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# Topology of Infinite-Dimensional Manifolds

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*There is no limit for our thoughts and imaginations since we are made "in God's image" (Genesis 1:27). Mathematical ability is a gift from God, which enable us to comprehend and to create various abstract concepts. I always thank and praise Jehovah, our Creator. I agree with the following words in the Scriptures:*

*"The fear of Jehovah is*
*the beginning of wisdom,*
*And knowledge of the Most Holy One*
*is understanding." —Proverbs 9:10*

*Note: Scripture quotations are from*
*New World Translation of the Holy Scriptures—Revised 2013*
*(Watchtower Bible and Tract Society of New York, Inc.)*

# Preface

An infinite-dimensional manifold is a (topological) manifold modeled on a given infinite-dimensional (homogeneous) space $E$ (called a model space), that is, a paracompact space covered by open sets that are homeomorphic to open sets in $E$. A manifold modeled on $E$ is simply called an $E$-manifold. Hilbert space $\ell^2$ and the Hilbert cube $\boldsymbol{Q} = [-1,1]^{\mathbb{N}}$ are typical examples of infinite-dimensional model spaces, where an $\ell^2$-manifold and a $\boldsymbol{Q}$-manifold are also called a Hilbert manifold and a Hilbert cube manifold, respectively.[1] We can also take any infinite-dimensional topological linear space as a model space. The direct limit $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ of Euclidean spaces is one of them. It is known that the direct limit $\boldsymbol{Q}^\infty = \varinjlim \boldsymbol{Q}^n$ of Hilbert cubes is homeomorphic to some topological linear space. Thus, there are various kinds of infinite-dimensional manifolds, which are research objects of Infinite-Dimensional Topology.[2]

Infinite-dimensional manifolds are more than just generalizations of usual manifolds modeled on Euclidean space $\mathbb{R}^n$. There are many unexpected special phenomena different from finite-dimensional manifolds. From research of infinite-dimensional manifolds, various useful tools, techniques, and ideas have been developed, which are attractive and exciting. In addition, there have been many applications to other fields of Topology.[3] Let us give some examples of outstanding results related to other fields.

In 1928, Fréchet asked whether Hilbert space $\ell^2$ is homeomorphic to the countable product of lines $\mathbb{R}^{\mathbb{N}}$. This question was positively answered in 1966 by R.D. Anderson. It was the dawn of Infinite-Dimensional Topology.[4] In 1939,

[1] We consider not only $\ell^2$ but also non-separable Hilbert spaces as model spaces.

[2] Infinite-Dimensional Topology is a branch of Geometric Topology, which studies infinite-dimensional spaces arising naturally in Topology and Functional Analysis.

[3] For the history of Infinite-Dimensional Topology, refer to the article of T. Koetsier and J. van Mill [96, Sect. 4].

[4] Due to Anderson's essay [9], the starting point is when he answered in [3] affirmatively to a question posed by V. Klee, that is, he proved in 1964 that the product of a triod, $T$, and the Hilbert cube $\boldsymbol{Q}$ is homeomorphic to $\boldsymbol{Q}$.

M. Wojdysławski conjectured that the hyperspace of a Peano continuum is homeomorphic to $\boldsymbol{Q}$. This conjecture was finally proved in 1978 and it is known as the Curtis–Schori–West Hyperspace Theorem. Simple Homotopy Theory had been established by J.H.C. Whitehead during 1939–1952, where the Whitehead group and Whitehead torsion were introduced. The topological invariance of Whitehead torsion had been a longstanding problem, but T.A. Chapman proved it in 1973 by using $\boldsymbol{Q}$-manifolds. In 1954, K. Borsuk conjectured that a compact absolute neighborhood retract (ANR) has the homotopy type of a finite simplicial complex. This conjecture was proved in 1977 by J.E. West, where $\boldsymbol{Q}$-manifolds were applied. Shape Theory was founded by K. Borsuk in 1968 as the homotopy theory for spaces without a good local behavior. In 1972, Chapman showed that two compacta $X$ and $Y$ in the pseudo-interior $(-1, 1)^{\mathbb{N}}$ of the Hilbert cube $\boldsymbol{Q}$ have the same shape type if and only if their complements $\boldsymbol{Q} \setminus X$ and $\boldsymbol{Q} \setminus Y$ are homeomorphic.

Meanwhile, the theory of infinite-dimensional manifolds was developed and outstanding results had been obtained. In 1969, D.W. Henderson proved the Open Embedding Theorem, that is, every separable $\ell^2$-manifold can be embedded into $\ell^2$ as an open set. In 1970, he joined with R.M. Schori and J.E. West in establishing the Classification and the Triangulation Theorems for Hilbert manifolds, respectively. Namely, it was shown that two Hilbert manifolds are homeomorphic if they have the same homotopy type and that every Hilbert manifold is homeomorphic to the product of Hilbert space and a locally finite-dimensional simplicial complex with the metric topology. For (compact) $\boldsymbol{Q}$-manifolds, the Triangulation and the Classification Theorems were established in 1973 by Chapman, that is, every compact $\boldsymbol{Q}$-manifold is homeomorphic to the product of $\boldsymbol{Q}$ and a finite simplicial complex, and two compact $\boldsymbol{Q}$-manifolds are homeomorphic if and only if finite simplicial complexes triangulating them have the same simple homotopy type. In 1980 and 1981, H. Toruńczyk succeeded in characterizing Hilbert cube manifolds and Hilbert manifolds topologically.

This book is designed as a textbook for graduate students and researchers in various branches related to Topology to acquire the fundamental results on infinite-dimensional manifolds and various techniques treating infinite-dimensional spaces. This can also be used as a reference book. Up to now, the following six books have been available for the same purpose:

(1) C. Bessaga and A. Pełczyński, *Selected Topics in Infinite-Dimensional Topology*, MM **58** (Polish Sci. Publ., Warsaw, 1975)
(2) T.A. Chapman, *Lectures on Hilbert Cube Manifolds*, CBMS Regional Conf. Ser. in Math. **28** (Amer. Math. Soc., Providence, 1975)
(3) J. van Mill, *Infinite-Dimensional Topology: Prerequisites and Introduction*, North-Holland Math. Library **43** (Elsevier Sci. Publ. B.V., Amsterdam, 1989)
(4) A. Chigogidze, *Inverse Spectra*, North-Holland Math. Library **53** (Elsevier Sci. Publ. B.V., Amsterdam, 1996)
(5) T. Banakh, T. Radul, and M. Zarichnyi, *Absorbing Sets in Infinite-Dimensional Manifolds*, Math. Studies, Monog. Ser. **1** (VNTL Publ., Lviv, 1996)

(6) J. van Mill, *The Infinite-Dimensional Topology of Function Spaces*, North-Holland Math. Library **64** (Elsevier Sci. Publ. B.V., Amsterdam, 2002)

We selected materials from the above books and compiled them into the present book and also included the new results that are not presented in those books. In addition, we have covered the manifolds modeled on the direct limits $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ and $\boldsymbol{Q}^\infty = \varinjlim \boldsymbol{Q}^n$. As an infinite-dimensional version of a combinatorial $n$-manifold, we have defined a combinatorial $\infty$-manifold, which triangulates an $\mathbb{R}^\infty$-manifold. This is the first book presenting such manifolds.

This book is written to be fairly self-contained if combined with the following book, which is cited as [GAGT]:

- K. Sakai, *Geometric Aspects of General Topology*, Springer Monog. in Math. (Springer, Tokyo, 2013)

To read [GAGT], the readers are required to have the fundamental knowledge on General Topology. For example, it is enough to finish Part I of the following popular textbook:

- J.R. Munkres, *Topology, 2nd ed.* (Prentice Hall, Inc., Upper Saddle River, 2000)

The book [GAGT] also contains some outstanding results in Infinite-Dimensional Topology different from applications mentioned above, that is, the existence of the following spaces or maps: a hereditary infinite-dimensional compact metrizable space (a compactum containing no subspaces of finite-dimension except zero-dimension); an infinite-dimensional compact metrizable space with finite cohomological dimension (Alexandroff's Problem); a cell-like map of a finite-dimensional compactum onto an infinite-dimensional compactum (Cell-Like Mapping Problem); a separable metrizable topological linear space that is not an absolute retract (AR). The first one is contained in Chapter 4 and the other three are in Chapter 7. Those are not necessary for reading the present book.

Almost all required background knowledge is listed in Chap. 1, whose detailed information is founded in [GAGT]. Besides, we need some additional results, for example, some fundamental results in PL Topology, which are also contained with their proofs. Taking a brief look at this first chapter, the reader can recognize what knowledge he or she should take in and where he or she can. Possibly skipping the first chapter, one can start with the second chapter and go back over necessary parts of Chap. 1 when needed. For Chap. 4, we need some results in Simple Homotopy Theory and Wall's Obstruction Theory, covering spaces and Algebraic Topology,[5] which are written in the preliminary part of Chap. 4, not in the first chapter.

