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Wavelet Analysis and Applications

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Dedicated to

Professor Rui Paulo da Silva Martins

Preface

The 41 articles collected in this volume are selected from 170 submissions to the conference Wavelet Analysis and Applications 2005 (WAA2005) held during the 29th November to the 2nd December, 2005, at University of Macau. The articles selected are the outgrowth and further development of the talks presented at the conference by international participants from 22 different countries and areas, including Australia, Belgium, Brazil, China, Ethiopia, France, Germany, India, Iran, Hong Kong, Japan, Korea, Macao, Malaysia, Mexico, Portugal, Russia, Taiwan, Thailand, Tunisia, UK, United States, and in both the applied and pure mathematics fields. Most of them are up-to-date new research. We include a number of comprehensive surveys, also containing new results, in several particular areas of research. All the papers are strictly refereed. This volume reflects some of the latest development in the area of wavelet analysis and its applications. It contains two major components: Part I - Wavelet Theory, and Part II - Wavelet Applications. We note that for the reader's convenience the book contains a colored-printed RAM disc although the book itself is in black and white.

There are four chapters in Part I on wavelet theory. In Chapter one, we include seven articles on approximation theory and Fourier analysis. In a paper by S. K. Bloshanskaya and I. L. Bloshanskii some local smoothness conditions are obtained in order to guarantee convergence almost everywhere on some sets of positive measure of the double Walsh-Fourier series summed over rectangles. We also include a paper by the latter in which the problem on convergence of Fourier series of composed function $f \circ m$, where m is a linear transformation, is studied in terms of smoothness of the function f and properties of the transformation m. N. A. Sheikh in his article generalizes the Sidon inequality for the trigonometric system to wavelets and obtains convergence of wavelet series in the L^1 norm. The article of G-B. Ren and H. R. Malonek formulates and proves an extension of the Almansi decomposition for the iterated Dunkl-Helmholtz equation. Included in this chapter the article by M. G. Cowling and M. Sandari, and another by E. S. M. Hitzer and B. Mawardi, study Uncertainty Principles in different contexts. The former proves the Hardy's Uncertainty Principle for operators, and the latter proves an Uncertainty Principle for some Clifford geometric algebras based on Clifford Fourier Transformation.

Chapter two contains ten articles on frame theory and construction of wavelets. In the paper by H-X. Cao and B-M. Yu, wavelet theory for general Hilbert spaces is formulated. In the paper of C-Y. Li and H-X. Cao close relationship between operator frames for bounded linear operators on a Hilbert space and the usual frames for the Hilbert space is studied. D. R. Larson in his paper presents, as an application of operator algebra, a profound operator-interpolation approach to wavelet theory in separable Hilbert spaces by using the local commutant of a unitary system. In other articles G. Wang and Z-X. Cheng study the stability of multi-wavelet frames; J-W. Yang, Y-Y. Tang, Z-X. Cheng and X-G. You construct bi-orthogonal wavelets from two-dimensional interpolatory functions; X-X. Feng, Z-X. Cheng and Z-P. Yang obtain a complete parametrization for the M-channel FIR orthogonal filter bank with linear phase while the number of the required parameters is reduced to $(N = 2)\binom{\frac{M}{2}}{2}$; Y. Li, Z-D. Deng and Y-C. Liang study multivariate orthonormal wavelets with trigonometric vanishing moments and propose a practical construction algorithm; Z. Yao, N. Rajpoot and R. Wilson study multiscale directional cosine transform and multiscale Fourier transform in order to effectively describe oriented features and linear discountinuities in image processing; P. Cerejeiras, M. Ferreira and U. Kähler present a group-theoretical approach for the continuous wavelet transform on the sphere S^{n-1} based on the Lorentz group $\mathrm{Spin}(1, n)$ that provides different representations for the Hilbert space $L^2(S^{n-1})$ and the Hardy space $H^2(S^{n-1})$; finally, F. Brackx, N.D. Schepper and F. Sommen present their study on Clifford-Jacobi polynomials and the associated continuous wavelet transform in Euclidean spaces within the Clifford analysis framework.

Chapter three deals with fractal and multi-fractal theory, wavelet algorithms and wavelets in numerical analysis. In their comprehensive article S. Jaffard, B. Lashermes and P. Abry compare several multifractal formalisms based on wavelet coefficients from mathematical and numerical points of view, and show that the formalism has to be based on wavelet leaders in order to yield the entire and correct spectrum of Hölder singularities. K. Markwardt in his paper studies discrete embedding of system operators in identification models on the base of Fast Wavelet Transform. J. Bai and X-C. Feng in their paper propose a digital curvelet reconstruction algorithm to detect singularities in anisotropic images. H. Diao and Y. Wei study structured condition numbers for Toeplitz under-determined systems with full row rank, compared in the probability sense with unstructured condition numbers. J. Maes and A. Bultheel present their study on Powell-Sabin spline prewavelets on the hexagonal lattice, providing an explicit construction of compactly supported, two-dimensional, piecewise quadratic finite element space of $L^2(\mathbb{R}^2)$.

Chapter four is on time-frequency Analysis and adaptive representation of nonlinear and non-stationary signals. In his paper N. E. Huang introduces his empirical mode decomposition algorithm (EMD) and Hilbert spectral analysis (HHT), and briefly reviews the recent developments. He appeals for a mathematical foundation of the invented method. The article of Q-H. Chen, L-Q. Li and T. Qian shows that the non-linear Fourier atoms $e^{i\theta_a(t)}$, $|a| < 1$, which are the boundary values of the normalized Möbius transforms parameterized by the zeros of the transforms, form a Riesz basis, and possess a number of good properties including Shannon sampling. In his paper T. Qian reviews recent developments aiming to establish mathematical foundation of EDM and HHT, and presents his new results on starlike mapping and constructing mono-components of the form $\rho(t)e^{i\theta_a(t)}$ for non-trivial $\rho(t) \geq 0$ without using Bedrosian's theorem.

