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The Fourfold Way In Real Analysis

An Alternative to the Metaplectic Representation

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Introduction

The n-dimensional metaplectic group $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ is the twofold cover of the symplectic group $\mathrm{Sp}(n,\mathbb{R})$, which is the group of linear transformations of $\mathcal{X} = \mathbb{R}^n \times \mathbb{R}^n$ that preserve the bilinear (alternate) form

$$[(\begin{smallmatrix} x \\ \xi \end{smallmatrix}), (\begin{smallmatrix} y \\ \eta \end{smallmatrix})] = -\langle x, \eta\rangle + \langle y, \xi\rangle. \tag{0.1}$$

There is a unitary representation of $\widetilde{\mathrm{Sp}}(n,\mathbb{R})$ in the Hilbert space $L^2(\mathbb{R}^n)$, called the metaplectic representation, the image of which is the group of transformations generated by the following ones: the linear changes of variables, the operators of multiplication by exponentials with pure imaginary quadratic forms in the exponent, and the Fourier transformation; some normalization factor enters the definition of the operators of the first and third species. The metaplectic representation was introduced in a great generality in [28] – special cases had been considered before, mostly in papers of mathematical physics – and it is of such fundamental importance that the two concepts (the group and the representation) have become virtually indistinguishable. This is not going to be our point of view: indeed, the main point of this work is to show that a certain finite covering of the symplectic group (generally of degree n) has another interesting representation, which enjoys analogues of most of the nicer properties of the metaplectic representation. We shall call it the anaplectic representation – other coinages that may come to your mind sound too medical – and shall consider first the one-dimensional case, the main features of which can be described in quite elementary terms.

It may not be an exaggeration to claim that among the foundational objects of classical analysis, the one-dimensional Gaussian function $e^{-\pi x^2}$ occupies one of the foremost positions: it is central in Fourier analysis and special function theory, everywhere in probability and, through its appearance in theta functions, it is basic in modular form theory as well. With the help of some of its satellites – the Heisenberg representation and Bargmann–Fock transform, the metaplectic representation, the Weyl calculus – it lies again at the core of fundamental methods of harmonic analysis or partial differential equations; it is also the basis of some mathematical techniques used in quantum field theory.

A starting point of the present work might be the fact that there is an alternative to this function, leading to a different kind of analysis but with a possibly wide range of influence too: this is the Bessel function $|x|^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi\, x^2)$, which lies in the null space of the (formal) harmonic oscillator. It has at infinity the considerable growth of the more obvious function $|x|^{-\frac{1}{2}}\, e^{\pi x^2}$: therefore, it cannot, in general, occur in integrals on the real line of the usual type. Actually, the development of the present analysis requires that we stray away from the usual one in several aspects. Possibly the only mathematical object which will remain as it stands, at least formally, is the Heisenberg representation: but a new notion of integral – not destroying the invariance under translations – will be needed, and the Fourier transformation and associated Weyl calculus of operators will be replaced by some different, quite parallel objects; finally, the usual L^2 scalar product will have to be changed to an indefinite pseudoscalar product.

Turning to the n-dimensional case, let us first recall that the role of the homogeneous space $\mathrm{Sp}(n,\mathbb{R})/U(n)$ in analysis is well documented. On one hand, it is the set of *complex* polarizations of $\mathcal{X}$, *i.e.,* the set of complex structures on this space such that the symplectic form appears as the imaginary part of some (Hilbert) scalar product on $\mathcal{X}$; on the other hand, it is a Hermitian domain (Siegel's domain), a natural place for analysis in Bergman's style. What is more important here is that one may realize the space $L^2(\mathbb{R}^n)$ as a space of vector-valued functions on Siegel's domain, in a way that makes the metaplectic representation appear as quite natural. To introduce the anaplectic representation, we substitute for Siegel's domain a finite covering $\Sigma^{(n)}$ of the space $U(n)/O(n)$ of *real* polarizations of $\mathcal{X}$, *i.e.,* the space of Lagrangian subspaces of $\mathcal{X}$. Again, we consider a certain space of vector-valued functions on $\Sigma^{(n)}$, getting in a natural way a new representation of some covering of the symplectic group as a result. These functions can in turn be identified with scalar functions on $\mathbb{R}^n$: however, in contradiction to the metaplectic case, the class of functions on $\mathbb{R}^n$ which enter the new analysis consists only of functions which extend as entire functions on $\mathbb{C}^n$. The one–dimensional case of this analysis coincides with the one hinted at above. A common point of the metaplectic and anaplectic representations is that each of the two groups of operators normalizes the group of operators arising from the Heisenberg representation: the latter one is formally the same in both cases. The anaplectic representation (only) can be enriched by a rotation of ninety degrees in the complex coordinates on $\mathbb{C}^n$, an operation that corresponds to the matrix $\left(\begin{smallmatrix} -i\,I & 0 \\ 0 & i\,I \end{smallmatrix}\right)$.