Chapter 2 is devoted to the fundamental results on manifolds modeled on an infinite-dimensional normed linear space, the Stability Theorem, the Unknotting Theorem, the Open Embedding Theorem, the Classification Theorem, etc., which are discussed in Bessaga and Pełczynski's book (1). Here, are also proved the fundamental results on Hilbert cube manifolds, which are discussed in van Mill's book (3) and Chapman's lecture notes (2). Furthermore, we prove the Toruńczyk

[5] It is not required for the reader to be familiar with these objects. Only elementary knowledge of Algebraic Topology is necessary.

Factor Theorem, that is, for each completely metrizable ANR $X$ with weight $\leqslant \tau$, the product of $X$ and Hilbert space of weight $\tau$ is a Hilbert manifold, which has not been proved in any other book. Combining this theorem with the Classification Theorem, we can easily obtain the Triangulation Theorem for Hilbert manifolds.

Toruńczyk's characterizations of Hilbert manifolds and $\boldsymbol{Q}$-manifolds are proved in Chap. 3. For the characterization of compact $\boldsymbol{Q}$-manifolds, a readable proof is provided in van Mill's book (3), but the non-compact version is not easily derived from the compact case. However, we discuss the non-compact case too. For Hilbert manifolds, we treat not only $\ell^2$-manifolds but also non-separable Hilbert manifolds. In this chapter, some applications of the characterization of Hilbert manifolds and $\boldsymbol{Q}$-manifolds are also given. In particular, it is proved that every Fréchet space (= locally convex completely metrizable topological space) is homeomorphic to Hilbert space with the same weight and that every infinite-dimensional compact convex set in a metrizable topological space is homeomorphic to the Hilbert cube $\boldsymbol{Q}$ if it is an AR. It is also proved that the space of all continuous map from a non-discrete compactum to a separable completely metrizable ANR is an $\ell^2$-manifold.

As mentioned above, Chap. 2 contains the Triangulation Theorem for Hilbert manifolds but not for $\boldsymbol{Q}$-manifolds. Chapter 4 is devoted to proving the Triangulation Theorem for $\boldsymbol{Q}$-manifolds, which is contained in Chapman's lecture notes (2) but is not in van Mill's book (3). To prove this theorem, we use some algebraic results concerning the Whitehead group and the Wall's finiteness obstruction. They are contained in the first section, but their proofs are not given. From the Triangulation Theorem for compact $\boldsymbol{Q}$-manifolds, we have derived the Borsuk conjecture mentioned above. The topological invariance of Whitehead torsion is also proved in this chapter.

In Chap. 5, we discuss f.d.cap sets and cap sets for $\ell^2$-manifolds and $\boldsymbol{Q}$-manifolds, which are manifolds modeled on the following incomplete normed linear spaces:

$$\ell^2_f = \left\{(x_i)_{i\in\mathbb{N}} \in \ell^2 \;\middle|\; x_i = 0 \text{ except for finitely many } i \in \mathbb{N}\right\},$$
$$\ell^2_Q = \left\{(x_i)_{i\in\mathbb{N}} \in \ell^2 \;\middle|\; \sup_{i\in\mathbb{N}} 2^i|x_i| < \infty\right\},$$

which are the linear spans of the orthonormal basis of $\ell^2$ and the Hilbert cube $Q = \prod_{n\in\mathbb{N}}[-2^{-n}, 2^{-n}] \subset \ell^2$, respectively. We also consider non-separable versions of f.d.cap sets and cap sets, named absorption bases. As a generalization of (f.d.)cap sets, introducing absorbing sets in Hilbert manifolds, M. Bestvina and J. Mogilski gave characterizations of manifolds modeled on the universal spaces of absolute Borel spaces. Since then, there have been many works on absorbing sets. In this chapter, we also intend to generalize the work of Bestvina and Mogilski to non-separable spaces. Although there are many related results, we treat only a small number of them. For results on absorbing sets, the book of Banakh, Radul, and Zarichinyi (5) can be referred to but non-separable absorbing sets are not treated. Related results also treated in the second book of J. van Mill (6).

Chapter 6 is devoted to manifolds modeled on $\mathbb{R}^\infty$ and $\boldsymbol{Q}^\infty$, which are the direct limits of Euclidean spaces and Hilbert cubes, respectively. Here, we prove Heisey's Theorem, that is, $\boldsymbol{Q}^\infty$ is homeomorphic to a locally convex linear topological space. Characterizations of these manifolds are given and their Classification Theorems are proved. We also discuss simplicial complexes triangulating $\mathbb{R}^\infty$-manifolds. Such a simplicial complex is an infinite-dimensional generalization of combinatorial manifolds, which is called an infinite-dimensional combinatorial manifold (or a combinatorial $\infty$-manifold). We prove the so-called Hauptvermutung[6] for them. Furthermore, a combinatorial $\infty$-manifold is characterized as a simplicial complex $K$, such that every simplex $\sigma \in K$ is a $Z$-set in $|K|$, equivalently every $\sigma \in K$ has the non-empty contractible link. We also prove that a countable simplicial complex $K$ is a combinatorial $\infty$-manifold if and only if $|K|$ is an $\mathbb{R}^\infty$-manifold. In the last section, we introduce bi-topological infinite-dimensional manifolds modeled on $(\mathbb{R}^\infty, \mathbb{R}^{\mathbb{N}}_f)$ and $(\boldsymbol{Q}^\infty, \boldsymbol{Q}^{\mathbb{N}}_f)$, which are called $(\mathbb{R}^\infty, \mathbb{R}^{\mathbb{N}}_f)$-manifolds and $(\boldsymbol{Q}^\infty, \boldsymbol{Q}^{\mathbb{N}}_f)$-manifolds, respectively. Every combinatorial $\infty$-manifold with the weak topology and the metric topology is a manifold modeled on $(\mathbb{R}^\infty, \mathbb{R}^{\mathbb{N}}_f)$.

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Tsukuba, Japan<br>April 2020

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[6] For meaning of this word, see p. 44.

**Note** In the text, numbers in brackets [ ] and [( )], respectively, refer to papers in the References and books or texts in the Bibliography at the end of the book. However, the author's first book [(15)] is cited as [GAGT]. Moreover, for a theorem (or proposition, etc.) quoted from [GAGT], its corresponding theorem number in [GAGT] is indicated in a frame box at the end of the statement. For example, $\boxed{2.9.4}$ means (Theorem) 2.9.4 in [GAGT].

# Contents

# Chapter 1
# Preliminaries and Background Results

In this chapter, we first introduce the terminology and notation, and then list the background results. The reader may skip this chapter and can read necessary parts later when needed. Several results might be learned in graduate courses, but others are advanced and special. Almost all results are listed without proofs, which can be found in the following book:

- K. Sakai, *Geometric Aspects of General Topology,* Springer Monog. in Math. (Springer, Tokyo, 2013) — [(15)]

That is cited as [GAGT] instead of [(15)], by which the reader can confirm proofs and details. When a theorem (or proposition, etc.) is quoted from [GAGT], each corresponding theorem number in [GAGT] is indicated in a frame box at the end of statement.

> Sections 1.1 and 1.2 are almost identical to the same sections of [GAGT]. Contents of Sects. 1.3, 1.4, 1.12, 1.13, and 1.15 come from Chapters 2, 3, 5, 6, and 7 of [GAGT], respectively. Almost all contents of Sections 1.5–1.11 except Sect. 1.9 are contained in Chapter 4 of [GAGT]. Sections 1.9 and 1.14 are supplements for Chapters 4 and 6 of [GAGT], respectively.

Fundamental results on simplicial complexes are described in Chapter 4 of [GAGT]. Besides, additional preliminary results in PL Topology (Combinatorial Topology) are required for $\boldsymbol{Q}$-manifolds and $\mathbb{R}^\infty$-manifolds, which are not covered by [GAGT]. In Sect. 1.8, a version of the PL embedding approximation theorem 1.8.11 is added together with its proof. Section 1.9 is devoted to regular neighborhoods that are effectively used in PL Topology. Although we refer to them too little, PL $n$-manifolds (or combinatorial $n$-manifolds) are main subjects in PL Topology. So we provide an appendix for basic results on them at the end of the book. The following is a good textbook:

- C.P. Rourke and B.J. Sanderson, *Introduction to Piecewise-Linear Topology,* Springer Study Edition (Springer-Verlag, Berlin, 1972, 1982) — [(17)]



K. Sakai, *Topology of Infinite-Dimensional Manifolds*, Springer Monographs in Mathematics, https://doi.org/10.1007/978-981-15-7575-4_1

## 1.1 Terminology and Notation

With respect to terminology and notation, we follow the book [GAGT]. For the standard sets, we use the following notation:

- $\mathbb{N}$ — the set of natural numbers (i.e., positive integers);
- $\omega = \mathbb{N} \cup \{0\}$ — the set of non-negative integers;
- $\mathbb{Z}$ — the set of integers;
- $\mathbb{Q}$ — the set of rationals;
- $\mathbb{R} = (-\infty, \infty)$ — the real line with the usual topology;
- $\mathbb{C}$ — the complex plane;
- $\mathbb{R}_+ = [0, \infty)$ — the half (real) line;
- $\mathbf{I} = [0, 1]$ — the unit closed interval.