In Part II on wavelet applications, in the paper by X-L. Tian, X-K. Li, Y-K. Sun and Z-S. Tang a new algorithm based on wavelet transform to transfer colors from images of Chinese Virtual Human Dada (CVHD) to Magnetic Resonance Images (MRI) is proposed and implemented. In their second paper a novel algorithm for the multimodalities medical images fusion based on wavelet transform is proposed and implemented. The paper by Y-Y. Qu, C-H. Li, N-N. Zheng, Z-J. Yuan and C-Y. Ye describes how wavelet transform may be used to detect salient building from a single nature image. In the paper of Y. Wu, X. Wang and G-S. Liao a despeckling method is proposed based on stationary wavelet transform (SWT) for synthetic aperture radar (SAR) images. In a paper by C-S. Tong and K-T. Leung, to reconstruct a high resolution image from a set of shifted and blurred low resolution images, a direct method based on Haar wavelet transform is proposed. In the paper of F-X. Yan, L-Z. Cheng and H-X. Wang, a design scheme for biorthogonal dual tree complex wavelet transform filter is proposed, and its implementation to iris image enhancement is presented. The other subjects include that the paper of S-K. Choy and C-S. Tong studies supervised learning using characteristic generalized Gaussian density and its applications to Chinese materia medica identification; T-Z. Tan and J-W. Huang propose an algorithm of singular points detection for fingerprint images by the Poincaré index method; G-J. Shi and S-L. Peng present a new receiver scheme for doubly-selective channels to combat the annoying Doppler diversity; by using the support vector machine method (SVM) C-F. Wong, J-K. Zhu, M-I. Vai, P-U. Mak and W-K. Ye present a face retrieval scheme based on lifting wavelets features; S-W. Pei, H-Y. Feng and M-H. Du propose a method based on a wavelet lifting scheme to increase the order of vanishing moments for high-resolution image reconstruction; B. Pradhan, K. Sandeep, S. Mansor, A.R. Ramli and A.R.B.M. Sharif in their paper study multiresolution spatial data compression using the lifting scheme; Y-Y. Ren, S. Wang, S-Y. Yang and L-C. Jiao put forward a method making use of ridgelet transform in remote sensing image recognition; Z-C. Cai, H. Ma, W. Sun and D-X. Qi present their analysis on frequency spectrum for geometric modelling of digital geometry; and, in the paper of M-H. Yang, Z-Y. Xiao and S-L. Peng they demonstrate a Hidden Markov Tree (HMT) model with localized parameters and a fast parameter estimation algorithm. Two papers on implementation of EMD and HHT are included of which one is by Z-H. Yang, L-H. Yang and D-X. Qi on detection of spindles in sleep EEGs; and the other by M. J. Brenner, S. L. Kukreja and R. J. Prazenica on the utility of the Hilbert-Huang algorithm for the analysis of aeroelastic flight data.

Since the corner-stone lecture of Yves Meyer presented in ICM1990, Kyoto, in some extent wavelet analysis in the last 15 years may be said to have been an applied and theoretical-applied area. Yet, we gladly noted that among the attendances of the conference a significant percentage were prominent mathematicians working mainly in pure mathematical areas. This indicates that the concept of wavelets is one that stretches continuously across various disciplines of mathematics.

The idea of organizing the conference at University of Macau was first initialized by Daniel Chi Wai Tse, Chairman of University Council, and Rui Paulo da Silva Martins, Vice Rector of the university, that was endowed through Vai Pan Iu, Rector of the university, whose support made possible the success of the conference. The editors wish to sincerely thank the mentioned university leaders for their kind and generous support. This volume is specially designed to be dedicated to Rui Paulo da Silva Martins, for his unflagging support to mathematics in the university, including the conference. We are grateful to all the university staff members and those in the scientific and organization committees who made this conference possible. Finally, we sincerely thank the referees for their extremely valuable assistance in creating this volume. The publication of this volume is partially supported by Macao Science and Technology development Fund (FDCT) 051/2005/A.

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Part 1. Wavelet Theory

Chapter 1: Approximation and Fourier Analysis

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Local Smoothness Conditions on a Function Which Guarantee Convergence of Double Walsh-Fourier Series of This Function

S.K. Bloshanskaya and I.L. Bloshanskii

Abstract. The local smoothness conditions on a function are obtained, which guarantee convergence almost everywhere on some set of positive measure of the double Walsh-Fourier series of this function summed over rectangles.

Mathematics Subject Classification (2000). Primary 42C10; Secondary 42B05.

Keywords. Double Walsh-Fourier series, summation over rectangles, convergence almost everywhere, localization principle.

1. Discussion and Setting of the Problem

Studies on convergence (including convergence almost everywhere) of series with respect to the classical orthonormal systems (in particular, the trigonometric and the Walsh systems) is one of the central problems in the modern theory of Fourier series.

In the present paper we shall consider Fourier series with respect to the Walsh-Paley system (which have different applications and, in particular, are used in the digital data processing).

As it is known, in 1961 E.Stein [1] proved that the one-dimensional Walsh-Fourier series of a function $f \in L_1(\mathbb{I}^1)$, where $\mathbb{I}^1 = [0, 1)$, can unboundedly diverge almost everywhere (a.e.) on $\mathbb{I}^1$. Moreover, in 2004 S.V.Bochkarev [2] obtained the following result: there exists a function $f \in \Phi(\mathbb{I}^1) = \Phi_L(\mathbb{I}^1)$ (where $\Phi_u = u\varphi(u)$, and $\varphi(u)$ is a non-decreasing on $[0, \infty)$ function, $\varphi(0) = 1$ and $\varphi(u) = o((\log u)^{\frac{1}{2}})$ as $u \to \infty$), whose Walsh-Fourier series unboundedly diverges everywhere on $\mathbb{I}^1$. On the other hand, as it was proved in 2003 by P.Sjolin and F.Soria [3], if a

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function $f \in \mathcal{F}_1(\mathbb{I}^1) = L(\log^+ L)(\log^+ \log^+ \log^+ L)(\mathbb{I}^1)$ then Walsh-Fourier series of this function already converges a.e. on $\mathbb{I}^1$.

The question arises: if for some measurable set $E \subset \mathbb{I}^1$, $\mu E > 0$ (μ is the Lebesgue measure on line) a function $f \in \mathcal{F}_1(E) \cap \Phi(\mathbb{I}^1)$, or (in the "scale" of Lebesgue classes) a function $f \in L_p(E) \cap L_1(\mathbb{I}^1)$, $p > 1$, then what can be said about convergence a.e. of the one-dimensional Walsh-Fourier series of this function, in particular, about convergence a.e. on the set E (where the function f is "sufficiently smooth") or on some of its subsets $E_1 \subset E$, $\mu E_1 > 0$?

In this case, the following question naturally arises: what must be the structure of the set E – open, closed, G_δ, etc., what must be its boundary.

The analogous question can be posed as well for the N - dimensional ($N > 1$) Walsh-Fourier series, namely: on what measurable subsets $E \subset \mathbb{I}^N$, where $\mathbb{I}^N = [0,1)^N$ is the N-dimensional cube, it is possible to "localize" these or those conditions on a function f, defined on the whole $\mathbb{I}^N$, which "guarantee" convergence a.e. on the whole $\mathbb{I}^N$ of the multiple Walsh-Fourier series summed over rectangles. In the multiple case besides the question concerning the structural characteristics of the set E the question arises concerning the geometric characteristics of this set.

Denote as $\mathcal{F}(\mathbb{I}^N)$ the class of summable (on $\mathbb{I}^N$) functions such that for any f in this class ($f \in \mathcal{F}(\mathbb{I}^N)$) the multiple Walsh-Fourier series (summed over rectangles) of the function f converges a.e. on $\mathbb{I}^N$. So, we are interested in the question concerning correlation between the *structural and geometric characteristics of the set* E and the *smoothness of the function* in the framework of these or those subspaces $\mathcal{F}$ of the space L_1.

In the present paper we shall give some solutions of the posed question for double Walsh-Fourier series summed over rectangles.