The development of anaplectic analysis calls for mathematical techniques rather different from the usual ones, as it depends as much on elementary real algebraic geometry as on Hilbert space methods. Some of the main questions that have to be tackled concern the analytic continuation of functions, and depend on a careful examination of the singularities of certain fractional-linear transformations; homotopy considerations often play a role too.

Except in the one-dimensional case, it seems unlikely that one could define a space of functions on $\mathbb{R}^n$, invariant under the full anaplectic representation, and on which an invariant pseudoscalar product could be defined. However, anaplectic analysis is not concerned solely with representation by the same name. In anaplectic analysis, the spectrum of the harmonic oscillator L is $\mathbb{Z}$ rather than $\frac{n}{2} + \mathbb{N}$, and the usual creation and annihilation operators become raising and lowering operators; also, unless $n = 1$, all the eigenspaces of L are infinite-dimensional. Provided that $n \not\equiv 0 \mod 4$, one can build, in a way unique up to normalization, a pseudoscalar product on the space generated by the eigenfunctions of L just alluded to, with respect to which the infinitesimal generators of the Heisenberg representation are self-adjoint.

Despite its many similarities with the usual analysis, anaplectic analysis differs from it in two major respects. First, there is no natural embedding of, say, the group of one-dimensional anaplectic transformations into the group of two-dimensional ones, that would generalize what is obtained, in the usual analysis, by regarding one of a pair of variables as a parameter. On the other hand, there is in the usual analysis a class of quite simple functions, to wit the exponentials with a second-order polynomial (the real part of which has a positive-definite top-order part) in the exponent, which resists all operations taken from the Heisenberg representation or the metaplectic representation. No comparable class can be described in such simple terms in anaplectic analysis. This is why non-trivial identities can sometimes be obtained by calculations the analogues of which, in the usual analysis, would not produce anything interesting: examples will occur in Section 10.

In the last chapter, we imbed the one-dimensional anaplectic analysis into a one-parameter family of analyses. There is one such analysis for every complex number $\nu \mod 2$, $\nu \notin \mathbb{Z}$: the case when ν is an integer should be regarded as leading to the usual analysis, the case when $\nu = -\frac{1}{2} \mod 2$ is that considered in Section 1. In each case, there is a translation-invariant concept of integral, an associated Fourier transformation and ν-anaplectic representation. When ν is real, $\nu \notin \mathbb{Z}$, there is on the basic relevant space $\mathfrak{A}_\nu$ a pseudoscalar product, invariant both under the Heisenberg representation and under the ν-anaplectic representation: besides, this latter representation, when restricted to the space of even, or odd, functions on $\mathfrak{A}_\nu$ (this depends on whether $\nu \in]-1,\, 0[+2\,\mathbb{Z}$ or $\nu \in]0,\, 1[+2\,\mathbb{Z}$), is unitarily equivalent to one of the representations of the universal cover of $SL(2, \mathbb{R})$ as made explicit in [18]; not surprisingly, the series that occurs here is one which does not occur in the Plancherel theorem for the group under consideration.