A (topological) **space** is assumed to be **Hausdorff** and a **map** is a **continuous** function. A **singleton** is a space consisting of one point. A space is said to be **non-degenerate** if it has at least two points. A compact metrizable space is called a **compactum** and a connected compactum is called a **continuum**.[1] For a space $X$ and $A \subset X$, we use the following notation:

- $\mathrm{cl}_X A$ (or $\mathrm{cl}\, A$) — the closure of $A$ in $X$;
- $\mathrm{int}_X A$ (or $\mathrm{int}\, A$) — the interior of $A$ in $X$;
- $\mathrm{bd}_X A$ (or $\mathrm{bd}\, A$) — the boundary of $A$ in $X$;
- $\mathrm{id}_X$ (or id) — the identity map of $X$.

For a metrizable space $X$,

- $\mathrm{Metr}(X)$ — the set of all admissible metrics of $X$.

The cardinality of a set $\Gamma$ is denoted by $\mathrm{card}\,\Gamma$. The **weight** $w(X)$, the **density** $\mathrm{dens}\, X$, and the **cellularity** $c(X)$ of a space $X$ are defined as follows:

- $w(X) = \min\{\mathrm{card}\,\mathcal{B} \mid \mathcal{B}$ is an open basis for $X\}$;
- $\mathrm{dens}\, X = \min\{\mathrm{card}\, D \mid D$ is a dense set in $X\}$;
- $c(X) = \sup\{\mathrm{card}\,\mathcal{G} \mid \mathcal{G}$ is a pairwise disjoint open collection$\}$.

As is easily observed, $c(X) \leqslant \mathrm{dens}\, X \leqslant w(X)$ in general. In the case where $X$ is metrizable, all these cardinalities coincide (cf. p. 2 of [GAGT]).

For spaces $X$ and $Y$ with subspaces $X_1, \ldots, X_n \subset X$ and $Y_1, \ldots, Y_n \subset Y$,

- $X \approx Y$ means that $X$ and $Y$ are homeomorphic;
- $(X, X_1, \ldots, X_n) \approx (Y, Y_1, \ldots, Y_n)$ means that there exists a homeomorphism $h : X \to Y$ such that $h(X_1) = Y_1, \ldots, h(X_n) = Y_n$;
- $(X, x_0) \approx (Y, y_0)$ means $(X, \{x_0\}) \approx (Y, \{y_0\})$,

where $(X, x_0)$ is called a **pointed space** and $x_0$ its **base point**.

[1] Their plurals are **compacta** and **continua**, respectively.

For the product space $\prod_{\gamma\in\Gamma} X_\gamma$, the $\gamma$-coordinate of each point $x \in \prod_{\gamma\in\Gamma} X_\gamma$ is denoted by $x(\gamma)$, that is, $x = (x(\gamma))_{\gamma\in\Gamma}$. We can regard $x \in \prod_{\gamma\in\Gamma} X_\gamma$ as a function $x : \Gamma \to \bigcup_{\gamma\in\Gamma} X_\gamma$ such that $x(\gamma) \in X_\gamma$ for each $\gamma \in \Gamma$. For each $\gamma \in \Gamma$, the projection $\mathrm{pr}_\gamma : \prod_{\gamma\in\Gamma} X_\gamma \to X_\gamma$ is defined by $\mathrm{pr}_\gamma(x) = x(\gamma)$. For $\Lambda \subset \Gamma$, the projection $\mathrm{pr}_\Lambda : \prod_{\gamma\in\Gamma} X_\gamma \to \prod_{\lambda\in\Lambda} X_\lambda$ is defined by $\mathrm{pr}_\Lambda(x) = x|\Lambda$ $(= (x(\lambda))_{\lambda\in\Lambda})$. In the case $X_\gamma = X$ for every $\gamma \in \Gamma$, we write $\prod_{\gamma\in\Gamma} X_\gamma = X^\Gamma$. In particular, $X^{\mathbb{N}}$ is the product space of countable infinite copies of $X$. When $\Gamma = \{1, \dots, n\}$, $X^\Gamma = X^n$ is the product space of $n$ copies of $X$. For the product space $X \times Y$, $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ denote the projections.

Now, let $X = (X, d)$ be a metric space, $x \in X$, $\varepsilon > 0$, and $A, B \subset X$. We use the following notation:

- $\mathrm{B}_d(x, \varepsilon) = \{y \in X \mid d(x, y) < \varepsilon\}$ — the $\varepsilon$-neighborhood of $x$ in $X$ (or the open ball with center $x$ and radius $\varepsilon$);
- $\overline{\mathrm{B}}_d(x, \varepsilon) = \{y \in X \mid d(x, y) \leqslant \varepsilon\}$ — the closed $\varepsilon$-neighborhood of $x$ in $X$ (or the closed ball with center $x$ and radius $\varepsilon$);
- $\mathrm{diam}_d\, A = \sup\{d(x, y) \mid x, y \in A\}$ — the diameter of $A$;
- $d(x, A) = \inf\{d(x, y) \mid y \in A\}$ — the distance of $x$ from $A$;
- $\mathrm{dist}_d(A, B) = \inf\{d(x, y) \mid x \in A,\ y \in B\}$ — the distance of $A$ and $B$;
- $\mathrm{N}_d(A, \varepsilon) = \{x \in X \mid d(x, A) < \varepsilon\}$ — the $\varepsilon$-neighborhood of $A$ in $X$;
- $\overline{\mathrm{N}}_d(A, \varepsilon) = \{y \in X \mid d(x, A) \leqslant \varepsilon\}$ — the closed $\varepsilon$-neighborhood of $A$ in $X$.

It should be noticed that $d(x, A) = \mathrm{dist}_d(\{x\}, A)$, $\mathrm{N}_d(\{x\}, \varepsilon) = \mathrm{B}_d(x, \varepsilon)$, $\overline{\mathrm{N}}_d(\{x\}, \varepsilon) = \overline{\mathrm{B}}_d(x, \varepsilon)$, and $\mathrm{N}_d(A, \varepsilon) = \bigcup_{x\in A} \mathrm{B}_d(x, \varepsilon)$. For a collection $\mathcal{A}$ of subsets of $X$, the **mesh** of $\mathcal{A}$ is defined as follows:

- $\mathrm{mesh}_d\, \mathcal{A} = \sup\{\mathrm{diam}_d\, A \mid A \in \mathcal{A}\}$.

When there are no possible confusions, we can drop the subscript $d$ and write $\mathrm{B}(x, \varepsilon)$, $\overline{\mathrm{B}}(x, \varepsilon)$, $\mathrm{N}(A, \varepsilon)$, $\mathrm{diam}\, A$, $\mathrm{dist}(A, B)$ and $\mathrm{mesh}\, \mathcal{A}$.

The standard spaces are listed below:

- $\mathbb{R}^n$ — Euclidean $n$-space with the norm

$$\|x\| = \sqrt{x(1)^2 + \cdots + x(n)^2},$$

  $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^n$ — the origin, the zero vector or the zero element, $\mathbf{e}_i \in \mathbb{R}^n$ — the unit vector defined by $\mathbf{e}_i(i) = 1$ and $\mathbf{e}_i(j) = 0$ for $j \neq i$;
- $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x(n) \geqslant 0\} = \mathbb{R}^{n-1} \times \mathbb{R}_+$ — Euclidean half $n$-space;
- $\mathbf{S}^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ — the unit $(n-1)$-sphere;
- $\mathbf{B}^n = \{x \in \mathbb{R}^n \mid \|x\| \leqslant 1\}$ — the unit closed $n$-ball;
- $\Delta^n = \{x \in (\mathbb{R}_+)^{n+1} \mid \sum_{i=1}^{n+1} x(i) = 1\}$ — the standard $n$-simplex;
- $\boldsymbol{Q} = [-1, 1]^{\mathbb{N}}$ — the Hilbert cube;
- $\boldsymbol{s} = \mathbb{R}^{\mathbb{N}}$ — the space of sequences.

Note that $\mathbf{S}^{n-1}$, $\mathbf{B}^n$, and $\Delta^n$ are not product spaces even though the same notations are used for product spaces, where the indexes $n-1$ and $n$ represent their dimensions.