As to the one-dimensional case, taking account of the classical principle of localization (see [4] or [5, p. 70])[1], and the mentioned earlier result by P.Sjolin and F.Soria [3], we can give a partial answer to the posed above question: for any open a.e.[2] set E, $E \subset \mathbb{I}^1$, $\mu E > 0$ and for any function $f \in \mathcal{F}_1(E) \cap \Phi(\mathbb{I}^1)$ (for any function $f \in L_p(E) \cap L_1(\mathbb{I}^1)$, $p > 1$) the one-dimensional Walsh-Fourier series of this function converges a.e. on the set E.

Note that in the setting of the problem we posed the question about convergence a.e. of Walsh-Fourier series in the classes $\mathcal{F}(E) \cap L_1(\mathbb{I}^1)$ on the set E (or on some of its subsets $E_1 \subset E$), and this is connected with the fact that outside the set E the Walsh-Fourier series of a function $f \in \mathcal{F}(E) \cap L_1(\mathbb{I}^1)$ can, in general, diverge. For example, it is not difficult to prove (taking account of [2] and [4]), that for any open set E with boundary of measure zero or for any closed set E, $E \subset \mathbb{I}^1$, $\mu E > 0$ there exists a function $f \in L_\infty(E) \cap \Phi(\mathbb{I}^1)$, whose Walsh-Fourier series unboundedly diverges a.e. outside the set E.

[1] Walsh-Fourier series of a function $f \in L_1(\mathbb{I}^1), f(x) = 0$ on an open interval $J \subset \mathbb{I}^1$ converges uniformly to zero on each segment which is entirely contained in J.

[2] The set E is called *open a.e.*, if there exists an open set E_1 such that $\mu(E \triangle E_1) = 0$.

For trigonometric Fourier series investigations of this type were carried in the one-dimensional case by G.Alexits, N.K.Bari, S.B.Stechkin, P.L.Ul'yanov (see [6, p. 350-354]), and in the multi-dimensional case ($N > 1$) by I.L.Bloshanskii [7].

2. Notation

Let us denote as $\{\omega_n\}_{n=0}^{\infty} = \{\omega_n(x)\}_{n=0}^{\infty}$, $x \in [0,1) = \mathbb{I}^1$ the Walsh system in Paley enumeration (see, e.g., [5]), i.e. the system of functions constructed as follows. Let us consider the function

$$r_0(x) = \begin{cases} 1, & \text{for } x \in [0, \frac{1}{2}), \\ -1, & \text{for } x \in [\frac{1}{2}, 1). \end{cases}$$

Continue this function with period 1 to the entire number line and define the Rademacher system $\{r_k\}_{k=0}^{\infty}$ by setting $r_k(x) = r_0(2^k x)$, $k = 0, 1, \ldots$.

Next, we represent each positive integer m as the sum $m = \sum_{i=0}^{k} \varepsilon_i 2^i$, with $\varepsilon_i = 0$ or 1 for $i = 0, 1, \ldots, k-1$ and $\varepsilon_k = 1$.

The Walsh functions $\omega_m(x)$ are defined as follows: $\omega_0(x) \equiv 1$,

$$\omega_m(x) = \prod_{i=0}^{k} (r_i(x))^{\varepsilon_i}, \quad m = 1, 2, \ldots .$$

Note that the system $\{\omega_n\}_{n=0}^{\infty}$ is orthonormal on $\mathbb{I}^1$ and complete in the space $L_p(\mathbb{I}^1)$ for each p, $1 \leq p < \infty$.

Let $\mathbb{Z}^N$, $\mathbb{Z}^N \subset \mathbb{R}^N$, $N \geq 1$ be a set of all vectors with integer coordinates, assume $\mathbb{Z}_\alpha^N = \{n = (n_1, \ldots, n_N) \in \mathbb{Z}^N : n_j \geq \alpha, \ j = 1, \ldots, N\}$, $\alpha \in \mathbb{Z}^1$. Further, for $x = (x_1, \ldots, x_N) \in \mathbb{I}^N$, where $\mathbb{I}^N = [0,1)^N$ and $k = (k_1, \ldots, k_N) \in \mathbb{Z}_0^N$, denote as $\omega_k(x) = \omega_{k_1}(x_1) \times \cdots \times \omega_{k_N}(x_N)$ the multiple Walsh-Paley system. Let a function $f \in L_1(\mathbb{I}^N)$ be expanded into a multiple Walsh-Fourier series with respect to the system $\{\omega_k\}_{k \in \mathbb{Z}_0^N}$:

$$f(x) \sim \sum_{k \in \mathbb{Z}_0^N} c_k \, \omega_k(x),$$

where

$$c_k = c_{k_1, \ldots, k_N} = \int_{\mathbb{I}^N} f(x) \, \omega_k(x) \, dx \tag{1}$$

are Walsh-Fourier coefficients of the function f.

We consider the rectangular partial sum of this series

$$S_n(x; f) = \sum_{k_1=0}^{n_1-1} \cdots \sum_{k_N=0}^{n_N-1} c_k \, w_k(x), \quad n = (n_1, \ldots, n_N) \in \mathbb{Z}_1^N,$$

whose particular case is the square partial sum $S_{n_0}(x; f)$, when $n_1 = \ldots = n_N = n_0$.

Let E be an arbitrary measurable set, $E \subset \mathbb{I}^N$, $\mu E > 0$ ($\mu = \mu_N$ is the N-dimensional Lebesque measure), and let $\mathcal{F}(E)$ be a subspace of $L_1(E)$ such that the multiple Walsh-Fourier series (summed over rectangles) of any function $f \in \mathcal{F}(\mathbb{I}^N)$ converges a.e. on $\mathbb{I}^N$.

We study the behavior of $S_n(x;f)$ as $n \to \infty$, i.e. $\min_{1\le j\le N} n_j \to \infty$ (or $S_{n_0}(x;f)$ as $n_0 \to \infty$) on $\mathbb{I}^N$ depending on the smoothness of the function f (i.e. on the type of the space $\mathcal{F}(\mathbb{I}^N)$) and on the structural and geometric characteristics of the set E.

3. Some Results on Convergence of Double Walsh-Fourier Series

For square summation the double Walsh-Fourier series (as follows from the result of F. Móricz, [8]) converges a.e. on $\mathbb{I}^2$ for functions in the class $L_2(\mathbb{I}^2)$, whereas for rectangular summation the double Walsh-Fourier series can diverge a.e. on $\mathbb{I}^2$ even for the continuous on $\mathbb{I}^2$ function (see the result of R.D.Getsadze [9]). From the theorem of E.M.Nikishin [10] concerning the Weyl multipliers (for convergence over rectangles of the double Fourier series with respect to the system of the form $\{\psi_{n_1}(x_1)\cdot\psi_{n_2}(x_2)\}_{n_1,\,n_2=1}^{\infty}$, where $\{\psi_{n_s}(x_s)\}_{n_s=1}^{\infty}$, $s=1,2$ is the orthonormal on a segment system of functions) and the result of P.Billard [11] concerning convergence of the one-dimensional Walsh-Fourier series of functions in L_2 it follows: if the following condition on Fourier coefficients (1) of the function $f \in L_2(\mathbb{I}^2)$ is true:

$$\sum_{k_1,\,k_2=0}^{\infty} |c_{k_1,\,k_2}|^2 \cdot \log^2[\min(|k_1|,\,|k_2|)+2] < \infty, \tag{2}$$

then the double Walsh-Fourier series summed over rectangles of the function f converges a.e. on $\mathbb{I}^2$.

