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Extremum Problems for Eigenvalues of Elliptic Operators

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Preface

Problems linking the shape of a domain or the coefficients of an elliptic operator to the sequence of its eigenvalues are among the most fascinating of mathematical analysis. One of the reasons which make them so attractive is that they involve different fields of mathematics: spectral theory, partial differential equations, geometry, calculus of variations .... Moreover, they are very simple to state and generally hard to solve! In particular, one can find in the next pages more than 30 open problems!

In this book, we focus on extremal problems. For instance, we look for a domain which minimizes or maximizes a given eigenvalue of the Laplace operator with various boundary conditions and various geometric constraints. We also consider the case of functions of eigenvalues. We investigate similar questions for other elliptic operators, like Schrödinger, non-homogeneous membranes or composites.

The targeted audience is mainly pure and applied mathematicians, more particularly interested in partial differential equations, calculus of variations, differential geometry, spectral theory. More generally, people interested in properties of eigenvalues in other fields such as acoustics, theoretical physics, quantum mechanics, solid mechanics, could find here some answers to natural questions. For that purpose, I choose to recall basic facts and tools in the two first chapters (with only a few proofs). In chapters 3, 4 and 5, we present known results and open questions for the minimization problem of a given eigenvalue $\lambda_k(\Omega)$ of the Laplace operator with Dirichlet boundary conditions, where the unknown is here the domain Ω itself. In chapter 6, we investigate various functions of the Dirichlet eigenvalues, while chapter 7 is devoted to eigenvalues of the Laplace operator with other boundary conditions. In chapter 8, we consider the eigenvalues of Schrödinger operators: therefore, the unknown is no longer the shape of the domain but the potential V. Chapter 9 is devoted to non-homogeneous membranes and chapter 10 to more general elliptic operators in divergence form. At last, in chapter 11, we are interested in the bi-Laplace operator.

Of course no book can completely cover such a huge field of research. In making personal choices for inclusion of material, I tried to give useful complementary references, in the process certainly neglecting some relevant works. I would be grateful to hear from readers about important missing citations.

I would like to thank Benoit Perthame who suggested in September 2004 that I write this book. Many people helped me with the enterprise, answering my questions and queries or suggesting interesting problems: Mark Ashbaugh, Friedemann Brock, Dorin Bucur, Giuseppe Buttazzo, Steve Cox, Pedro Freitas, Antonio Greco, Evans Harrell, Francois Murat, Edouard Oudet, Gerard Philippin, Michel Pierre, Marius Tucsnak. I am pleased to thank them here.

Nancy, March 2006 *Antoine Henrot*

Chapter 1

Eigenvalues of elliptic operators

1.1 Notation and prerequisites

In this section, we recall the basic results of the theory of elliptic partial differential equations. The prototype of elliptic operator is the Laplacian, but the results that we state here are also valid for more general (linear) elliptic operators. For the basic facts we recall here, we refer to any textbook on partial differential equations and operator theory. For example, [36], [58], [75], [83] are good standard references.

1.1.1 Notation and Sobolev spaces

Let Ω be a bounded open set in $\mathbb{R}^N$. We denote by $L^2(\Omega)$ the Hilbert space of square summable functions defined on Ω and by $H^1(\Omega)$ the Sobolev space of functions in $L^2(\Omega)$ whose partial derivatives (in the sense of distributions) are in $L^2(\Omega)$:

$$H^1(\Omega) := \{u \in L^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} \in L^2(\Omega),\ i = 1, 2, \ldots, N\}.$$

This is a Hilbert space when it is endowed with the scalar product

$$(u, v)_{H^1} := \int_\Omega u(x)v(x)\, dx + \int_\Omega \nabla u(x).\nabla v(x)\, dx$$

and the corresponding norm:

$$\|u\|_{H^1} := \left(\int_\Omega u(x)^2\, dx + \int_\Omega |\nabla u(x)|^2\, dx\right)^{1/2}.$$

In the case of Dirichlet boundary conditions, we will use the subspace $H_0^1(\Omega)$ which is defined as the closure of C^∞ functions compactly supported in Ω (functions in $C_0^\infty(\Omega)$) for the norm $\| \ \|_{H^1}$. It is also a Hilbert space. At last, $H^{-1}(\Omega)$ denotes the dual space of $H_0^1(\Omega)$. For some non-linear problems, for example when we are interested in the p-Laplace operator, it is more convenient to work with the spaces $L^p, p \geq 1$ instead of L^2. In this case, the Sobolev spaces, defined exactly in the same way, are denoted by $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ respectively. These are Banach spaces.