It is our hope, and belief, that anaplectic analysis will prove useful in several domains: in quantum mechanics (especially in relativistic quantum mechanics), in partial differential equations, in special function theory. Let us only observe to start with that a mathematical analysis based on a harmonic oscillator unbounded from below cannot fail to help in questions in which we would like to have time circulate just as well in two directions. Also, the pseudoscalar product which occurs in the one-dimensional anaplectic analysis has a striking similarity to that which plays a role in the covariant formulation [5, p. 384] or [3, p. 68] of quantum electrodynamics. Concerning the possibility of using anaplectic analysis in partial differential equations, this only has, as yet, the status of wishful thinking. We have, however, initiated the study of the anaplectic Weyl calculus: though we have mostly dealt, up to now, with its more formal aspects only, one may expect that some kind of new pseudodifferential analysis will eventually emerge. Under the name of “Krein spaces”, the subject of linear spaces with an indefinite metric is currently under much scrutiny, in particular in connection with spectral problems of an unusual type (*cf.* for instance [19]); such a kind of problems has also been considered by several authors [1, 2] for reasons having to do with PT-symmetry. Anaplectic analysis certainly provides a special domain of research related to this question, with a rich harmonic analysis of its own. Also, when it is completed, the anaplectic pseudodifferential analysis might be a useful tool for this kind of problems in general. Some possible connection between the one-dimensional anaplectic pseudodifferential analysis and a variant of the Lax–Phillips scattering theory for the automorphic wave equation has been briefly hinted at at the end of Section 10. Finally, but this goes beyond our current projects, there is the question whether some version of the anaplectic representation could be developed in the case of local fields such as the fields of p-adic numbers or their quadratic extensions, thus following in the steps of Weil’s celebrated paper [28] on the metaplectic representation.

Let me apologize to M. Gell–Mann and Y. Ne’eman [8] for my choice of a title: I simply could not resist its poetic appeal. On the other hand, the first section of this volume will show that no other choice was possible.

Chapter 1

The One-dimensional Anaplectic Representation

In this chapter, we introduce one-dimensional anaplectic analysis in an elementary, though probably somewhat puzzling, way. The trick is to relate the functions u on the real line to be considered – they all extend as entire functions – to uniquely defined 4-tuples of functions. This is not as strange as it might seem, especially in connection with the study of the Fourier transformation: in mathematical tables dealing with this transform, functions are always split into their even and odd parts. Here, the introduction of the four functions f_0, f_1, $f_{i,0}$, $f_{i,1}$ (*cf.* Definition 1.1) is up to some point a matter of convenience, since the last two can be obtained from the first two by analytic continuation. The first ones are not exactly the even and odd parts of u: however, f_0 (*resp.* f_1) characterizes the even (*resp.* odd) part of u, while enjoying better estimates near $+\infty$. The first example, in Proposition 1.2, will make matters clear. A fundamental definition is that (Proposition 1.16) of the linear form Int which substitutes for the notion of integral: in connection with the Heisenberg representation – which is formally defined in the usual way – it makes it possible to define the anaplectic Fourier transformation, from which it is easy (Theorem 1.20) to obtain the anaplectic representation in general.

However, the proof of some major facts (including the characterization, given in Theorem 1.8, of the space $\mathfrak{A}$), requires that one should construct the anaplectic representation as the direct sum of two representations taken from the full non-unitary principal series of $SL(2,\mathbb{R})$. This is the object of Section 2: also, the decomposition (*cf.* Proposition 2.3) of analytic vectors of such a representation into their entire and ramified parts will play a role in several parts of this work. It is the characterization given in Theorem 1.8 that prepares the way for the definition of the anaplectic representation in the n-dimensional setting, to be developed in Chapter 2. We suggest that the reader satisfy himself with a look at the definition ((2.3) and (2.6)) of the representation $\pi_{\rho,\varepsilon}$, at the statements of Proposition

2.3 and of Theorems 2.9–2.11, otherwise jump directly from Section 1 to Section 3 or even Section 4, using the technical Section 2 mostly for reference. Another possibility is to continue the reading of Section 1 with that of Sections 11 and 12, coming back only later to the n-dimensional case.