Let us note, that in solution of the problem (considered in the present paper) for one-dimensional Walsh-Fourier series we used (see section 1) the validity (for $N=1$ in the class L_1) of the principle of the classical localization, which permits to state that for any open (nonempty) set $E \subset \mathbb{I}^1$ and for any function $f \in L_1(\mathbb{I}^1)$ such that $f(x)=0$ on E

$$\lim_{n\to\infty} S_n(x;f) = 0 \quad \text{uniformly on any compact set} \quad K \subset E. \tag{3}$$

Unfortunately, for multiple (i.e. for $N \ge 2$) Fourier series (both with respect to the trigonometric system and to Walsh system) such localization is not true even for continuous functions (for more details see our papers [12], [13]).

Being in the framework of the classes $L_p(\mathbb{I}^N)$, $p \ge 1$ we “replaced” in (3) the uniform convergence by the convergence a.e., introducing the following concept of *the generalized localization almost everywhere* (see [14], [15][3]).

Let E, $E \subset \mathbb{I}^N$, $N \ge 1$ be an arbitrary set of positive measure. On the set E for multiple Fourier series of functions in the classes $L_p(\mathbb{I}^N)$, $p \ge 1$ *the generalized*

[3]In the paper [14] the concept of *the generalized localization a.e.* was introduced for trigonometric Fourier series.

localization almost everywhere is valid if for any function $f \in L_p(\mathbb{I}^N),\ f(x) = 0$ on E the multiple Fourier series of the function f converges a.e. to zero on the set E.

In 1995 in [12] for $N = 2$ we proved the validity of the generalized localization a.e. for the double Walsh-Fourier series summed over rectangles on arbitrary open (open a.e.) set in the classes $L_p(\mathbb{I}^2),\ p > 1$ (see [12, Theorem 1]).

Concerning the cases $N = 2,\ p = 1$ and $N > 2,\ p > 1$, in the same paper [12] (see also [15]) we ascertained the invalidity of the generalized localization a.e. in the indicated cases not only on the open sets, but also on any non-dense in $\mathbb{I}^N$ set.

Later in [16] (see also [17] and [18]) we (extending the notion of generalized localization a.e. on the Lebesgue-Orlicz classes) strengthened the result (of Theorem 1) of the paper [12], proving the following theorem

Theorem A. *Let $E,\ E \subset \mathbb{I}^2$ be an arbitrary open a.e. set, $\mu E > 0$. For any function $f \in L(\log^+ L)^2(\mathbb{I}^2),\ f(x) = 0$ on E*

$$\lim_{n\to\infty} S_n(x; f) = 0 \quad \textit{almost everywhere on} \quad E.$$

Thus, for double Walsh-Fourier series summed over rectangles of the function in the classes $L(\log^+ L)^2(\mathbb{I}^2)$ the generalized localization a.e. is true on the open a.e. sets, but, as it was already said, the generalized localization a.e. is not true in the class $L_1(\mathbb{I}^2)$ on the wide class of sets, in particular, it is not true on the open sets.

Being again in the framework of classes $L_p(\mathbb{I}^N),\ p \geq 1$, it was natural (the same way, as for the trigonometric system, see [15]) to pass to a more refined apparatus for studying the behavior of the Fourier series of a function f on the sets where f equals zero, namely, to the concept of "*the weak generalized localization a.e.*" (on the set E *the weak generalized localization almost everywhere* is true, if for any function $f \in L_p(\mathbb{I}^N),\ f(x) = 0$ on E the multiple Fourier series of the function f converges a.e. to zero on some subset $E_0,\ E_0 \subset E,\ \mu E_0 > 0$).

In the paper [13] (see also [15], [18]) we obtained the criteria of the weak generalized localization a.e. in the class $L_1(\mathbb{I}^N),\ N \geq 1$. For $N = 2$ let us formulate the particular case of this result (see [13, Theorem $2'$]), and for this let us give the following definitions.

Let us consider on the axis Ox_j an arbitrary (nonempty) open set $\Omega_j \subset \mathbb{I}^1,\ j = 1, 2$, and denote as W^0 and W the sets

$$W^0 = \left(\Omega_1 \times \mathbb{I}^1\right) \cap \left(\mathbb{I}^1 \times \Omega_2\right) \tag{4}$$

and

$$W = W(W^0) = \left(\Omega_1 \times \mathbb{I}^1\right) \cup \left(\mathbb{I}^1 \times \Omega_2\right). \tag{5}$$

We shall say that a set E possesses property $\mathbb{B}_1$ if there exists a set W of the form (5) such that $\mu\,(W \setminus E) = 0$; property $\mathbb{B}_1$ is property $\mathbb{B}_1(W^0)$ if $W = W(W^0)$.

Further, let us denote by $pr_{(x_j)}\{P\}$ the orthogonal projection of the set $P,\ P \subset I^2$ onto the axis $Ox_j,\ j = 1, 2$; by $int\,(P)$ the set of interior points of P; by $\overline{P}$ the closure of the set P and by $Fr\,P$ the boundary of P.

Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$. Let us denote $G = \mathbb{I}^2 \setminus E$ and consider the following two conditions on $Fr\, E$:

$$\mu\,(G \setminus \overline{int\,(G)}\,) \;=\; 0,\,^{4} \tag{6}$$

$$\mu_1\left(Fr\, pr_{(x_j)}\{int\,(G)\}\right) \;=\; 0, \quad j = 1,\, 2, \tag{7}$$

where $\mu = \mu_2$ is the measure on the plane, μ_1 is the measure on the line.

Theorem B. *Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$, and let $G = \mathbb{I}^2 \setminus E$.*

1. *If for some set W^0 of the form (4) the set E possesses property $\mathbb{B}_1(W^0)$, then for any function $f \in L_1(\mathbb{I}^2)$, $f(x) = 0$ on E*

$$\lim_{n\to\infty} S_n(x; f) = 0 \quad \textit{almost everywhere on} \quad W^0.$$

2. *Let in addition the set E satisfy conditions (6) and (7). If the set E does not possess property $\mathbb{B}_1$, then there exists a function $f^{(0)} \in L_1(\mathbb{I}^2)$, $f^{(0)}(x) = 0$ on E such that*

$$\overline{\lim_{n\to\infty}}\, |S_n(x; f^{(0)})| = +\infty \quad \textit{almost everywhere on} \quad \mathbb{I}^2.$$

4. Main Results

In the present paper, basing on Theorem A, we have obtained the result which shows possibility "to localize on an open a.e. subset" $E \subset \mathbb{I}^2$ condition (2) of convergence a.e. on the whole cube $\mathbb{I}^2$ of double Walsh-Fourier series.