When Ω is bounded (or bounded in one direction), we have the Poincaré inequality:

$$\exists C = C(\Omega) \text{ such that } \forall u \in H_0^1(\Omega), \ \int_\Omega u(x)^2\, dx \leq C \int_\Omega |\nabla u(x)|^2\, dx\,. \tag{1.1}$$

Actually the constant C which appears in (1.1) is closely related to the eigenvalues of the Laplacian since we will see later (cf (1.36)) that the best possible constant C is nothing other than $1/\lambda_1(\Omega)$ where $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian with Dirichlet boundary conditions.

By definition, $H_0^1(\Omega)$ and $H^1(\Omega)$ are continuously embedded in $L^2(\Omega)$, but we will need later a compact embedding. This is the purpose of the following theorem.

Theorem 1.1.1 (Rellich).

- *For any bounded open set Ω, the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*
- *If Ω is a bounded open set with Lipschitz boundary, the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact.*

Remark 1.1.2. We can weaken the assumption of Lipschitz boundary but not too much, see e.g. the book [148] for more details.

1.1.2 Partial differential equations

Elliptic operator

Let $a_{ij}(x)$, $i, j = 1, \dots, N$ be bounded functions defined on Ω and satisfying the usual ellipticity assumption:

$$\begin{array}{c} \exists \alpha > 0, \text{ such that } \forall \xi = (\xi_1, \xi_2, \dots, \xi_N) \in \mathbb{R}^N, \ \forall x \in \Omega \\ \sum_{i,j=1}^N a_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2 \end{array} \tag{1.2}$$

where $|\xi| = \left(\xi_1^2 + \xi_2^2 + \cdots + \xi_N^2\right)^{1/2}$ denotes the euclidean norm of the vector ξ. We will also assume a symmetry assumption for the a_{ij} namely:

$$\forall x \in \Omega, \forall i, j \quad a_{ij}(x) = a_{ji}(x)\ . \tag{1.3}$$

Let $a_0(x)$ be a bounded function defined on Ω. We introduce the linear elliptic operator L, defined on $H^1(\Omega)$ by:

$$Lu := -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0(x)u \tag{1.4}$$

(derivatives are to be understood in the sense of distributions). The prototype of elliptic operator is the Laplacian:

$$-\Delta u := -\sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \tag{1.5}$$

which will be considered in the main part of this book (chapters 3 to 7). In chapter 8, we consider the Schrödinger operator $L_V u = -\Delta u + V(x)u$ where V (the potential) is a bounded function, while chapters 9 and 10 deal with more general elliptic operators. In that case, we will keep the notation L when we want to consider general operators given by (1.4). At last, in chapter 11, we consider operators of fourth order.

Remark 1.1.3. Let us remark that, since we are only interested in eigenvalue problems, we do not put any sign condition on the function $a_0(x)$ which appears in (1.4). Indeed, since $a_0(x)$ is bounded, we can always replace the operator L by $L + (\|a_0\|_\infty + 1)Id$, i.e. replace the function $a_0(x)$ by $a_0(x) + \|a_0\|_\infty + 1$ if we need a positive function in the term of order 0 of the operator L. For the eigenvalues, that would just induce a translation of $\|a_0\|_\infty + 1$ to the right.

Dirichlet boundary condition

Let f be a function in $L^2(\Omega)$. When we call u a solution of the Dirichlet problem

$$\begin{array}{rcll} Lu &=& f & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \tag{1.6}$$

we actually mean that u is the unique solution of the variational problem

$$\left\{ \begin{array}{c} u \in H_0^1(\Omega) \text{ and } \forall v \in H_0^1(\Omega), \\ \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}\,dx + \int_\Omega a_0(x)u(x)v(x)\,dx = \int_\Omega f(x)v(x)\,dx\ . \end{array} \right. \tag{1.7}$$

Existence and uniqueness of a solution for problem (1.7) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the Poincaré inequality (1.1). Note that, according to Remark 1.1.3, we can restrict ourselves to the case $a_0(x) \geq 0$. In the sequel, we will denote by A_L^D (or $A_L^D(\Omega)$ when we want to emphasize the dependence on the domain Ω) the linear operator defined by:

$$\begin{array}{rcl} A_L^D : L^2(\Omega) & \to & H_0^1(\Omega) \subset L^2(\Omega), \\ f & \mapsto & u \text{ solution of (1.7).} \end{array} \tag{1.8}$$

Neumann boundary condition

In the same way, if f is a function in $L^2(\Omega)$, we will also consider u a solution of the Neumann problem

$$\begin{array}{rcl} Lu & = & f \quad \text{in } \Omega, \\ \sum_{i,j=1}^N a_{ij} \frac{\partial u}{\partial x_j} n_i & = & 0 \quad \text{on } \partial\Omega \end{array} \tag{1.9}$$