A separable metrizable space $M$ is called an ***n*-manifold** (or an ***n*-dimensional manifold**)[2] if each $x \in M$ has a neighborhood that is homeomorphic to (an open set in) $\mathbf{I}^n$, where $\mathbf{I}^n$ can be replaced with $\mathbb{R}^n_+$. We call $x \in M$ an **interior point** if it has a neighborhood homeomorphic to (an open set in) $(0,1)^n$ ($\approx \mathbb{R}^n$). The set $\operatorname{Int} M$ of all interior points of $M$ is called the **interior** of $M$, which is open in $M$. We also call $x \in M$ a **boundary point** if it is not an interior point, that is, any neighborhood of $x$ is not homeomorphic to (an open set in) $\mathbb{R}^n$. In other words, $x \in M$ is a boundary point if and only if $x$ has a neighborhood $N$ such that $(N, x) \approx (\mathbf{I}^n, \mathbf{0})$ (or $(N, x) \approx (\mathbb{R}^n_+, \mathbf{0})$). The set $\partial M$ consisting of all boundary points of $M$ is called the **boundary** of $M$, which is an $(n-1)$-manifold and closed in $M$. When $\partial M = \emptyset$, $M$ is called an ***n*-manifold without boundary**. A **closed *n*-manifold** is a compact $n$-manifold without boundary. The closed $n$-ball $\mathbf{B}^n$ is an $n$-manifold with $\partial \mathbf{B}^n = \mathbf{S}^{n-1}$ and the $n$-sphere $\mathbf{S}^n$ is a closed $n$-manifold.

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of $X$ and $Y \subset X$. We define

- $\mathcal{A} \wedge \mathcal{B} = \{A \cap B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}$;
- $\mathcal{A}|Y = \{A \cap Y \mid A \in \mathcal{A}\}$;
- $\mathcal{A}[Y] = \{A \in \mathcal{A} \mid A \cap Y \neq \emptyset\}$;
- $\mathcal{A}^{\mathrm{cl}} = \{\operatorname{cl} A \mid A \in \mathcal{A}\}$.

The **star** of $Y$ with respect to $\mathcal{A}$ is defined as follows:

- $\operatorname{st}(Y, \mathcal{A}) = Y \cup \bigcup \mathcal{A}[Y]$ $(= Y \cup \bigcup_{A \in \mathcal{A}[Y]} A)$.

When each $A \in \mathcal{A}$ is contained in some $B \in \mathcal{B}$, it is said that $\mathcal{A}$ **refines** $\mathcal{B}$, which is denoted by

$$\mathcal{A} \prec \mathcal{B} \ \text{ or } \ \mathcal{B} \succ \mathcal{A}.$$

It is said that $\mathcal{A}$ **covers** $Y$ (or $\mathcal{A}$ is a **cover** of $Y$ in $X$) if $Y \subset \bigcup \mathcal{A}$ $(= \bigcup_{A \in \mathcal{A}} A)$. When $\mathcal{A}$ is a cover of $Y$ in $X$, $\operatorname{st}(Y, \mathcal{A}) = \bigcup \mathcal{A}[Y]$. A cover of $X$ in $X$ is simply called a **cover** of $X$.

A cover of $Y$ in $X$ is said to be **open** (resp. **closed**) in $X$ depending on whether its members are open (resp. closed) in $X$. If $\mathcal{A}$ is an open cover of $X$, then $\mathcal{A}|Y$ is an open cover of $Y$ and $\mathcal{A}[Y]$ is an open cover of $Y$ in $X$. When $\mathcal{A}$ and $\mathcal{B}$ are open covers of $X$, $\mathcal{A} \wedge \mathcal{B}$ is also an open cover of $X$. For covers $\mathcal{A}$ and $\mathcal{B}$ of $X$, it is said that $\mathcal{A}$ is a **refinement** of $\mathcal{B}$ if $\mathcal{A} \prec \mathcal{B}$, where $\mathcal{A}$ is an **open** (resp. **closed**) **refinement** if $\mathcal{A}$ is an open (resp. closed) cover.

For a space $X$, we write

- $\operatorname{cov}(X)$ — the collection of all open covers of $X$.

[2] The (topological) dimension of $M$ is equal to $n$.

For $\mathcal{U}, \mathcal{V} \in \mathrm{cov}(X)$, we define

$$\mathrm{st}(\mathcal{U}, \mathcal{V}) = \{\mathrm{st}(U, \mathcal{V}) \mid U \in \mathcal{U}\}.$$

In the case of $\mathcal{V} = \mathcal{U}$, $\mathrm{st}(\mathcal{U}, \mathcal{U})$ is denoted by $\mathrm{st}\,\mathcal{U}$. When $\mathrm{st}\,\mathcal{U} \prec \mathcal{V}$, we call $\mathcal{U}$ a **star-refinement** of $\mathcal{V}$. We inductively define $\mathrm{st}^n \mathcal{U}$, $n \in \mathbb{N}$, as follows:

$$\mathrm{st}^n \mathcal{U} = \mathrm{st}(\mathrm{st}^{n-1} \mathcal{U}, \mathcal{U}),$$

where $\mathrm{st}^0 \mathcal{U} = \mathcal{U}$ (so $\mathrm{st}^1 \mathcal{U} = \mathrm{st}\,\mathcal{U}$).

Let $(X_\gamma)_{\gamma\in\Gamma}$ be a family of (topological) spaces and $X = \bigcup_{\gamma\in\Gamma} X_\gamma$. The **weak topology** on $X$ with respect to $(X_\gamma)_{\gamma\in\Gamma}$ is the topology defined as follows:

$$U \subset X \text{ is open in } X \Leftrightarrow \forall\gamma \in \Gamma, U \cap X_\gamma \text{ is open in } X_\gamma$$
$$\Big(A \subset X \text{ is closed in } X \Leftrightarrow \forall\gamma \in \Gamma, A \cap X_\gamma \text{ is closed in } X_\gamma\Big).$$ [3]

Suppose that $X$ has the weak topology with respect to $(X_\gamma)_{\gamma\in\Gamma}$ and the topologies of $X_\gamma$ and $X_{\gamma'}$ agree on $X_\gamma \cap X_{\gamma'}$ for any $\gamma, \gamma' \in \Gamma$. If $X_\gamma \cap X_{\gamma'}$ is closed (resp. open) in $X_\gamma$ for any $\gamma, \gamma' \in \Gamma$, then each $X_\gamma$ is closed (resp. open) in $X$ and the original topology of each $X_\gamma$ is a subspace topology inherited from $X$. In the case $X_\gamma \cap X_{\gamma'} = \emptyset$ for $\gamma \neq \gamma'$, $X$ is called the **topological sum** of $(X_\gamma)_{\gamma\in\Gamma}$ and denoted by $X = \bigoplus_{\gamma\in\Gamma} X_\gamma$. In the case $X_\gamma \cap X_{\gamma'} = \{x_0\}$ for $\gamma \neq \gamma'$, $X$ is called the **wedge sum** (or **wedge**) of $(X_\gamma)_{\gamma\in\Gamma}$ at $x_0$ and denoted by $X = \bigvee_{\gamma\in\Gamma} X_\gamma$.[4] When $\Lambda = \{1, \ldots, n\}$, we write as follows:

$$\bigoplus_{i=1}^{n} X_i = X_1 \oplus \cdots \oplus X_n \quad \text{and} \quad \bigvee_{i=1}^{n} X_i = X_1 \vee \cdots \vee X_n.$$

Let $f : A \to Y$ be a map from a closed set $A$ in a space $X$ to another space $Y$. The **adjunction space** $Y \cup_f X$ is the quotient space $(X \oplus Y)/\sim$, where $X \oplus Y$ is the topological sum and $\sim$ is the equivalence relation corresponding to the decomposition of $X \oplus Y$ into singletons $\{x\}$, $x \in X \setminus A$, and sets $\{y\} \cup f^{-1}(y)$, $y \in Y$ (the latter is a singleton $\{y\}$ if $y \in Y \setminus f(A)$). When $Y$ is a singleton, $Y \cup_f X \approx X/A$. One should note that the adjunction spaces are *not Hausdorff* in general. It is necessary to require some conditions for the adjunction space to be Hausdorff.

[3] I.e., the finest (or largest) topology such that each inclusion $X_\gamma \subset X$ is continuous. (The term "weak topology" is used with a different meaning by functional analysts, etc.)

[4] The wedge (sum) is used for a family of pointed spaces with the common base point.

Let $f : X \to Y$ be a map. For $A \subset X$ and $B \subset Y$, we write

$$f(A) = \{f(x) \mid x \in A\} \text{ and } f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

For collections $\mathcal{A}$ and $\mathcal{B}$ of subsets of $X$ and $Y$, respectively, we write

$$f(\mathcal{A}) = \{f(A) \mid A \in \mathcal{A}\} \text{ and } f^{-1}(\mathcal{B}) = \{f^{-1}(B) \mid B \in \mathcal{B}\}.$$

The restriction of $f$ to $A \subset X$ is denoted by $f|A$. It is said that a map $g : A \to Y$ **extends over** $X$ if there is a map $f : X \to Y$ such that $f|A = g$. Such a map $f$ is called an **extension** of $g$.