Let E, $E \subset \mathbb{I}^2$ be an arbitrary set of positive measure. Assume

$$\mathcal{F}(E) \;=\; \Bigg\{ f \in L_2(E) : \sum_{k_1,\, k_2 = 0}^{\infty} \left| \iint_E f(x_1,\, x_2)\, \omega_{k_1}(x_1)\, \omega_{k_2}(x_2)\, dx_1\, dx_2 \right|^2 \times \log^2[\min(|k_1|,\, |k_2|) + 2] < +\infty \Bigg\}.$$

Theorem 4.1. *Let E be an arbitrary open a.e. set, $E \subset \mathbb{I}^2$, $\mu E > 0$. For any function $f \in \mathcal{F}(E) \cap L_p(\mathbb{I}^2)$, $1 < p \le 2$*

$$\lim_{n\to\infty} S_n(x; f) = f(x) \quad \textit{almost everywhere on} \quad E.$$

Further, taking into account geometry of the set $E \subset \mathbb{I}^2$, and basing on Theorem B, we can get the following result, which shows under what conditions it is possible "to localize on some subset" of the set E condition (2) (of convergence a.e. on the whole cube $\mathbb{I}^2$ of the double Walsh-Fourier series) in the case when on the whole $\mathbb{I}^2$ the function is in the class L_1 only.

[4] In particular, the sets G such that $\mu\,\{int\, G\} = \mu\, G$ satisfy this condition; in it's turn the, last condition is true, for example, for an arbitrary open set.

Theorem 4.2. *Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$, with conditions on the boundary $Fr\, E$ – (6) and (7), and let the set E have an open (nonempty) subset E^0. For any function $f \in \mathcal{F}(E^0) \cap L(\log^+ L)^2(E) \cap L_1(\mathbb{I}^2)$,*

$$\lim_{n\to\infty} S_n(x;f) = f(x) \quad \textit{almost everywhere on} \quad E^0 \subset W^0$$

if and only if the set E possesses property $\mathbb{B}_1(W^0)$, where

$$W^0 = \left(pr_{(x_1)}\{E^0\} \times \mathbb{I}^1\right) \cap \left(\mathbb{I}^1 \times pr_{(x_2)}\{E^0\}\right). \tag{8}$$

Remark 4.3. *In the part of sufficiency the result of Theorem 4.2 is true without the restrictions* (6) *and* (7).

Taking into account "more fine" structural and geometric characteristics of the sets E and E^0 (which appear in Theorem 4.2), it is possible to obtain the following result

Theorem 4.4. *Let E be an arbitrary measurable set, $E \subset \mathbb{I}^2$, $\mu E > 0$, with conditions on the boundary $Fr\, E$ – (6) and (7), and let E^0 be an open (nonempty) subset of E. If the set E possesses property $\mathbb{B}_1(W^0)$, where the set W^0 is defined in (8), but for any set $\widetilde{W^0}$ of the form (4) such that $\mu(\widetilde{W^0} \setminus W^0) > 0$ the set E does not possess property $\mathbb{B}_1(\widetilde{W^0})$, then*

1. *If $\mu(\mathbb{I}^2 \setminus W^0) > 0$, then there exists a function $f \in \mathcal{F}(E^0) \cap L(\log^+ L)^2(E) \cap L_1(\mathbb{I}^2)$ such that*

$$\overline{\lim_{n\to\infty}} |S_n(x;f)| = +\infty \quad \textit{almost everywhere on} \quad \mathbb{I}^2 \setminus W^0.$$

2. *If $\mu(W^0 \setminus E^0) > 0$, and $\mu\, Fr\, E^0 = 0$, then there exists a function $f^{(1)} \in \mathcal{F}(E^0) \cap L(\log^+ L)^2(E) \cap L_1(\mathbb{I}^2)$ such that*

$$\overline{\lim_{n\to\infty}} |S_n(x;f^{(1)})| = +\infty \quad \textit{almost everywhere on} \quad \mathbb{I}^2 \setminus E^0.$$

And finally, let us once more turn our attention to the questions of convergence a.e. of Walsh-Fourier series in the classes $\mathcal{F}(E) \cap L_p(\mathbb{I}^N)$, $p \geq 1$, $N \geq 1$ outside the set E (see section 1), this time for $N = 2$. Basing on the result concerning general properties of sequences of linear operators obtained by I.L.Bloshanskii in [19] (see [19, Theorem 1]) and using the function constructed by R.D.Getsadze in [9] we can get the following result.

Theorem 4.5. *For any closed set $E \subset \mathbb{I}^2$, $\mu E > 0$ there exists a function $f \in L_\infty(\mathbb{I}^2)$, $f(x) = 0$ on E such that*

1. $\lim_{n\to\infty} S_n(x;f) = 0$ *almost everywhere on* E,
2. $\overline{\lim_{n\to\infty}} |S_n(x;f)| = +\infty$ *almost everywhere on* $\mathbb{I}^2 \setminus E$.

Remark 4.6. *For any (nonempty) open set E, $E \subset \mathbb{I}^2$ with the boundary of measure zero the result similar to the result of Theorem 4.5 directly follows from* [9] *and* [15].

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Linear Transformations of $\mathbb{R}^N$ and Problems of Convergence of Fourier Series of Functions Which Equal Zero on Some Set

I.L. Bloshanskii

Abstract. Let $\mathfrak{M}$ be a class of (all) linear transformations of $\mathbb{R}^N$, $N \geq 1$. Let $\mathcal{A} = \mathcal{A}(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N$ be some linear subspace of $L_1(\mathbb{T}^N)$, and let $\mathfrak{A}$ be an arbitrary set of positive measure $\mathfrak{A} \subset \mathbb{T}^N$.

We consider the problem: how are the sets of convergence and divergence everywhere or almost everywhere (a.e.) of trigonometric Fourier series (in case $N \geq 2$ summed over rectangles) of function $(f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x))$, $f \in \mathcal{A}$, $f(x) = 0$ on $\mathfrak{A}$, $\mathfrak{m} \in \mathfrak{M}$, changed depending on the smoothness of the function f (i.e. on the space $\mathcal{A}$), as well as on the transformation $\mathfrak{m}$.

In the paper a (wide) class of spaces $\mathcal{A}$ is found such that for each $\mathcal{A}$ the system of classes (of nonsingular linear transformations) Ψ_k, $\Psi_k \subset \mathfrak{M}$ $(k = 0, 1, \ldots, N)$, which "change" the sets of convergence and divergence everywhere or a.e. of the indicated Fourier expansions is defined.

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1. Discussion of the Problem

In the theory of Fourier expansions the following problem plays an important role: how properties of Fourier expansions are affected by modifying the function that generates these expansions?

In the one-dimensional case to this range of problems, for example, the following result belongs obtained in 1940 by D.E.Men'shov [1] (for trigonometric Fourier series): any measurable function finite almost everywhere on $\mathbb{T}^1 = [-\pi, \pi)$ (in particular, any continuous function f, $f \in \mathbb{C}(\mathbb{T}^1)$), can be changed on a set of

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arbitrary small measure so that the obtained function has the uniformly convergent Fourier series.