(where n stands for the exterior unit normal vector to $\partial\Omega$ and n_i is its ith coordinate). For example, when $L = -\Delta$, the boundary condition reads (formally)

$$\frac{\partial u}{\partial n} = 0 \, .$$

It means that u is the unique solution in $H^1(\Omega)$ of the variational problem

$$\left\{ \begin{array}{c} u \in H^1(\Omega) \text{ and } \forall v \in H^1(\Omega), \\ \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} \, dx + \int_\Omega a_0(x) u(x) v(x) \, dx = \int_\Omega f(x) v(x) \, dx \, . \end{array} \right. \tag{1.10}$$

Existence and uniqueness of a solution for problem (1.10) follows from the Lax-Milgram Theorem, the ellipticity assumption (1.2) and the fact that we can assume that $a_0(x) \geq 1$ (according to Remark 1.1.3). In the sequel, we will denote by A_L^N the linear operator defined by:

$$\begin{array}{rcl} A_L^N : L^2(\Omega) & \rightarrow & H^1(\Omega) \subset L^2(\Omega), \\ f & \mapsto & u \text{ solution of (1.10).} \end{array} \tag{1.11}$$

Remark 1.1.4. We will also consider later, for example in chapter 7, other kinds of boundary conditions like Robin or Stekloff boundary conditions.

1.2 Eigenvalues and eigenfunctions

1.2.1 Abstract spectral theory

Let us now give the abstract theorem which provides the existence of a sequence of eigenvalues and eigenfunctions. Let H be a Hilbert space endowed with a scalar product $(.,.)$ and recall that an operator T is a linear continuous map from H into H. We say that:

- T is positive if, $\forall x \in H, \; (Tx, x) \geq 0$,
- T is self-adjoint, if $\forall x, y \in H, \; (Tx, y) = (x, Ty)$,
- T is compact, if the image of any bounded set is relatively compact (i.e. has a compact closure) in H.

Theorem 1.2.1. *Let H be a separable Hilbert space of infinite dimension and T a self-adjoint, compact and positive operator. Then, there exists a sequence of real positive eigenvalues (ν_n), $n \geq 1$ converging to 0 and a sequence of eigenvectors (x_n), $n \geq 1$ defining a Hilbert basis of H such that $\forall n,\ T\, x_n = \nu_n\, x_n$.*

Of course, this theorem can be seen as a generalization to Hilbert spaces of the classical result in finite dimension for symmetric or normal matrices (existence of real eigenvalues and of an orthonormal basis of eigenvectors).

1.2.2 Application to elliptic operators

Dirichlet boundary condition

We apply Theorem 1.2.1 to $H = L^2(\Omega)$ and the operator A_L^D defined in (1.8).

- A_L^D is positive: let $f \in L^2(\Omega)$ and $u = A_L^D\, f$ be the solution of (1.7). We get

  $$(f, A_L^D\, f) = \int_\omega f(x)u(x)\, dx = \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j}\, dx + \int_\Omega a_0(x) u^2(x)\, dx\ .$$

  Now, we recall that $a_0(x)$ can be taken as a positive function and then the ellipticity condition (1.2) yields the desired result. Moreover, we see that $(f, A_L^D\, f) > 0$ as soon as $f \neq 0$ (strict positivity).

- A_L^D is self-adjoint: let $f, g \in L^2(\Omega)$ and $u = A_L^D\, f$, $v = A_L^D\, g$. We have:

  $$(f, A_L^D\, g) = \int_\omega f(x)v(x)dx = \sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega a_0(x)u(x)v(x)dx. \tag{1.12}$$

  Now, according to the symmetry assumption (1.3) and the equation (1.7) satisfied by v, the right-hand side in (1.12) is equal to $\int_\Omega u(x)g(x)\, dx = (A_L^D\, f, g)$.

- A_L^D is compact: it is an immediate consequence of the Rellich Theorem 1.1.1.

As a consequence of Theorem 1.2.1, there exists (u_n) a Hilbert basis of $L^2(\Omega)$ and a sequence $\nu_n \geq 0$, converging to 0, such that $A_L^D\, u_n = \nu_n\, u_n$. Actually, the ν_n are positive, since the strict positivity of A_L^D yields $\nu_n \|u_n\|_{L^2} = (u_n, A_L^D\, u_n) > 0$.