Possibly the most specific feature of the one-dimensional anaplectic representation (which extends to the higher-dimensional case) is that it includes the complex rotation $\mathcal{R}$ such that $(\mathcal{R}\,u)(x) = u(ix)$: note that rotations by angles $\neq \frac{\pi n}{2}$, $n \in \mathbb{Z}$, are not permitted *in general*. Since the conjugate, under $\mathcal{R}$, of the operator $A = \pi^{\frac{1}{2}}\,(x + \frac{1}{2\pi}\,\frac{d}{dx})$ – also called the annihilation operator in the usual analysis because of its effect on the ground state $x \mapsto e^{-\pi x^2}$ of the harmonic oscillator – is the "creation" operator A^*, the distinction between A and A^*, usually so essential, blurs out, and the spectrum of the anaplectic harmonic oscillator is $\mathbb{Z}$ instead of $\frac{1}{2} + \mathbb{N}$.

1 The one-dimensional case

A representation π of a Lie group G in some complex linear space $\mathfrak{H}$ is a homomorphism π from G to the group of linear automorphisms of $\mathfrak{H}$: we shall usually concern ourselves with non-unitary representations.

Consider the Hilbert space $\mathfrak{H} = L^2(\mathbb{R})$. Given $u \in \mathfrak{H}$ and $(y, \eta) \in \mathbb{R}^2$, the function $\pi(y, \eta)\,u$ defined as

$$(\pi(y, \eta)\,u)(x) = u(x - y)\,e^{2i\pi(x - \frac{y}{2})\eta} \tag{1.1}$$

still lies in $L^2(\mathbb{R})$. An elementary calculation shows that one has

$$\pi(y, \eta)\,\pi(y', \eta') = \pi(y + y', \eta + \eta')\,e^{i\pi\,(-y\,\eta' + y'\,\eta)}. \tag{1.2}$$

Enlarging the group $\mathbb{R}^2$ to the so-called *Heisenberg group* which is the set-theoretic product $\mathbb{R}^2 \times S^1$ endowed with the law of composition defined as

$$(y, \eta\,;\, e^{i\theta}).\,(y', \eta'\,;\, e^{i\theta'}) = (y + y', \eta + \eta'\,;\, e^{i(\theta + \theta' - y\,\eta' + y'\,\eta)}), \tag{1.3}$$

one gets a unitary representation, the Heisenberg representation. Denoting as Q and P the (unbounded) self-adjoint operators on $L^2(\mathbb{R})$ that consist respectively in multiplying by x or taking $(2i\pi)^{-1}$ times the first-order derivative, one may also write, in the sense of Stone's theorem relative to one-parameter groups of unitary operators,

$$\pi(y, \eta) = e^{2i\pi\,(\eta\,Q - y\,P)}\ : \tag{1.4}$$

we shall also use this notation later, outside the context of unitary operators, then *taking it as a definition of the operator on the right-hand side.*

Still with the same Hilbert space $\mathfrak{H} = L^2(\mathbb{R})$ as before, consider instead of $\mathbb{R}^2$ the group $SL(2,\mathbb{R})$: it is generated by the elements

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.5}$$

where a is an arbitrary positive number and c is an arbitrary real number. It is impossible to find a representation of $SL(2,\mathbb{R})$ in $L^2(\mathbb{R})$ such that the automorphisms $\pi(g)$ associated to the three transformations above should be respectively:

(i) the transformation $u \mapsto v$, $v(x) = a^{-\frac{1}{2}}\, u(a^{-1}x)$;
(ii) the multiplication by the exponential $\exp(i\pi c x^2)$;
(iii) $e^{-\frac{i\pi}{4}}$ times the Fourier transformation $\mathcal{F}$, normalized as

$$(\mathcal{F}u)(\xi) = \int_{-\infty}^{\infty} u(x)\, e^{-2i\pi x\xi}\, dx. \tag{1.6}$$