Let us also note the classical problem posed by N.N.Luzin: does a continuous function exist such that, after the continuous transformation of variable, it becomes a function with absolutely convergent Fourier series. As it is known the answer to this question turned out to be negative: in 1981 A.M.Olevskii [2] proved the existence of a function f, $f \in \mathbb{C}(\mathbb{T}^1)$, such that for any homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, – the Fourier series of superposition $(f \circ \varphi)(x) = f(\varphi(x))$ is not absolutely convergent. Let us mention, as well, the result of 1935 by H.Bohr [3], who proved that for any continuous function f there exists a homeomorphism $\varphi : \mathbb{T}^1 \to \mathbb{T}^1$, such that the Fourier series of superposition $f \circ \varphi$ is uniformly convergent. (The detailed survey of the concerning results in the one-dimensional case see in the papers of J.P.Kahane [4] and A.M.Olevskii [5, 6].)

As for the multiple case, in 1998 A.A.Saakyan [7], generalizing the result of H.Bohr, proved that for any function $f \in \mathbb{C}(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N, N \geq 2$ (and therefore, for the continuous function with trigonometric Fourier series rectangularly divergent everywhere on $\mathbb{T}^N$ – see the example of Ch.Fefferman [8]), there exists a homeomorphism $\varphi : \mathbb{T}^N \to \mathbb{T}^N$ such that trigonometric Fourier series of superposition $f \circ \varphi$ uniformly rectangularly converges. The same year S.Galstyan and G.Karagulyan [9] proved an "opposite" (in a certain sense) result, namely: for any function $f \in \mathbb{C}(\mathbb{T}^N)$, $N \geq 2$ (which has no intervals of constancy in $\mathbb{T}^N$) there exists a homeomorphism $\varphi : \mathbb{T}^N \to \mathbb{T}^N$ such that the Fourier series of $f \circ \varphi$ rectangularly diverges almost everywhere (a.e.).

In 2000 O.S. Dragoshanskii [10] published the following result: there exists a function $f \in \mathbb{C}(\mathbb{T}^2)$ (whose support belongs to the square $[\frac{1}{2}, \frac{3}{4}]^2$) such that the double trigonometric Fourier series of f converges rectangularly a.e. on $\mathbb{T}^2$, while the same series but of the function $f \circ \tau$, where τ is a rotation of the coordinate system $\mathbb{R}^2$ on an angle $\frac{\pi}{4}$ diverges rectangularly on its support. In the same paper it was proved that rotation on the angle $\frac{\pi}{4}$ can "spoil", as well, the uniform convergence of the series under consideration.

We [11] in 2002 studied the problem concerning convergence everywhere and a.e. of multiple trigonometric Fourier series (summed over rectangles) of the function $f \circ \tau$, when $f \in L_1(\mathbb{T}^N)$, $N \geq 2$, $f(x) = 0$ on some subset (of positive measure) of $\mathbb{T}^N$, and τ is a rotation of the coordinate system $\mathbb{R}^N$ on an arbitrary angle.

In its turn, the results earlier obtained by us (see, e.g., [12]– [18]) which describe *the structural and geometric characteristics* (**SGC**) of sets of convergence and divergence a.e. and everywhere for multiple trigonometric Fourier series, multiple Walsh-Fourier series (summed over rectangles) and multiple Fourier integrals of functions f from various functional spaces $\mathcal{A}$ (e.g., L_1, Orlicz classes $L(\log^+ L)^s$, $s > 1$, the classes L_p, $1 < p < \infty$, $\mathbb{C}$, H^ω, etc.), f equals zero on some set $\mathfrak{A}$ of positive measure, permit to make some conclusions concerning convergence a.e. and everywhere of multiple Fourier expansions of the superposition $f \circ \psi$, when $f \in \mathcal{A}$, $f(x) = 0$ on $\mathfrak{A}$, and ψ belongs to some class Ψ of linear (e.g., orthogonal) transformations of $\mathbb{R}^N$, $\Psi \subset \mathfrak{M}$, where $\mathfrak{M}$ is the set of (all) linear transformations of $\mathbb{R}^N$.

It is a matter of interest to find (describe) all those classes of transformations Ψ, which for the given space $\mathcal{A}$ "change" the sets of convergence and divergence everywhere or a.e. of the multiple Fourier expansion of a function f in the space $\mathcal{A}$ ($f(x) = 0$ on some set of positive measure), i.e. to give description of pairs $(\mathcal{A}, \Psi)$.

2. Notation

Consider the N–dimensional Euclidean space $\mathbb{R}^N$, whose elements will be denoted as $x = (x_1, \ldots, x_N)$, and set $kx = k_1x_1 + \cdots + k_Nx_N$, $|x| = (x_1^2 + \ldots + x_N^2)^{1/2}$.

Let $\mathbb{Z}^N$, $\mathbb{Z}^N \subset \mathbb{R}^N$ be a set of all vectors with integer coordinates, let us also define the set $\mathbb{Z}^N_\alpha = \{n = (n_1, \ldots, n_N) \in \mathbb{Z}^N : n_j \geq \alpha,\ j = 1, \ldots, N\}$, $\alpha \in \mathbb{Z}^1$.

Let $S_n(x; f)$, $n \in \mathbb{Z}^N_1$, $N \geq 1$ be the rectangular partial sum of trigonometric Fourier series of a function $f \in L_1(\mathbb{T}^N)$, $\mathbb{T}^N = [-\pi, \pi)^N$, whose particular case is the square partial sum $S_{n_0}(x; f)$, when $n_1 = \ldots = n_N = n_0$. Let $\mathcal{A} = \mathcal{A}(\mathbb{T}^N)$ be some linear subspace of the space $L_1(\mathbb{T}^N)$, $\mathfrak{A}$ – an arbitrary measurable set, $\mathfrak{A} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > 0$ ($\mu = \mu_N$ is the N-dimensional Lebesgue measure), and let $f(x) = 0$ on $\mathfrak{A}$.

We investigate how does the behavior of $S_n(x; f)$ as $n \to \infty$, i.e. $\min\limits_{1 \leq j \leq N} n_j \to \infty$ (or $S_{n_0}(x; f)$ as $n_0 \to \infty$) on $\mathbb{T}^N$ depend on the smoothness of the function f (i.e. on the type of the space $\mathcal{A}$), on the "modification" of the function f, and, finally, on *the structural and geometric characteristics* of the set $\mathfrak{A}$ (**SGC**($\mathfrak{A}$)).

3. Definition of the System of Functional Spaces

Denote as $\mathbb{F} = \mathbb{F}_N = \left\{\mathcal{A}^{(j)}_k\right\}_{k,j}$ a matrix $N \times 6$, whose elements are functional spaces $\mathcal{A}^{(j)}_k = \mathcal{A}^{(j)}_k(\mathbb{T}^N)$, $k = 1, \ldots, N$; $N \geq 1$ and $j \in \{0\} \bigcup J$, where $J = \{1, 2, \ldots, 5\}$. The spaces $\mathcal{A}^{(j)}_k$ will be defined as follows. For $k = 1, 2$ we set:

$$\mathcal{A}^{(j)}_1 = \mathcal{A}^{(0)}_1 = L_1\,,\ j \in J\,; \quad \mathcal{A}^{(0)}_2 = \mathcal{A}^{(1)}_2 = L_\infty\,; \quad \mathcal{A}^{(2)}_2 = \mathcal{A}^{(3)}_2 = L_2\,;$$
$$\mathcal{A}^{(4)}_2 = L_p\,,\ 1 < p < 2; \quad \mathcal{A}^{(5)}_2 = L(\log^+ L)^2. \quad (3.1)$$