Let $[a, b]$ be a closed interval, where $a < b$ $(\in \mathbb{R})$. A map $f : [a, b] \to X$ is called a **path** (from $f(a)$ to $f(b)$) in $X$, where it is said that two points $f(a)$ and $f(b)$ are connected by the path $f$ in $X$. An embedding (i.e., an injective path) $f : [a, b] \to X$ is called an **arc** (from $f(a)$ to $f(b)$) in $X$, and the image $f([a, b])$ is also called an **arc**. Namely, a space is called an **arc** if it is homeomorphic to $\mathbf{I}$. A space $X$ is **path-connected** (or **arcwise connected**) if each pair of *distinct* points $x, y \in X$ are connected by a path (or an arc). It is said that $X$ is **locally path-connected** (or **locally arcwise connected**) if any neighborhood $U$ of each point $x \in X$ contains a neighborhood $V$ of $x$ such that each pair of distinct points in $V$ are connected by a path (or an arc) in $U$ (i.e., for each $y, z \in V$, there is a path (or an arc) $f : \mathbf{I} \to U$ such that $f(0) = y$ and $f(1) = z$). In this definition, as is easily observed, $V$ may be a path-connected (or an arcwise connected). The (local) arcwise connectedness looks to be stronger than the (local) path-connectedness, but they are the same concepts, that is:

**Proposition 1.1.1** *An arbitrary space $X$ is path-connected if and only if $X$ is arcwise connected. Moreover, $X$ is locally path-connected if and only if $X$ is locally arcwise connected.* $\boxed{5.14.7}$

For spaces $X$ and $Y$, we write

- $\mathrm{C}(X, Y)$ — the set of (continuous) maps from $X$ to $Y$.

Given subspaces $X_1, \dots, X_n \subset X$ and $Y_1, \dots, Y_n \subset Y$, a map $f : X \to Y$ is said to be a map from $(X, X_1, \dots, X_n)$ to $(Y, Y_1, \dots, Y_n)$ and is written

$$f : (X, X_1, \dots, X_n) \to (Y, Y_1, \dots, Y_n)$$

if $f(X_1) \subset Y_1, \dots, f(X_n) \subset Y_n$. We write

- $\mathrm{C}((X, X_1, \dots, X_n), (Y, Y_1, \dots, Y_n))$ — the set of maps from $(X, X_1, \dots, X_n)$ to $(Y, Y_1, \dots, Y_n)$;
- $\mathrm{C}((X, x_0), (Y, y_0)) = \mathrm{C}((X, \{x_0\}), (Y, \{y_0\}))$.

For maps $f, g : X \to Y$ (i.e., $f, g \in \mathrm{C}(X, Y)$),

- $f \simeq g$ means that $f$ and $g$ are homotopic (or $f$ is homotopic to $g$), that is, there is a map $h : X \times \mathbf{I} \to Y$ such that $h_0 = f$ and $h_1 = g$, where $h_t : X \to Y$, $t \in \mathbf{I}$, are defined by $h_t(x) = h(x, t)$, and $h$ is called a **homotopy** from $f$ to $g$ (between $f$ and $g$). When $g$ is a constant map, it is said that $f$ is **null-homotopic**, which is denoted by $f \simeq 0$. For a homotopy $h : X \times \mathbf{I} \to Y$, we call $h(\{x\} \times \mathbf{I})$, $x \in X$, the **tracks** of $h$, where each $h(\{x\} \times \mathbf{I})$ is the **track** of $x \in X$ by $h$.

For spaces $X$ and $Y$,

- $X \simeq Y$ means that $X$ and $Y$ are homotopy equivalent,

that is, there are maps $f : X \to Y$ and $g : Y \to X$ such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, where $f$ is called a **homotopy equivalence** and $g$ is a **homotopy inverse** of $f$. Then, we also say that $X$ and $Y$ have the **same homotopy type** or $X$ has the **homotopy type** of $Y$. For each $f, f' \in \mathrm{C}(X, Y)$ and $g, g' \in \mathrm{C}(Y, Z)$, we have the following:

$$f \simeq f',\ g \simeq g' \Rightarrow gf \simeq g'f'.$$

A homotopy $h$ between maps $f, g \in \mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))$ requires the condition that $h_t \in \mathrm{C}((X, X_1, \ldots, X_n), (Y, Y_1, \ldots, Y_n))$ for every $t \in \mathbf{I}$, that is, $h$ is regarded as the map

$$h : (X \times \mathbf{I}, X_1 \times \mathbf{I}, \ldots, X_n \times \mathbf{I}) \to (Y, Y_1, \ldots, Y_n).$$

When there are maps

$$f : (X, X_1, \ldots, X_n) \to (Y, Y_1, \ldots, Y_n),$$
$$g : (Y, Y_1, \ldots, Y_n) \to (X, X_1, \ldots, X_n)$$

such that $gf \simeq \mathrm{id}_X$ and $fg \simeq \mathrm{id}_Y$, we write

- $(X, X_1, \ldots, X_n) \simeq (Y, Y_1, \ldots, Y_n)$;
- $(X, x_0) \simeq (Y, y_0)$ means $(X, \{x_0\}) \simeq (Y, \{y_0\})$.

For $A \subset X$, a homotopy $h : X \times \mathbf{I} \to Y$ is called a **homotopy relative to** $A$ if $h(\{x\} \times \mathbf{I})$ is a singleton for every $x \in A$. When a homotopy from $f$ to $g$ is a homotopy relative to $A$ (where $f|A = g|A$), it is said that $f$ and $g$ are **homotopic relative to** $A$, which is written as follows:

$$f \simeq g \ \text{ rel. } A.$$

Let $f, g : X \to Y$ be maps and $\mathcal{U}$ a collection of subsets of $Y$ (as usual, $\mathcal{U} \in \mathrm{cov}(Y)$). It is said that $f$ and $g$ are **$\mathcal{U}$-close** (or $f$ is **$\mathcal{U}$-close** to $g$) if

$$\big\{\{f(x), g(x)\} \bigm| x \in X\big\} \prec \mathcal{U} \cup \big\{\{y\} \bigm| y \in Y\big\},$$

which implies that $\mathcal{U}$ covers the set $\{f(x), g(x) \mid f(x) \neq g(x)\}$. A homotopy $h$ is called a **$\mathcal{U}$-homotopy** if the collection of non-degenerate tracks of $h$ refines $\mathcal{U}$, that is,

$$\big\{h(\{x\} \times \mathbf{I}) \bigm| x \in X\big\} \prec \mathcal{U} \cup \big\{\{y\} \bigm| y \in Y\big\}.$$

In this case, $\mathcal{U}$ covers the set

$$\bigcup \big\{h(\{x\} \times \mathbf{I}) \bigm| h(\{x\} \times \mathbf{I}) \text{ is non-degenerate}\big\}.$$

When a homotopy from $f$ to $g$ is a $\mathcal{U}$-homotopy, it is said that $f$ and $g$ are **$\mathcal{U}$-homotopic** (or $f$ is **$\mathcal{U}$-homotopic** to $g$), which is written as follows:

$$f \simeq_{\mathcal{U}} g.$$

When $Y = (Y, d)$ is a metric space, we can define a metric $d$ called the **sup-metric** on the set $\mathrm{C}(X, Y)$ as follows:

$$d(f, g) = \sup_{x \in X} \min\{d(f(x), g(x)),\ 1\}.$$[5]

The metric space $(\mathrm{C}(X, Y), d)$ is denoted by $\mathrm{C}_d(X, Y)$. The topology of $\mathrm{C}_d(X, Y)$ is called the **uniform convergence topology**, where each $f \in \mathrm{C}_d(X, Y)$ has a neighborhood basis consisting of the following:

$$\mathrm{B}_d(f, \varepsilon) = \big\{g \in \mathrm{C}(X, Y) \bigm| d(f, g) < \varepsilon\big\},\ \varepsilon > 0.$$

For $\varepsilon > 0$, it is said that $f$ and $g$ are **$\varepsilon$-close** or $f$ is **$\varepsilon$-close** to $g$ if $d(f, g) < \varepsilon$. A homotopy $h$ is called an **$\varepsilon$-homotopy** if $\operatorname{mesh}\{h(\{x\} \times \mathbf{I}) \mid x \in X\} < \varepsilon$. When a homotopy from $f$ to $g$ is an $\varepsilon$-homotopy, it is said that $f$ and $g$ are **$\varepsilon$-homotopic**, which is written as follows:

$$f \simeq_{\varepsilon} g.$$

The **compact-open topology** on $\mathrm{C}(X, T)$ is generated by the sets

$$\langle K; U \rangle = \big\{f \in \mathrm{C}(X, Y) \bigm| f(K) \subset U\big\},$$