To see this is immediate, since the fourth power of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the unit matrix, while $\mathcal{F}^4 = I$: despite appearances, dropping the factor $e^{-\frac{i\pi}{4}}$ in the definition of the transformation (iii) would only make matters worse, though it is a little bit harder to see. The difficulty is that if some matrix $g \in SL(2,\mathbb{R})$ can be written as $g = g_1 \dots g_k$, where all factors are of the special type described in (1.5), the product $\pi(g_1)\dots\pi(g_k)$ depends on the decomposition chosen, not only on g: however, the corresponding indeterminacy in such a definition is not that bad, since the *unordered* pair $\pm\,\pi(g_1)\dots\pi(g_k)$ depends only on g. To remedy it completely, one constructs a group "more precise" than $SL(2,\mathbb{R})$, namely the *metaplectic group* $\widetilde{SL}(2,\mathbb{R})$, a twofold covering of $SL(2,\mathbb{R})$: this means a connected Lie group together with a homomorphism: $\widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$, the kernel of which has two elements. That such a group exists is a consequence of the fact that the fundamental group, in the topological sense, of $SL(2,\mathbb{R})$, is $\mathbb{Z}$ (since $SL(2,\mathbb{R})$ has the homotopy type of its compact subgroup $SO(2)$), of which $\mathbb{Z}/2\mathbb{Z}$ is a quotient group: the two elements of $\widetilde{SL}(2,\mathbb{R})$ which are sent to some given $g \in SL(2,\mathbb{R})$ by the homomorphism in question are said to lie above g. One can then show that there exists a unitary representation Met of $\widetilde{SL}(2,\mathbb{R})$ in $L^2(\mathbb{R})$, the *metaplectic representation*, such that, given $g \in SL(2,\mathbb{R})$, the unordered pair $\pm\,\pi(g)$ as defined above should coincide with the pair $\{\mathrm{Met}(\gamma_1),\, \mathrm{Met}(\gamma_2)\}$, where $\{\gamma_1,\, \gamma_2\}$ is the pair of points in the metaplectic group lying above g.

To proceed towards the anaplectic representation, we may start from a complexification of the Heisenberg representation (1.1): that is, we want to substitute for the generic pair $(y,\eta) \in \mathbb{R}^2$ a pair of complex numbers; elements of the complexified Heisenberg group will then be triples $(y,\eta\,;\,\omega)$ with $(y,\eta) \in \mathbb{C}^2$ and $\omega \in \mathbb{C}^\times$. Of course, it is clear that, in this case, $\pi(y,\eta)$ can no longer operate within the space $L^2(\mathbb{R})$, and that we must substitute for this space an appropriate space $\mathfrak{A}$ of entire functions of one variable; also, it is impossible to preserve unitarity. So as to introduce the anaplectic representation, and above all to connect it to the Heisenberg representation, it is suitable to introduce first the definition of a certain space

$\mathfrak{A}$, which is to play the role of a set of analytic vectors of the anaplectic representation. It will be clearly explained in the remark following the proof of Theorem 2.9 why the use of analytic vectors, at least at this first stage, is essential.

Definition 1.1. Let us say that an entire function f of one variable is nice if on one hand $f(z)$ is bounded by a constant times some exponential $\exp(\pi R|z|^2)$, on the other hand the restriction of f to the *positive* half-line is bounded by a constant times some exponential $\exp(-\pi\varepsilon x^2)$: here, R and ε are assumed to be positive. The space $\mathfrak{A}$ consists of all entire functions u of one variable with the following properties:

(i) the even part u_{even} of u coincides with the even part of some nice function f_0 satisfying the property that the function $z \mapsto f_0(iz) + i\, f_0(-iz)$ is nice too;

(ii) the odd part u_{odd} of u coincides with the odd part of some nice function f_1 such that the function $z \mapsto f_1(iz) - i\, f_1(-iz)$ is nice as well.

It will be proven below (Corollary 1.7) that given $u \in \mathfrak{A}$, a pair $(f_0,\, f_1)$ satisfying the above properties is of necessity unique: for short, we shall refer to the pair $(f_0,\, f_1)$ as the $\mathbb{C}^2$-realization of u. We shall go one step further, associating with u the $\mathbb{C}^4$-valued function (indifferently written in line or column form), called the the $\mathbb{C}^4$-realization of u,

$$\boldsymbol{f} = (f_0,\, f_1,\, f_{i,0},\, f_{i,1}) \tag{1.7}$$

with

$$\begin{aligned} f_{i,0}(z) &= \frac{1-i}{2}\,(f_0(iz) + i\, f_0(-iz)),\\ f_{i,1}(z) &= \frac{1+i}{2}\,(f_1(iz) - i\, f_1(-iz)). \end{aligned} \tag{1.8}$$

All four components of $\boldsymbol{f}$ are thus nice functions in the sense of Definition 1.1.