Coming back to (1.7), we see that u_n satisfies, $\forall v \in H_0^1(\Omega)$:

$$\nu_n \left(\sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial u_n}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega a_0(x) u_n(x) v(x)\, dx \right) = \int_\Omega u_n(x)v(x)\, dx$$

which means

$$L\, u_n = \frac{1}{\nu_n}\, u_n\ .$$

Setting $\lambda_n = \frac{1}{\nu_n}$, we have proved:

Theorem 1.2.2. *Let Ω be a bounded open set in $\mathbb{R}^N$. There exists a sequence of positive eigenvalues (going to $+\infty$) and a sequence of corresponding eigenfunctions (defining a Hilbert basis of $L^2(\Omega)$) that we will denote respectively $0 < \lambda_1^D(L,\Omega) \leq \lambda_2^D(L,\Omega) \leq \lambda_3^D(L,\Omega) \leq \dots$ and $u_1^D, u_2^D, u_3^D, \cdots$ satisfying:*

$$\begin{cases} Lu_n^D = \lambda_n^D(L,\Omega)\, u_n^D & \text{in } \Omega, \\ \quad u_n^D = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.13}$$

Remark 1.2.3. For uniformly elliptic operators (those satisfying (1.2)), the compactness of the operator A_L^D follows simply from Rellich's Theorem 1.1.1. In some other cases, particularly for degenerate operators, one generally needs to use weighted Sobolev spaces. For example, if the operator is $Lu := -\text{div}(\sigma\nabla u)$ with $\sigma \geq 0$ allowed to vanish into Ω, one needs to introduce the space $H_\sigma^1(\Omega)$ (endowed with the norm $\|u\|^2 = \int_\Omega u^2\,dx + \int_\Omega \sigma|\nabla u|^2\,dx$). Then, it happens that the previous theory works as soon as the function $1/\sigma$ belongs to some $L^p(\Omega)$ space with $p > N/2$. We refer e.g. to [155] or [205]. At last, there is an interesting alternative to prove compactness. Thanks to the Green function, one can usually write the operator A_L^D (or A_L^N) with an integral representation. Then, it suffices to prove that it is a Hilbert-Schmidt operator. We will see an example of this strategy in section 10.2.3.

When $L = -\Delta$ is the Laplacian, we will simply denote the eigenvalues by $\lambda_n(\Omega)$ (or λ_n when no confusion is possible) and the corresponding eigenfunctions by u_n.

Since the eigenfunctions are defined up to a constant, we decide to normalize the eigenfunctions by the condition

$$\int_\Omega u_n(x)^2\,dx = 1\,. \tag{1.14}$$

Of course, it can occur that some eigenvalues are multiple (especially when the domain has symmetries). In this case, the eigenvalues are counted with their multiplicity.

Remark 1.2.4. When Ω is non-connected, for example if Ω has two connected components $\Omega = \Omega_1 \cup \Omega_2$, we obtain the eigenvalues of Ω by collecting and reordering the eigenvalues of each connected components

$$\begin{array}{lcl} \lambda_1^D(L,\Omega) & = & \min(\lambda_1^D(L,\Omega_1), \lambda_1^D(L,\Omega_2)), \\ \lambda_2^D(L,\Omega) & = & \min(\max(\lambda_1^D(L,\Omega_1), \lambda_1^D(L,\Omega_2)), \lambda_2^D(L,\Omega_1), \lambda_2^D(L,\Omega_2)), \\ & \vdots & \end{array} \tag{1.15}$$

More generally, we can always choose every eigenfunction of a disconnected open set Ω to vanish on all but one of the connected components of Ω. In particular, when the two connected components are the same, we will have $\lambda_1^D(L,\Omega) = \lambda_2^D(L,\Omega)$, i.e. λ_1 is a *double* eigenvalue.

That cannot happen when Ω is connected:

Theorem 1.2.5. *Let us assume that Ω is a regular connected open set. Then the first eigenvalue $\lambda_1^D(L,\Omega)$ is simple and the first eigenfunction u_1^D has a constant sign on Ω. Usually, we choose it to be positive on Ω.*

Actually, this theorem is a consequence of the Krein-Rutman Theorem which is an abstract result that we recall here (see [181] for a proof).

Theorem 1.2.6 (Krein-Rutman). *Let E be a Banach space and C be a closed convex cone in E with vertex at O, non-empty interior $Int(C)$ and satisfying $C \cap (-C) = \{O\}$. Let T be a compact operator in E which satisfies $T(C \setminus \{O\}) \subset Int(C)$; then the greatest eigenvalue of T is simple, and the corresponding eigenvector is in $Int(C)$ (or in $-Int(C)$).*

To prove Theorem 1.2.5, we apply the Krein-Rutman Theorem with $E = C^0(\overline{\Omega})$, $T = A_L^D$ and $C = \{v \in C^0(\overline{\Omega}),$ such that $v(x) \geq 0.\}$. Then, the assumption $T(C \setminus \{O\}) \subset Int(C)$ comes from the strong maximum principle. The fact that T can be defined as an operator from E to E and the fact that it is compact comes from classical regularity results (if the right-hand side of f is continuous, the solution u of (1.6) is also continuous and even Hölderian (De Giorgi-Stampacchia Theorem, see [36], [94]).