[5] In the case where $Y$ is bounded or $X$ is compact, we can employ the definition $d(f, g) = \sup_{x \in X} d(f(x), g(x))$. But, in general, the case $d(f, g) = \infty$ might occur for this definition.

where $K$ is any compact set in $X$ and $U$ is any open set in $Y$. With respect to the compact-open topology, we have the following:

**Proposition 1.1.2 (Properties of the Compact-Open Topology)**

(1) *Every map* $f : Z \times X \to Y$ *induces the map* $\tilde{f} : Z \to \mathrm{C}(X, Y)$ *defined by* $\tilde{f}(z)(x) = f(x, y)$. $\boxed{1.1.1}$

(2) *For each* $f \in \mathrm{C}(Z, X)$ *and* $g \in \mathrm{C}(Y, Z)$*, the following are continuous:*

$$f^* : \mathrm{C}(X, Y) \to \mathrm{C}(Z, Y),\ f^*(h) = h \circ f;$$

$$g_* : \mathrm{C}(X, Y) \to \mathrm{C}(X, Z),\ g_*(h) = g \circ h. \qquad \boxed{1.1.3(1)}$$

(3) *When* $Y$ *is locally compact, the following composition is continuous:*

$$\mathrm{C}(X, Y) \times \mathrm{C}(Y, Z) \ni (f, g) \mapsto g \circ f \in \mathrm{C}(X, Z). \qquad \boxed{1.1.3(2)}$$

(4) *When* $X$ *is locally compact, the following (evaluation) is continuous:*

$$\mathrm{ev} : \mathrm{C}(X, Y) \times X \to Y,\ \mathrm{ev}(f, x) = f(x).$$

*So, every map* $f : Z \to \mathrm{C}(X, Y)$ *induces the map* $\tilde{f} : Z \times X \to Y$ *defined by* $\tilde{f}(z, x) = f(z)(x)$. $\boxed{1.1.3(4)}$

(5) *When* $X$ *is locally compact, the following inequalities hold:*

$$w(Y) \leqslant w(\mathrm{C}(X, Y)) \leqslant \aleph_0 w(X) w(Y).$$

*In particular, if* $X$ *is separable locally compact and* $Y$ *has infinite, then* $w(\mathrm{C}(X, Y)) = w(Y)$. $\boxed{1.1.3(5)}$

(6) *When* $X$ *is compact and* $Y = (Y, d)$ *is a metric space, the sup-metric on* $\mathrm{C}(X, Y)$ *is admissible, that is, the compact-open topology is induced by the sup-metric.* $\boxed{1.1.3(6)}$

Regarding $\mathrm{C}(X, Y)$ as a subspace of the product space $Y^X$, we can introduce another topology on $\mathrm{C}(X, Y)$, which is called the **pointwise convergence topology**. The space $\mathrm{C}(X, Y)$ with the pointwise convergence topology is written as $\mathrm{C}_p(X, Y)$. For each $x \in X$, the **evaluation** $\mathrm{ev}_x : \mathrm{C}(X, Y) \to Y$ is defined by $\mathrm{ev}_x(f) = f(x)$, which is the restriction of the projection $\mathrm{pr}_x : Y^X \to X$. Thus, the pointwise convergence topology is the coarsest topology such that the evaluations $\mathrm{ev}_x$, $x \in X$, are continuous and it is generated by the sets

$$\langle x; U \rangle = \left\{ f \in \mathrm{C}(X, Y) \;\middle|\; f(x) \in U \right\},$$

where $x$ is any point of $X$ and $U$ is any open set in $Y$. Hence, every open set in $\mathrm{C}_p(X, Y)$ is open in $\mathrm{C}(X, Y)$, that is, the pointwise convergence topology is not finer than the compact-open topology.

*Remark 1.1* Proposition 1.1.2(4) does not hold for the space $C_p(X, Y)$ even if $X$ is compact and $Y = \mathbb{R}$. For example, let $X = \{0, 1/n \mid n \in \mathbb{N}\}$ and $c_0 : X \to \mathbb{R}$ be the constant map with $c_0(X) = \{0\}$. For any neighborhood $\mathcal{U}$ of $c_0$ in $\mathrm{C}_p(X, \mathbb{R})$ and any neighborhood $V$ of 0 in $X$, we can choose a finite set $F \subset X$, $0 < \varepsilon < 1$, and $x_0 \in V \setminus F$ so that $\bigcap_{x \in F} \langle x; (-\varepsilon, \varepsilon)\rangle \subset \mathcal{U}$. Let $f_0 : X \to \mathbb{R}$ be the map defined by $f_0(x_0) = 1$ and $f_0(x) = 0$ for any $x \in X \setminus \{x_0\}$. Then, $(f_0, x_0) \in \mathcal{U} \times V$ but $\mathrm{ev}(f_0, x_0) = f_0(x_0) = 1 \notin (-1, 1)$.

## 1.2 Banach Spaces in the Product of Real Lines

Throughout this section, let $\Gamma$ be an infinite set. Here, we review Banach spaces[6] being linear subspaces of the product $\mathbb{R}^\Gamma$. We write

- $\mathrm{Fin}(\Gamma)$ — the set of all non-empty finite subsets of $\Gamma$.

Then, note that $\operatorname{card} \mathrm{Fin}(\Gamma) = \operatorname{card} \Gamma$. The product space $\mathbb{R}^\Gamma$ is a linear space with the following scalar multiplication and addition:

$$\mathbb{R}^\Gamma \times \mathbb{R} \ni (x, t) \mapsto tx = (tx(\gamma))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma;$$

$$\mathbb{R}^\Gamma \times \mathbb{R}^\Gamma \ni (x, y) \mapsto x + y = (x(\gamma) + y(\gamma))_{\gamma \in \Gamma} \in \mathbb{R}^\Gamma.$$

These operations are continuous with respect to the product topology of $\mathbb{R}^\Gamma$. Namely, $\mathbb{R}^\Gamma$ with the product topology is a topological linear space.[7] Note that $w(\mathbb{R}^\Gamma) = \aleph_0 \operatorname{card} \mathrm{Fin}(\Gamma) = \operatorname{card} \Gamma$.

For each $\gamma \in \Gamma$, we define the unit vector $\mathbf{e}_\gamma \in \mathbb{R}^\Gamma$ by $\mathbf{e}_\gamma(\gamma) = 1$ and $\mathbf{e}_\gamma(\gamma') = 0$ for $\gamma' \neq \gamma$. It should be noticed that $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is not a Hamel basis for $\mathbb{R}^\Gamma$ and its linear span[8] is the following:

$$\mathbb{R}^\Gamma_f = \left\{x \in \mathbb{R}^\Gamma \;\middle|\; x(\gamma) = 0 \text{ except for finitely many } \gamma \in \Gamma \right\},$$

which is a dense linear subspace of $\mathbb{R}^\Gamma$. The subspace $\mathbb{R}^\mathbb{N}_f$ of $\boldsymbol{s} = \mathbb{R}^\mathbb{N}$ is also denoted by $\boldsymbol{s}_f$, which is consisting of all finite sequences.

As is easily observed, the following are equivalent:

(a) $\mathbb{R}^\Gamma$ is metrizable;
(b) $\mathbb{R}^\Gamma_f$ is metrizable;
(c) $\mathbb{R}^\Gamma_f$ is first countable;
(d) $\operatorname{card} \Gamma \leqslant \aleph_0$.

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[6]A **Banach space** is a complete normed linear space.

[7]Refer to p. 23.

[8]The **linear span** of $B$ is the linear subspace generated by a set $B$.

Thus, when $\Gamma$ is uncountable, every linear subspace $L$ of $\mathbb{R}^\Gamma$ with $\mathbb{R}^\Gamma_f \subset L$ is non-metrizable. Moreover, $\mathbb{R}^\Gamma$ (or $\mathbb{R}^\Gamma_f$) is metrizable only when $\Gamma$ is countable. In the case $\operatorname{card}\Gamma = \aleph_0$, $\mathbb{R}^\Gamma$ is linearly homeomorphic to the space of sequences $\boldsymbol{s} = \mathbb{R}^\mathbb{N}$, that is, there exists a linear homeomorphism between $\mathbb{R}^\Gamma$ and $\boldsymbol{s}$. On the other hand, we have the following proposition:

**Proposition 1.2.1** *Let $\Gamma$ be an infinite set. Then, any norm on $\mathbb{R}^\Gamma_f$ does not induce the topology inherited from the product topology of $\mathbb{R}^\Gamma$. Consequently, every linear subspace $L$ of $\mathbb{R}^\Gamma$ with $\mathbb{R}^\Gamma_f \subset L$ is not normable.* $\boxed{1.2.1}$

We can consider various norms defined on linear subspaces of $\mathbb{R}^\Gamma$, which are not compatible with the product topology as in Proposition 1.2.1 above. In general, the unit closed ball and the unit sphere of a normed linear space $X = (X, \|\cdot\|)$ are denoted by $\mathbf{B}_X$ and $\mathbf{S}_X$ respectively, that is,

$$\mathbf{B}_X = \{x \in X \mid \|x\| \leqslant 1\} \text{ and } \mathbf{S}_X = \{x \in X \mid \|x\| = 1\}.$$

The zero vector (the zero element) of $X$ is denoted by $\mathbf{0}_X$, or simply by $\mathbf{0}$ if there is no possible confusion.