For $k = 3, \ldots, N$ we set:

$$\mathcal{A}^{(0)}_k = H^{\overline{\omega}^{(k)}},$$

where $\overline{\omega}^{(k)}(\delta)$ is the modulus of continuity $\overline{\omega}^{(k)}(\delta) = \overline{\omega}^{(k)}_\lambda(\delta) = \lambda(\delta) \cdot \left(\log \frac{1}{\delta}\right)^{-\left[\frac{k}{2}\right]}$, where $[\xi]$ is the integral part of ξ, and $\lambda(\delta)$ is a function increasing to $+\infty$ as $\delta \to +0$ and $\lambda(\delta) = o(\log\log \frac{1}{\delta})$, $\delta \to +0$;

$$\mathcal{A}^{(1)}_k = H^{\omega^{(k-1)}} \quad \text{and} \quad \mathcal{A}^{(2)}_k = H_2^{\omega^{(k-1)}}\,,$$

where $\omega^{(k-1)}(\delta) = \omega_\varepsilon^{(k-1)}(\delta) = (\log \frac{1}{\delta})^{-\frac{k-1}{2}-\varepsilon}$, $0 < \varepsilon < \frac{1}{2}$;[1]

$$\mathcal{A}_k^{(3)} = \left\{ f \in L_2(\mathbb{T}^N) : \sum_{n \in \mathbb{Z}^N} |c_n|^2 \cdot \max_{1 \le j_1 < \ldots < j_{k-1} \le N} \prod_{s=1}^{k-1} \log(|n_{j_s}| + 2) < +\infty \right\},$$

where $c_n = c_n(f)$ are Fourier coefficients of function f; and, besides, we set

$$\mathcal{A}_3^{(4)} = \left\{ f \in L_2(\mathbb{T}^N) : \sum_{n \in \mathbb{Z}^N} |c_n|^2 \log^2 [\max_{s,l=1,2,\ldots,N} \min_{s \ne l} (|n_s|, |n_l|) + 2] < +\infty \right\},$$

$$\mathcal{A}_3^{(5)} = H^{\omega^{(1)}}, \quad \text{where} \quad \omega^{(1)}(\delta) = o([\log \frac{1}{\delta} \log\log\log \frac{1}{\delta}]^{-1}), \quad \delta \to +0.$$

For $k = 4, \ldots, N$ we set:

$$\mathcal{A}_k^{(4)} = \mathcal{A}_k^{(5)} = \mathcal{A}_k^{(1)}.$$

Let us note that "smoothness" of functions $f \in \mathcal{A}_k^{(j)}(\mathbb{T}^N)$ certainly "increases" with the growth of the number k, i.e. $\mathcal{A}_k^{(j)} \supset \mathcal{A}_{k+1}^{(j)}$, $j \in \{0\} \bigcup J$, $k = 1 \ldots, N-1$.

Let us also note that the classes $\mathcal{A}_k^{(j)}(\mathbb{T}^N)$, $j \in J$ have the following property: in the case $k > 1$ for any function $f \in \mathcal{A}_k^{(j)}(\mathbb{T}^{k-1})$ convergence of $(k-1)$ -multiple trigonometric Fourier series summed over rectangles takes place a.e. on $\mathbb{T}^{k-1}$ (see results of L.Carleson [19], R.Hunt [20] ($k = 2$); K.I.Oskolkov [21], P.Sjölin [22] ($k = 3$); L.V.Zhizhiashvili [23] and [24], F.Móricz [25] ($k \ge 4$)).

The indicated ("functional") matrix $\mathbb{F}$ was introduced by us in the paper [26].

4. Definition of the Classes of Linear Transformations of $\mathbb{R}^N$

Let $\mathfrak{M}$ be a class of (all) linear transformations of $\mathbb{R}^N$, $N \ge 1$. Denote as Ψ_1, $\Psi_1 \subset \mathfrak{M}$ the class of linear nonsingular transformations, whose inverse transformations have matrices $\mathbb{A} = \{a_{l,m}\}_{l,m=1}^N$, satisfying condition: there exists s, $1 \le s \le N$ such that

$$\max_{1 \le l \le N} |a_{l,s}| < 1. \tag{4.1}$$

Further, in the case of dimension of the space $N \ge 2$, we define the following N subsets of Ψ_1.

First, for any k, $2 \le k \le N$, we define the class of transformations Ψ_k: $\psi \in \Psi_k$ if the matrix $\mathbb{A}$ of inverse (to ψ) transformation ψ^{-1} satisfies condition: there exist $m_1, \ldots, m_k$, $1 \le m_1 < \ldots < m_k \le N$ such that

$$\max_{1 \le l \le N} \{|a_{l,m_1}| + \ldots + |a_{l,m_k}|\} < 1. \tag{4.2}$$

For classes of transformations $\Psi_1, \ldots, \Psi_N$, the embeddings are obvious: $\Psi_1 \supset \Psi_2 \supset \ldots \supset \Psi_N$.

[1]Note that $\mathcal{A}_k^{(0)} \subset \mathcal{A}_k^{(1)}$ if k is even, $k \ge 4$.

Second, for $N \geq 2$ we define the class of transformations $\Psi_0 \subset \Psi_1$. Let $\mathcal{F}$ be a group of rotations of $\mathbb{R}^N$ about the origin, and let $\mathcal{F}_0$ be a set of rotations from $\mathcal{F}$, that are compositions of rotations in all the two-dimensional coordinate planes by angles which are integer multiple of $\frac{\pi}{2}$. Set[2]

$$\Psi_0 = \mathcal{F} \setminus \mathcal{F}_0. \tag{4.3}$$

5. Setting of the Problem and Approaches to Its Solution

We pose and study the problem: how are the sets of convergence and divergence (everywhere or a.e.) of trigonometric Fourier series (in case $N \geq 2$ summed over rectangles) of function f, belonging to one of the spaces $\mathcal{A}$ (elements of the matrix $\mathbb{F}$) and vanishing on some measurable set $\mathfrak{A} \subset \mathbb{T}^N$, $0 < \mu\mathfrak{A} < (2\pi)^N$, $N \geq 1$, ($\mu = \mu_N$ is the Lebesgue measure) changed (if changed) in dependence on the transformation $\psi \in \Psi$, where $\Psi = \Psi_k$, $0 \leq k \leq N$? Thus, we want to "describe" a pair $(\mathcal{A}, \Psi)$.

Further, for any set $E \subset \mathbb{R}^N$ and any $\mathfrak{m} \in \mathfrak{M}$ we define the set $\mathfrak{m}(E) = \{y \in \mathbb{R}^N : y = \mathfrak{m}(x),\ x \in E\}$. Analogously the set $\mathfrak{m}^{-1}(E)$ is defined, where transformation $\mathfrak{m}^{-1}$ is such that: $\mathfrak{m}^{-1} \cdot \mathfrak{m} = 1$ (if $\mathfrak{m}^{-1}$ exists). It is obvious that for any $E \subset \mathbb{T}^N$ there exists $\mathfrak{m} \in \mathfrak{M}$ such that $\mathfrak{m}(E) \not\subset \mathbb{T}^N$.