Remark 1.2.7. We will see two other proofs of the non-negativity of the first eigenfunction (and simplicity of the first eigenvalue) in section 1.3.3. In particular, no regularity assumptions are actually needed for this result.

Neumann boundary condition

In the same way, when Ω is a bounded Lipschitzian open set in $\mathbb{R}^N$, we can prove the following theorem (the Lipschitz regularity of Ω is necessary to have the compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$):

Theorem 1.2.8. *Let Ω be a bounded open Lipschitzian set in $\mathbb{R}^N$. There exists a sequence of non-negative eigenvalues (going to $+\infty$) and a sequence of corresponding eigenfunctions (defining a Hilbert basis of $L^2(\Omega)$) that we will denote respectively $0 \leq \mu_1^N(L,\Omega) \leq \mu_2^N(L,\Omega) \leq \mu_3^N(L,\Omega) \leq \cdots$ and $u_1^N, u_2^N, u_3^N, \ldots$ satisfying:*

$$\begin{cases} Lu_n^N = \mu_n^N(L,\Omega)\, u_n^N & \text{in } \Omega, \\ \frac{\partial u_n^N}{\partial n} = 0 & \text{on } \partial\Omega\,. \end{cases} \tag{1.16}$$

When $L = -\Delta$ is the Laplacian, we will simply denote the eigenvalues by $\mu_n(\Omega)$ (or μ_n when no confusion is possible) and the corresponding eigenfunctions by u_n.

We observe that, for the Laplacian and, more generally in the case $a_0(x) = 0$, the first eigenvalue is always $\mu_1 = 0$ and a corresponding eigenfunction is a non-zero constant (on a connected component of Ω).

In the case of Neumann boundary condition, we also decide to normalize the eigenfunctions by the condition

$$\int_\Omega u_n(x)^2\, dx = 1\ . \tag{1.17}$$

At last, when Ω is non-connected, we have the same property as described in Remark 1.2.4.

1.2.3 First Properties of eigenvalues

In this section, we only consider the eigenvalues of the Laplacian operator. It is well known that this operator is invariant for translations and rotations. More precisely, let us denote by τ_{x_0} the translation of vector x_0: $\tau_{x_0}(x) = x + x_0$. If v is a function defined on a set Ω, we define the function $\tau_{x_0}v$ on $\tau_{x_0}(\Omega)$ by the formula $\tau_{x_0}v(x) := v(x - x_0)$. Then, it is clear that

$$\tau_{x_0} \circ \Delta = \Delta \circ \tau_{x_0}$$

from which we can deduce

$$\lambda_n(\tau_{x_0}(\Omega)) = \lambda_n(\Omega)\ . \tag{1.18}$$

In the same way, denoting by R any isometry, we have

$$\lambda_n(R(\Omega)) = \lambda_n(\Omega)\ . \tag{1.19}$$

Let us also look at the effect of homothety. Let $k > 0$ and H_k be a homothety of origin O and ratio k: $H_k(x) = kx$. If v is a function defined on Ω, we define the function H_kv on $H_k(\Omega)$ by the formula $H_kv(x) := v(x/k)$. Since $H_k \circ \Delta = k^2 \Delta \circ H_k$, we deduce

$$\lambda_n(H_k(\Omega)) = \frac{\lambda_n(\Omega)}{k^2}\ . \tag{1.20}$$

An important consequence of (1.20) is the following. In the sequel, we will often consider minimization problems with a volume constraint, such as

$$\min\{\lambda_n(\Omega),\ |\Omega| = c\}\ . \tag{1.21}$$

Then , it could be convenient to replace Problem (1.21) by:

$$\min\ |\Omega|^{2/N}\, \lambda_n(\Omega)\ . \tag{1.22}$$

Proposition 1.2.9. *Problems* (1.21) *and* (1.22) *are equivalent.*

By equivalent, we just mean that there exists a bijective correspondence between solutions of these two problems. Actually, since the functional $\Omega \mapsto |\Omega|^{2/N}\, \lambda_n(\Omega)$ is invariant by homothety (thanks to (1.20)), the correspondence is simply:

- every solution of (1.21) is a solution of (1.22),
- if Ω is a solution of (1.22) with volume c', then $H_k(\Omega)$, with $k = (c/c')^{1/N}$, is a solution of (1.21).

Of course, the result of Proposition 1.2.9 is also true when we add most of the supplementary geometric constraints. For functions of eigenvalues, it will obviously depend on the homogeneity of the function.

1.2.4 Regularity of eigenfunctions

Interior regularity

Due to the hypo-ellipticity of the Laplacian, the eigenfunctions of the Laplacian are known to be analytic inside the domain, see e.g. [75], [58]. For more general operators, it depends on the regularity of the coefficients of the operator L. A good reference to handle such cases is [94].