The Banach space $\ell^\infty(\Gamma)$ and its closed linear subspaces $\boldsymbol{c}(\Gamma) \supset \boldsymbol{c}_0(\Gamma)$ are defined as follows:

- $\ell^\infty(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \sup_{\gamma\in\Gamma} |x(\gamma)| < \infty\}$ with the sup-norm
$$\|x\|_\infty = \sup_{\gamma\in\Gamma} |x(\gamma)|;$$ [9]
- $\boldsymbol{c}(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \exists t \in \mathbb{R} \text{ such that } \forall \varepsilon > 0,\ |x(\gamma) - t| < \varepsilon \text{ except for finitely many } \gamma \in \Gamma\}$;
- $\boldsymbol{c}_0(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \forall \varepsilon > 0,\ |x(\gamma)| < \varepsilon \text{ except for finitely many } \gamma \in \Gamma\}$.

These are linear subspaces of $\mathbb{R}^\Gamma$ but not topological ones as seen above. The space $\boldsymbol{c}(\Gamma)$ is linearly homeomorphic to $\boldsymbol{c}_0(\Gamma) \times \mathbb{R}$ by the following correspondence:

$$\boldsymbol{c}_0(\Gamma) \times \mathbb{R} \ni (x, t) \mapsto (x(\gamma) + t)_{\gamma\in\Gamma} \in \boldsymbol{c}(\Gamma).$$

This correspondence and its inverse are Lipschitz with respect to the norm $\|(x, t)\| = \max\{\|x\|_\infty, |t|\}$.

Furthermore, $\ell^\infty_f(\Gamma)$ denotes $\mathbb{R}^\Gamma_f$ with this norm. Then,

$$\ell^\infty_f(\Gamma) \subset \boldsymbol{c}_0(\Gamma) \subset \boldsymbol{c}(\Gamma) \subset \ell^\infty(\Gamma).$$

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[9] In some literature, this space is denoted by $\boldsymbol{m}(\Gamma)$.

For the weight of these spaces, we have the following:

$$w(\ell^\infty(\Gamma)) = 2^{\operatorname{card}\Gamma} \text{ but } w(\boldsymbol{c}(\Gamma)) = w(\boldsymbol{c}_0(\Gamma)) = w(\ell^\infty_f(\Gamma)) = \operatorname{card}\Gamma$$

(cf. Proposition 1.2.2 in [GAGT]). The topology of $\ell^\infty_f(\Gamma)$ is different from the topology inherited from the product topology. Indeed, $\{\mathbf{e}_\gamma \mid \gamma \in \Gamma\}$ is discrete in $\ell^\infty_f(\Gamma)$ but $\mathbf{0}$ is a cluster point of this set with respect to the product topology.

In the case $\Gamma = \mathbb{N}$, we write:

- $\ell^\infty(\mathbb{N}) = \ell^\infty$ — the space of bounded sequences;[10]
- $\boldsymbol{c}(\mathbb{N}) = \boldsymbol{c}$ — the space of convergent sequences;
- $\boldsymbol{c}_0(\mathbb{N}) = \boldsymbol{c}_0$ — the space of null-sequences (= sequence tending to 0).

We also write $\ell^\infty_f(\mathbb{N}) = \ell^\infty_f$, where $\ell^\infty_f = \boldsymbol{s}_f$ as sets (linear spaces) but they have different topologies. It should be noted that $\boldsymbol{c}$ and $\boldsymbol{c}_0$ are separable but $\ell^\infty$ is non-separable. When $\operatorname{card}\Gamma = \aleph_0$, the spaces $\ell^\infty(\Gamma)$, $\boldsymbol{c}(\Gamma)$ and $\boldsymbol{c}_0(\Gamma)$ are linearly isometric to these spaces $\ell^\infty$, $\boldsymbol{c}$ and $\boldsymbol{c}_0$, respectively.

Here, we regard $\operatorname{Fin}(\Gamma)$ as a directed set by $\subset$. For $x \in \mathbb{R}^\Gamma$, we say that $\sum_{\gamma\in\Gamma} x(\gamma)$ is **convergent** if $\left(\sum_{\gamma\in F} x(\gamma)\right)_{F\in\operatorname{Fin}(\Gamma)}$ is convergent and define

$$\sum_{\gamma\in\Gamma} x(\gamma) = \lim_{F\in\operatorname{Fin}(\Gamma)} \sum_{\gamma\in F} x(\gamma).$$

When $x(\gamma) \geqslant 0$ for all $\gamma \in \Gamma$, in order that $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent, it is necessary and sufficient that $\left(\sum_{\gamma\in F} x(\gamma)\right)_{F\in\operatorname{Fin}(\Gamma)}$ is upper bounded, and then

$$\sum_{\gamma\in\Gamma} x(\gamma) = \sup_{F\in\operatorname{Fin}(\Gamma)} \sum_{\gamma\in F} x(\gamma).$$

Thus, by $\sum_{\gamma\in\Gamma} x(\gamma) < \infty$, we mean that $\sum_{\gamma\in\Gamma} x(\gamma)$ is convergent.

For $x \in \mathbb{R}^\mathbb{N}$, $\sum_{i\in\mathbb{N}} x(i)$ should be distinguished from $\sum_{i=1}^\infty x(i)$. When the sequence $\left(\sum_{i=1}^n x(i)\right)_{n\in\mathbb{N}}$ is convergent, we say that $\sum_{i=1}^\infty x(i)$ is **convergent** and define

$$\sum_{i=1}^\infty x(i) = \lim_{n\to\infty} \sum_{i=1}^n x(i).$$

Evidently, if $\sum_{i\in\mathbb{N}} x(i)$ is convergent then $\sum_{i=1}^\infty x(i)$ is also convergent and $\sum_{i=1}^\infty x(i) = \sum_{i\in\mathbb{N}} x(i)$. However, $\sum_{i\in\mathbb{N}} x(i)$ is not convergent even if $\sum_{i=1}^\infty x(i)$ is convergent. In fact, due to Proposition 1.2.3 in [GAGT], the following equiva-

[10] In some literature, this space is denoted by $\boldsymbol{m}$.

lence holds:

$$\sum_{i\in\mathbb{N}} x(i) \text{ is convergent } \Leftrightarrow \sum_{i=1}^{\infty} |x(i)| \text{ is convergent.}$$

For each $p \geqslant 1$, the Banach space $\ell^p(\Gamma)$ is defined as follows:

- $\ell^p(\Gamma) = \left\{x \in \mathbb{R}^\Gamma \;\middle|\; \sum_{\gamma\in\Gamma} |x(\gamma)|^p < \infty\right\}$ with the norm

$$\|x\|_p = \left(\sum_{\gamma\in\Gamma} |x(\gamma)|^p\right)^{1/p}.$$

Like $\ell_f^\infty(\Gamma)$, the space $\mathbb{R}_f^\Gamma$ with this norm is denoted by $\ell_f^p(\Gamma)$.[11]

Similarly to $\boldsymbol{c}_0(\Gamma)$, we have $w(\ell^p(\Gamma)) = \operatorname{card}\Gamma$. When $\operatorname{card}\Gamma = \aleph_0$, the Banach space $\ell^p(\Gamma)$ is linearly isometric to $\ell^p = \ell^p(\mathbb{N})$, which is separable. The space $\ell^2(\Gamma)$ is the **Hilbert space** with the inner product

$$\langle x, y\rangle = \sum_{\gamma\in\Gamma} x(\gamma)y(\gamma),$$

which is well-defined because

$$\sum_{\gamma\in\Gamma} |x(\gamma)y(\gamma)| \leqslant \frac{\|x\|_2^2 + \|y\|_2^2}{2} < \infty.$$

For $1 \leqslant p < q < \infty$, we have

$$\ell^p(\Gamma) \subsetneqq \ell^q(\Gamma) \subsetneqq \boldsymbol{c}_0(\Gamma) \quad \text{as sets (or linear spaces).}$$

These inclusions are continuous because $\|x\|_\infty \leqslant \|x\|_q \leqslant \|x\|_p$ for every $x \in \ell^p(\Gamma)$. When $\Gamma$ is infinite, the topology of $\ell^p(\Gamma)$ is distinct from the one induced by the norm $\|\cdot\|_q$ or $\|\cdot\|_\infty$ (i.e., the topology inherited from $\ell^q(\Gamma)$ or $\boldsymbol{c}_0(\Gamma)$). In fact, the unit sphere $\mathbf{S}_{\ell^p(\Gamma)}$ is closed in $\ell^p(\Gamma)$ but not closed in $\ell^q(\Gamma)$ for any $q > p$ nor in $\boldsymbol{c}_0(\Gamma)$. — Refer to [GAGT, p. 17].