Thus, taking into account that (in the present paper) we consider 2π-periodic functions $f(x)$, the question arises: how the Fourier series should be understood for the function $(f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x))$, e.g., for rotation (of the coordinate system of $\mathbb{R}^N$), i.e. when $\mathfrak{m} = \tau \in \mathcal{F}$.[3]

Analogously to the paper [11], where we considered the group of rotations $\mathcal{F}$, we shall formulate two variants how the Fourier series of function $f \circ \mathfrak{m}$, $\mathfrak{m} \in \mathfrak{M}$ can be understood.

Let us fix an arbitrary $\mathfrak{m} \in \mathfrak{M}$. For any function $f \in L_1(\mathbb{T}^N)$[4] we define 2π-periodic (for $N \geq 2$ — in each argument) functions $g_{\mathfrak{m}}^{(l)}(x)$, $l = 1, 2$, so that on $\mathbb{T}^N$ these functions are defined by equalities:

$$g_{\mathfrak{m}}^{(1)}(x) = (f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x)), \quad x \in \mathbb{T}^N, \tag{5.1}$$

$$g_{\mathfrak{m}}^{(2)}(x) = (f \circ \mathfrak{m})(x) = f(\mathfrak{m}(x)) \cdot \chi_{\mathbb{T}^N}(\mathfrak{m}(x)), \quad x \in \mathbb{T}^N, \tag{5.2}$$

where $\chi_{\mathbb{T}^N}(\cdot)$ is the characteristic function of the cube $\mathbb{T}^N$.

Thus, the posed above problem is decomposed into two problems in dependence on the regard to Fourier series of function $f \circ \mathfrak{m}$. Further in the text: for $l = 1$ – the problem 1, and for $l = 2$ – the problem 2.

Earlier we have investigated [12]–[18] (see also [26]) the problem concerning changes of the structure and geometry of sets of convergence and divergence a.e.

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[2] It is obvious that rotations $\tau \in \mathcal{F}_0$ can not change the sets of convergence or divergence of multiple Fourier expansions.

[3] Let us note that for Fourier integrals $\int_{\mathbb{R}^N} \widehat{h}(\xi) e^{ix\xi}\, d\xi$, $x \in \mathbb{R}^N$, of function $h \in L_1(\mathbb{R}^N)$ the problem "in this sense" does not arise.

[4] Naturally, the function $f(x)$ is 2π -periodic in each argument.

and everywhere for (multiple) trigonometric Fourier series (for $N \geq 2$ summed over rectangles) of functions f in $\mathcal{A}_k^{(j)}(\mathbb{T}^N)$, $k = 1, \ldots, N$, $j \in \{0\} \bigcup J$, $f(x) = 0$ on some measurable set $\mathfrak{A} \subset \mathbb{T}^N$, in dependence on changes of structure and geometry of the set $\mathfrak{A}$. So, the both posed problems are reduced (in fact) to the study of the question concerning changes of structure and geometry of sets $\psi^{-1}(\mathfrak{A}) \cap \mathbb{T}^N$ and $\mathbb{T}^N \setminus supp(f \circ \psi)$, in dependence on $\psi \in \Psi_k$, $0 \leq k \leq N$.[5]

Let us note that problem 1, being a more complicated problem, is, at the same time, a more natural one for trigonometric Fourier series even in the study of such "unnatural" for these series "problem of rotations".

Let us show some particular solutions of problem 2, whose results give the description of the pairs $(\mathcal{A}, \Psi)$, more exactly, let us formulate the results describing (some) relation between the "smoothness" (in terms of the matrix $\mathbb{F}$) of the function f ($f(x) = 0$ on $\mathfrak{A}$) and the transformation ψ (in terms of the classes Ψ_k).

6. The Set of Transformations Ψ_k, $k = 1, \ldots, N$. Solution of Problem 2

Two following theorems give description of the pair $(\mathcal{A}_1^{(j)}, \Psi_1)$, $j \in \{0\} \bigcup J$, i.e., taking account of (3.1), – the pair (L_1, Ψ_1) (for $N = 1$ and for $N > 1$, respectively).

Theorem 6.1. *For any $\psi \in \Psi_1$ and ε, $0 < \varepsilon < 2\pi$, there exist the measurable sets $\Omega = \Omega(\varepsilon, \psi) \subset \mathbb{T}^1$, $\mathfrak{A} = \mathfrak{A}(\varepsilon, \psi) \subset \mathbb{T}^1$: $\mu\Omega > 0$, $\mu\mathfrak{A} > 2\pi - \varepsilon$ and a function $f = f_{\varepsilon, \psi} \in L_1(\mathbb{T}^1)$, $f(x) = 0$ on $\mathfrak{A}$, such that*

1. $$\overline{\lim_{n \to \infty}} |S_n(x; f)| = +\infty \quad \text{in each point} \quad x \in \mathbb{T}^1, \tag{6.1}$$
2. $$\lim_{n \to \infty} S_n(x; f \circ \psi) = 0 \quad \text{in each point} \quad x \in \Omega. \tag{6.2}$$

Here the notation $f \circ \psi$ is understood in the sense of equality (5.2), *i.e. $f \circ \psi = g_\psi^{(2)}$.*

Theorem 6.2. *Let $N > 1$. For any $\psi \in \Psi_1$ and ε, $0 < \varepsilon < (2\pi)^N$, there exist the open sets $\Omega = \Omega(\varepsilon, \psi)$, $\mathfrak{A} = \mathfrak{A}(\varepsilon, \psi) : \Omega \subset \mathfrak{A} \subset \mathbb{T}^N$, $\mu\mathfrak{A} > (2\pi)^N - \varepsilon$, $0 < \mu\Omega < \mu\mathfrak{A}$ such that*

1. *There exists a function $f^{(0)} = f^{(0)}_{\varepsilon, \psi} \in L_1(\mathbb{T}^N)$, $f^{(0)}(x) = 0$ on $\mathfrak{A}$, and*
$$\overline{\lim_{n_0 \to \infty}} |S_{n_0}(x; f^{(0)})| = +\infty \quad \text{in each point} \quad x \in \mathbb{T}^N. \tag{6.3}$$
2. *For any function $f \in L_1(\mathbb{T}^N)$, $f(x) = 0$ on $\mathfrak{A}$,*
$$\lim_{n \to \infty} S_n(x; f \circ \psi) = 0 \quad \text{in each point} \quad x \in \Omega. \tag{6.4}$$

Here the notation $f \circ \psi$ is understood in the sense of equality (5.2), *i.e. $f \circ \psi = g_\psi^{(2)}$.*

Analogous results are obtained for other pairs $(\mathcal{A}_r^{(j)}, \Psi_k)$, where $k \leq r \leq N$, $j \in \{0\} \bigcup J$ for $k = r = N$, if $N = 2$, and for $1 \leq k \leq 2 \cdot \left[\frac{N-1}{2}\right]$, if $N \geq 3$, namely, the following theorems are true

[5]Note that for singular transformations $\mathfrak{m} \in \mathfrak{M}$ the discussed problem becomes trivial.