Regularity up to the boundary

To have some regularity up to the boundary, we need to assume enough regularity of the domain. Standard results are the following, see [94] or [96]:

Theorem 1.2.10 (Sobolev regularity). *Let us assume that Ω is $C^{1,1}$ or convex and the coefficients a_{ij} are C^0 and $a_0 \in L^\infty$. Then each eigenfunction u of* (1.13) *belongs to the Sobolev space $H^2(\Omega)$.*

Remark 1.2.11. One can also get L^p regularity results. For example, using Theorem 9.15 in [94] together with a bootstrap argument, one can prove that the eigenfunctions are in $W^{2,p}(\Omega)$ with $p > N$. In particular, thanks to Sobolev embedding, the eigenfunctions are in $C^1(\overline{\Omega})$ as soon as the boundary of Ω is $C^{1,1}$.

Theorem 1.2.12 (Hölderian regularity). *Let us assume that Ω is $C^{2,\alpha}$ for some $\alpha > 0$ and the coefficients a_{ij} are $C^{1,\alpha}$ and a_0 in $C^{0,\alpha}$. Then each eigenfunction u of* (1.13) *belongs to $C^{2,\alpha}(\overline{\Omega})$.*

1.2.5 Some examples

In this section, we are interested in the eigenvalues of the Laplacian for some very simple domains. In one dimension, one can also choose explicit eigenvalues of some specific Sturm-Liouville operators. This leads to the huge theory of special functions; we refer e.g. to [3], [139] for examples of such functions.

Rectangles

In the 1-D case, i.e. for an interval like $\Omega = (0, L)$, it is very easy to solve *at hand* the differential equation

$$\begin{cases} -u'' = \lambda u, & x \in (0, L), \\ u(0) = u(L) = 0, \end{cases} \tag{1.23}$$

and the only non-trivial solutions are

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad u_n(x) = \sin\left(\frac{n\pi x}{L}\right), \; n \geq 1. \tag{1.24}$$

Now, for rectangles, using the classical trick of separation of variables, we prove

Proposition 1.2.13. *Let $\Omega = (0, L) \times (0, l)$ be a plane rectangle; then its eigenvalues and eigenfunctions for the Laplacian with Dirichlet boundary conditions are:*

$$\begin{array}{rl} \lambda_{m,n} = \pi^2 \left(\frac{m^2}{L^2} + \frac{n^2}{l^2}\right) & \\ u_{m,n}(x, y) = \frac{2}{\sqrt{Ll}} \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi y}{l}) & \end{array} \quad m, n \geq 1, \tag{1.25}$$

while its eigenvalues and eigenfunctions for the Laplacian with Neumann boundary conditions are:

$$\begin{array}{rl} \mu_{m,n} = \pi^2 \left(\frac{m^2}{L^2} + \frac{n^2}{l^2}\right) & \\ v_{m,n}(x, y) = \frac{2}{\sqrt{Ll}} \cos(\frac{m\pi x}{L}) \cos(\frac{n\pi y}{l}) & \end{array} \quad m, n \geq 0. \tag{1.26}$$

It is immediate to check that the pair $(\lambda_{m,n}, u_{m,n})$ given by (1.25) are eigenvalue and eigenfunction for the Laplacian with Dirichlet boundary condition. Of course, the difficulty is to prove that there are no other possibilities. Actually, it is due to the fact that the functions $\sin(\frac{m\pi x}{L}) \sin(\frac{n\pi y}{l})$ $m, n \geq 1$ form a complete orthogonal system in $L^2(\Omega)$, see [58].

Disks

Let us consider the disk B_R of radius R centered at O. In polar coordinates (r, θ), looking for an eigenfunction u of D of the kind $u(r, \theta) = v(r)w(\theta)$ leads us to solve the ordinary differential equations:

$$w''(\theta) + kw(\theta) = 0\,, \quad w \ 2\pi\text{-periodic},$$

$$v''(r) + \frac{1}{r}\, v'(r) + (\lambda - \frac{k}{r^2})v(r) = 0\,, \quad v'(0) = 0,\ v(R) = 0\,.$$

The periodicity condition for the first one implies that $k = n^2$ where n is an integer and $w(\theta) = a_n \cos n\theta + b_n \sin n\theta$. Then, replacing k by n^2 in the second equation allows us to recognize the classical Bessel differential equation. We can state

Proposition 1.2.14. *Let $\Omega = B_R$ be a disk of radius R; then its eigenvalues and eigenfunctions for the Laplacian with Dirichlet boundary conditions are:*

$$\begin{array}{l}
\lambda_{0,k} = \frac{j_{0,k}^2}{R^2},\ k \geq 1, \\
u_{0,k}(r,\theta) = \sqrt{\frac{1}{\pi}} \frac{1}{R|J_0'(j_{0,k})|} J_0(j_{0,k} r/R),\ k \geq 1, \\
\lambda_{n,k} = \frac{j_{n,k}^2}{R^2},\ n,k \geq 1,\ \textit{double eigenvalue} \\
u_{n,k}(r,\theta) = \begin{array}{l} \sqrt{\frac{2}{\pi}} \frac{1}{R|J_n'(j_{n,k})|} J_n(j_{n,k} r/R) \cos n\theta \\ \sqrt{\frac{2}{\pi}} \frac{1}{R|J_n'(j_{n,k})|} J_n(j_{n,k} r/R) \sin n\theta \end{array},\ n,k \geq 1,
\end{array} \tag{1.27}$$

where $j_{n,k}$ is the k-th zero of the Bessel function J_n.

For the Laplacian with Neumann boundary conditions, the eigenvalues and eigenfunctions of the disk B_R are:

$$\begin{array}{l}
\mu_{0,k} = \frac{{j'_{0,k}}^2}{R^2},\ k \geq 1, \\
v_{0,k}(r,\theta) = \sqrt{\frac{1}{\pi}} \frac{1}{R|J_0(j'_{0,k})|} J_0(j'_{0,k} r/R),\ k \geq 1, \\
\mu_{n,k} = \frac{{j'_{n,k}}^2}{R^2},\ n,k \geq 1,\ \textit{double eigenvalue} \\
v_{n,k}(r,\theta) = \begin{array}{l} \sqrt{\frac{2}{\pi}} \frac{j'_{n,k}}{R\sqrt{{j'_{n,k}}^2 - n^2}|J_n(j'_{n,k})|} J_n(j'_{n,k} r/R) \cos n\theta \\ \sqrt{\frac{2}{\pi}} \frac{j'_{n,k}}{R\sqrt{{j'_{n,k}}^2 - n^2}|J_n(j'_{n,k})|} J_n(j'_{n,k} r/R) \sin n\theta \end{array},\ n,k \geq 1,
\end{array} \tag{1.28}$$

where $j'_{n,k}$ is the k-th zero of J'_n (the derivative of the Bessel function J_n).

Here is an array of the first values of $j_{n,k}$ (left) and $j'_{n,k}$ (right):

$n\backslash k$	1	2	3	4
0	2.405	5.520	8.654	11.791
1	3.832	7.016	10.173	13.324
2	5.136	8.417	11.620	14.796
3	6.380	9.761	13.015	16.223

$n\backslash k$	1	2	3	4
0	0	3.832	7.016	10.173
1	1.841	5.331	8.536	11.706
2	3.054	6.706	9.969	13.170
3	4.201	8.015	11.346	14.586

Remark 1.2.15. Similarly, in dimension $N \geq 3$, the eigenvalues of the ball B_R of radius R involve the zeros of the Bessel functions $J_{N/2-1}, J_{N/2}, \ldots$. For example

$$\lambda_1(B_R) = \frac{j_{N/2-1,1}^2}{R^2} \qquad \lambda_2(B_R) = \lambda_3(B_R) = \cdots = \lambda_{N+1}(B_R) = \frac{j_{N/2,1}^2}{R^2}. \tag{1.29}$$

1.2.6 Fredholm alternative

Let L be an elliptic operator and λ one of its eigenvalues. Then, by definition, the linear operator $L - \lambda Id$ has a non-trivial kernel. Nevertheless, in some situations, we need to solve an equation like $(L - \lambda Id)v = f$. It is remarkable that we have the same result as in finite dimension.

Theorem 1.2.16 (Fredholm alternative). *Let L be an elliptic operator on a bounded open set Ω and λ one of its eigenvalues for Dirichlet boundary conditions. Let $f \in L^2(\Omega)$, then the problem*

$$\begin{array}{rcll} Lv - \lambda v &=& f & \text{in } \Omega, \\ v &=& 0 & \text{on } \partial\Omega \end{array} \tag{1.30}$$

has a solution if and only if f is orthogonal (for the L^2 scalar product) to any eigenfunction associated to λ.

For the proof, see e.g. [58], [36]. It is clear that if problem (1.30) has a solution v_0, we obtain all the solutions by adding any eigenfunction associated to λ: $v_0 + tu$.