For $1 \leqslant p \leqslant \infty$, we have

$$\mathbb{R}_f^\Gamma \subset \ell^p(\Gamma) \quad \text{as sets (or linear spaces).}$$

Let $\ell_f^p(\Gamma)$ denote the subspace of $\ell^p(\Gamma)$ with $\ell_f^p(\Gamma) = \mathbb{R}_f^\Gamma$ as sets.

[11] The triangle inequality for $\|x\|_p$ is known as the Minkowski inequality. The proof can be found on pp. 16–17 in [GAGT].

When $\Gamma = \mathbb{N}$, we write $\ell^p_f(\mathbb{N}) = \ell^p_f$. By Proposition 1.2.1, we know $\ell^p_f(\Gamma) \neq \mathbb{R}^\Gamma_f$ as spaces for any infinite set $\Gamma$. In the above, the sequence $(x_n)_{n\in\mathbb{N}}$ is contained in the unit sphere $\mathbf{S}_{\ell^p_f(\Gamma)}$ of $\ell^p_f(\Gamma)$, which means that $\mathbf{S}_{\ell^p_f(\Gamma)}$ is not closed in $\ell^q_f$, hence $\ell^p_f \neq \ell^q_f$ as spaces for $1 \leqslant p < q \leqslant \infty$. Note that $\mathbf{S}_{\ell^p_f(\Gamma)}$ is a closed subset of $\ell^q_f$ for $1 \leqslant q < p$.

Concerning the convergence of sequences in $\ell^p(\Gamma)$, we have the following:

**Proposition 1.2.2** *For each $p \in \mathbb{N}$ and $x \in \ell^p(\Gamma)$, a sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x$ in $\ell^p(\Gamma)$ if and only if*

$$\|x\|_p = \lim_{n\to\infty} \|x_n\|_p \ \textit{ and } \ \forall \gamma \in \Gamma, x(\gamma) = \lim_{n\to\infty} x_n(\gamma). \qquad \boxed{1.2.4}$$

*Remark 1.2* It should be noted that Proposition 1.2.2 is valid not only for sequences but also nets, which means that the unit spheres $\mathbf{S}_{\ell^p(\Gamma)}$, $p \in \mathbb{N}$, are subspaces of the product space $\mathbb{R}^\Gamma$, whereas neither $\mathbb{R}^\Gamma$ nor $\mathbb{R}^\Gamma_f$ is metrizable if $\Gamma$ is uncountable. Therefore, if $1 \leqslant p < q \leqslant \infty$, then $\mathbf{S}_{\ell^p(\Gamma)}$ is also a subspace of $\ell^q(\Gamma)$, while, as mentioned above, $\mathbf{S}_{\ell^p(\Gamma)}$ of $\ell^p(\Gamma)$ is not closed in the space $\ell^q(\Gamma)$. The unit sphere $\mathbf{S}_{\ell^p_f(\Gamma)}$ of $\ell^p_f(\Gamma)$ is a subspace of $\mathbb{R}^\Gamma_f$ ($\subset \mathbb{R}^\Gamma$) and also a subspace of $\ell^q(\Gamma)$ for $1 \leqslant q \leqslant \infty$.

*Remark 1.3* The "if" part of Proposition 1.2.2 does not hold for the space $\boldsymbol{c}_0(\Gamma)$ for any infinite set $\Gamma$ (but the "only if" part obviously does hold). — Refer to Remark 3 on p. 17 of [GAGT].

Concerning the topological classification of $\ell^p(\Gamma)$, we have the following theorem due to S. Mazur [104]:

**Theorem 1.2.3 (Mazur)** *For each $1 < p < \infty$, $\ell^p(\Gamma)$ is homeomorphic to $\ell^1(\Gamma)$. By the same homeomorphism, $\ell^p_f(\Gamma) \approx \ell^1_f(\Gamma)$, that is, the pair $(\ell^p(\Gamma), \ell^p_f(\Gamma))$ is homeomorphic to the pair $(\ell^1(\Gamma), \ell^1_f(\Gamma))$.* $\boxed{1.2.5}$

For each space $X$, we simply write $\mathrm{C}_d(X, \mathbb{R}) = \mathrm{C}_u(X)$, where the metric $d$ of $\mathbb{R}$ is the usual metric induced by the absolute value $|\cdot|$, that is, $\mathrm{C}_u(X)$ is the metric space with the sup-metric

$$d(f, g) = \sup_{x\in X} \min\big\{|f(x) - g(x)|,\ 1\big\}.$$[12]

This metric is not induced by a norm. The topology of $\mathrm{C}_u(X)$ is the uniform convergence topology. It should be noted that $\mathrm{C}_u(X)$ is a linear space but it is not a topological linear space in general. In fact, the scalar multiplication $\mathbb{R} \times \mathrm{C}_u((0, 1]) \to \mathrm{C}_u((0, 1])$ is not continuous with respect to the uniformly convergence topology.

[12] See Footnote 5 (p. 8).

Let $f \in \mathrm{C}_u((0,1])$ be defined by $f(x) = x^{-1}$ for each $x \in (0,1]$. Then, for any $t > 0$, if $x < t$, then $|tf(x) - 0f(x)| > 1$, that is, $d(tf, 0f) = 1$.

Among subspaces $\mathrm{C}_u(X)$, we have the following Banach space:

- $\mathrm{C}^B(X) = \{f \in \mathrm{C}(X) \mid \sup_{x\in X}|f(x)| < \infty\}$ with the sup-norm

$$\|f\| = \sup_{x\in X}|f(x)|.$$

As is easily observed, $\mathrm{C}^B(X)$ is clopen in $\mathrm{C}(X)$. Moreover, $\mathrm{C}^B(X)$ is a component of the space $\mathrm{C}_u(X)$ because $\mathrm{C}^B(X)$ is path-connected as a normed linear space. When $X$ is compact, we have $\mathrm{C}^B(X) = \mathrm{C}_u(X)$. If $X$ is discrete and infinite, then $\mathrm{C}^B(X) = \ell^\infty(X)$, so $\mathrm{C}^B(\mathbb{N}) = \ell^\infty$ in particular.

The space $\mathrm{C}_p(X) = \mathrm{C}_p(X, \mathbb{R})$ is a topological linear space as a subspace of the product space $\mathbb{R}^X$. The topology of $\mathrm{C}_p(X)$ is the pointwise convergence topology. The space $\mathrm{C}_p(\mathbb{N})$ is none other than the space of sequences $\boldsymbol{s} = \mathbb{R}^{\mathbb{N}}$.

## 1.3 Topological Spaces

The following TIETZE–URYSOHN EXTENSION THEOREM is very useful and applied in various fields:

**Theorem 1.3.1 (TIETZE–URYSOHN)** *Let $A$ be a closed set in a normal space $X$. Then, every map $f : A \to \mathbf{I}$ extends over $X$.* $\boxed{2.2.2}$

Let $\mathcal{A}$ be a collection of subsets of a space $X$. It is said that $\mathcal{A}$ is **locally finite** (resp. **discrete**) in $X$ if each point has a neighborhood $U$ in $X$ which meets only finitely many members (resp. at most one member) of $\mathcal{A}$, that is, $\operatorname{card}\mathcal{A}[U] < \aleph_0$ (resp. $\operatorname{card}\mathcal{A}[U] \leqslant 1$). If $\mathcal{A}$ is locally finite (resp. discrete) in $X$, then so is $\mathcal{A}^{\mathrm{cl}}$ $(= \{\operatorname{cl} A \mid A \in \mathcal{A}\})$. Moreover, we say that $\mathcal{A}$ is **$\sigma$-locally finite** (resp. **$\sigma$-discrete**) in $X$ if $\mathcal{A}$ is a countable union of locally finite (resp. discrete) subcollections.

**Theorem 1.3.2 (A.H. STONE)** *Every open cover of a metrizable space has a locally finite and $\sigma$-discrete open refinement.* $\boxed{2.3.1}$

**Theorem 1.3.3 (BING; NAGATA–SMIRNOV)** *For a regular space $X$, the following conditions are equivalent:*

(a) *$X$ is metrizable;*
(b) *$X$ has a $\sigma$-discrete open basis;*
(c) *$X$ has a $\sigma$-locally finite open basis.* $\boxed{2.3.4}$

The equivalence of (a) and (b) in the above theorem is called the BING METRIZATION THEOREM and the equivalence of (a) and (c) is called the NAGATA–SMIRNOV METRIZATION THEOREM. Separable metrizable spaces are characterized as follows: