* \\ -\widetilde{\boldsymbol{B}}^{*T} & \widetilde{\boldsymbol{D}}^* \end{bmatrix} \begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{\kappa} \end{Bmatrix}, \quad (1.56a)$$

where

$$\widetilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{B}, \quad \widetilde{\boldsymbol{B}} = \boldsymbol{B}\boldsymbol{D}^{-1}, \quad \widetilde{\boldsymbol{D}} = \boldsymbol{D}^{-1},$$
$$\widetilde{\boldsymbol{A}}^* = \boldsymbol{A}^{-1}, \quad \widetilde{\boldsymbol{B}}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}, \quad \widetilde{\boldsymbol{D}}^* = \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}. \quad (1.56b)$$

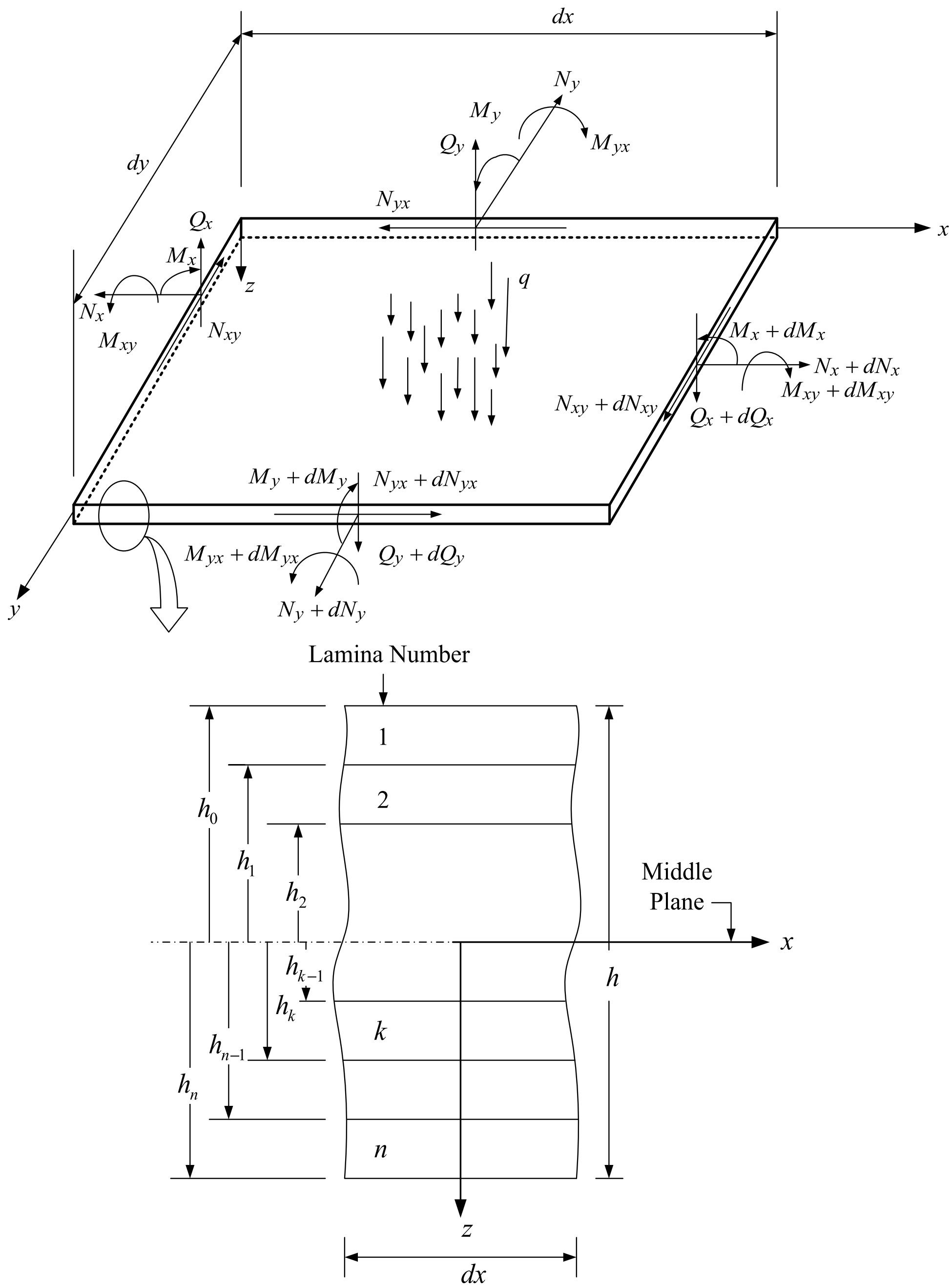

Fig. 1.1 Laminate geometry, resultant forces and moments

Because $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ defined in (1.54b) are symmetric matrices, $\widetilde{\boldsymbol{A}}$, $\widetilde{\boldsymbol{D}}$, $\widetilde{\boldsymbol{A}}^*$ and $\widetilde{\boldsymbol{D}}^*$ defined in (1.56b) are also symmetric, whereas $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{B}}^*$ are not necessarily symmetric.

1.3 Thermoelastic Problems

Considering the effects of temperature change, the stress-strain relations shown in (1.7) or (1.12) for the linear anisotropic elastic materials are generally modified as