1.3 Min-max principles and applications

1.3.1 Min-max principles

One very useful tool is the following variational characterization of the eigenvalues, known as the Poincaré principle or Courant-Fischer formulae, see [58]. Let us define the **Rayleigh quotient** of the operator L to be:

$$R_L[v] := \frac{\sum_{i,j=1}^N \int_\Omega a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx + \int_\Omega a_0(x) v^2(x)\, dx}{\int_\Omega v(x)^2\, dx} . \tag{1.31}$$

Then, we have

$$\lambda_k^D(L,\Omega) = \min_{\substack{E_k \subset H_0^1(\Omega), \\ \text{subspace of dim } k}} \max_{v \in E_k, v \neq 0} R_L[v] , \tag{1.32}$$

$$\mu_k^N(L,\Omega) = \min_{\substack{E_k \subset H^1(\Omega), \\ \text{subspace of dim } k}} \max_{v \in E_k, v \neq 0} R_L[v] . \tag{1.33}$$

In formulae (1.32) and (1.33), the minimum is achieved for choosing E_k the space spanned by the k-th first eigenfunctions. In particular, assuming that we have already computed $u_1, u_2, \ldots, u_{k-1}$ the $k-1$-th first eigenfunctions, we also have:

$$\lambda_k^D(L,\Omega) = \min_{\substack{v \in H_0^1(\Omega), \\ v \text{ orthogonal to } u_1, u_2, \ldots, u_{k-1}}} R_L[v] . \tag{1.34}$$

For example, in the case of the Laplacian, formulae (1.32) becomes

$$\lambda_k(\Omega) = \min_{\substack{E_k \subset H_0^1(\Omega), \\ \text{subspace of dim } k}} \max_{v \in E_k, v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2\, dx}{\int_\Omega v(x)^2\, dx} \tag{1.35}$$

and, in particular, for the first Dirichlet eigenvalue, we get

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega), v \neq 0} \frac{\int_\Omega |\nabla v(x)|^2 \, dx}{\int_\Omega v(x)^2 \, dx} , \tag{1.36}$$

while the first non-zero Neumann eigenvalue for the Laplacian is given by

$$\mu_2(\Omega) = \min_{v \in H^1(\Omega), v \neq 0, \int_\Omega v = 0} \frac{\int_\Omega |\nabla v(x)|^2 \, dx}{\int_\Omega v(x)^2 \, dx} . \tag{1.37}$$

In (1.36) and (1.37), the minimum is achieved by the corresponding eigenfunction(s).

There exist also similar variational characterizations for sums of consecutive eigenvalues or sums of inverses of consecutive eigenvalues (see e.g. [19] pp. 98-99 or [110]). We give here the case of the Laplacian with Dirichlet boundary conditions, but any other case can be handled in the same way. Let us denote by $u_1, u_2, \ldots, u_k$ the k-th first eigenfunctions. Then,

$$\sum_{i=k+1}^{k+n} \lambda_i(\Omega) = \min \left\{ \sum_{i=k+1}^{k+n} \int_\Omega |\nabla v_i(x)|^2 \, dx \right\}, \tag{1.38}$$

where (v_i) is an orthonormal family in $L^2(\Omega)$ satisfying $\int_\Omega v_i u_j \, dx = 0$, $j = 1, 2, \ldots, k$. Similarly

$$\sum_{i=k+1}^{k+n} \frac{1}{\lambda_i(\Omega)} = \max \left\{ \sum_{i=k+1}^{k+n} \int_\Omega v_i(x)^2 \, dx \right\}, \tag{1.39}$$

where (v_i) is a family in $H_0^1(\Omega)$ satisfying $\int_\Omega \nabla v_i . \nabla v_j \, dx = \delta_{ij}$ and $\int_\Omega v_i u_j \, dx = 0$, $j = 1, 2, \ldots, k$.

1.3.2 Monotonicity

Let us consider two open bounded sets such that $\Omega_1 \subset \Omega_2$. This inclusion induces a natural embedding $H_0^1(\Omega_1) \hookrightarrow H_0^1(\Omega_2)$ just by extending by zero functions in $H_0^1(\Omega_1)$. In particular, the min-max principle implies the following monotonicity for inclusion of eigenvalues with Dirichlet boundary conditions:

$$\Omega_1 \subset \Omega_2 \implies \lambda_k^D(L, \Omega_1) \geq \lambda_k^D(L, \Omega_2) \tag{1.40}$$

(since the minimum is taken over a larger class for $\lambda_k^D(L, \Omega_2)$). Moreover, the inequality is strict as soon as $\Omega_2 \setminus \Omega_1$ contains a set of positive capacity (since the first eigenfunction cannot vanish on such a set).

Let us also remark that this monotonicity formula **is not** valid in the Neumann case. For example, Figure 1.1 gives an elementary counter-example with rectangles. We use the fact that the first non-zero eigenvalue of a rectangle for the Neumann-Laplacian is given by $\mu_2(R) = \frac{\pi^2}{L^2}$ where L is the length of the rectangle (see (1.26)).