Here is a basic example.

Proposition 1.2. *Set, for x real,*

$$\phi(x) = (\pi\,|x|)^{\frac{1}{2}}\, I_{-\frac{1}{4}}(\pi\, x^2), \tag{1.9}$$

with [17, p. 66]

$$I_\nu(t) = \sum_{m\geq 0} \frac{(\frac{t}{2})^{\nu+2m}}{m\,!\,\Gamma(\nu+m+1)} \tag{1.10}$$

for $t > 0$. The function ϕ lies in $\mathfrak{A}$.

Proof. Clearly, ϕ extends as an entire even function. Note [17, p. 139] that it has the considerable growth of $|x|^{-\frac{1}{2}}\, e^{\pi x^2}$ as $|x| \to \infty$. Set, however, for $x > 0$,

$$\begin{aligned} \psi(x) &= 2^{\frac{1}{2}}\, \pi^{-\frac{1}{2}}\, x^{\frac{1}{2}}\, K_{\frac{1}{4}}(\pi\, x^2)\\ &= (\pi\, x)^{\frac{1}{2}}\, [I_{-\frac{1}{4}}(\pi\, x^2) - I_{\frac{1}{4}}(\pi\, x^2)]. \end{aligned} \tag{1.11}$$

From (1.10), this is the restriction to $(0,\infty)$ of an entire function, the even part of which coincides with ϕ: but now (*loc. cit.*), $\psi(x)$ goes to zero, as $x\to\infty$, just like $x^{-\frac{1}{2}}\,e^{-\pi x^2}$. On the other hand, for $x>0$,

$$\psi(\pm ix)=(\pi\,x)^{\frac{1}{2}}\,[I_{-\frac{1}{4}}(\pi\,x^2)\mp I_{\frac{1}{4}}(\pi\,x^2)], \tag{1.12}$$

as can be seen from a careful use of (1.10), so that

$$\psi(ix)+i\,\psi(-ix)=(1+i)\,\psi(x). \tag{1.13}$$

Consequently $\phi\in\mathfrak{A}$: note that the $\mathbb{C}^4$-realization of ϕ is $(\psi,\,0,\,\psi,\,0)$. $\square$

We shall prove presently that the map $(f_0,f_1)\mapsto u$ introduced in Definition 1.1 is one-to-one, and we take this opportunity to prove at the same time a few related lemmas which will be put to use later. All this is related to the Phragmén–Lindelöf lemma, an extension of the maximum principle to angular regions which can be found in many textbooks, including [26, p. 496]:

Lemma 1.3. *Let f be an entire function of one variable, let S be the sector defined by the inequality $|\mathrm{Arg}\,z|\leq\frac{\alpha\,\pi}{2}$ for some $\alpha\in]0,2[$, and let $\delta\in]0,\alpha^{-1}[$. Assume that one has $|f(z)|\leq\exp(|z|^\delta)$ if $z\in S$ and $|z|$ is sufficiently large. Then, if the restriction of f to the boundary of S is bounded, f is bounded in S. Moreover, if $f(z)$ goes to zero as z goes to infinity along any of the two sides of the sector, $f(z)$ goes to zero in a uniform way as z goes to infinity while staying in S.*

Lemma 1.4. *Let f be an entire function satisfying some estimate*

$$|f(z)|\leq C\,e^{\pi R|z|^2},\quad z\in\mathbb{C}, \tag{1.14}$$

together with some estimate

$$|f(x)|\leq C\,e^{-2\pi\delta x^2},\quad x>0. \tag{1.15}$$

Then there exists $\theta_0>0$ such that

$$|f(xe^{i\theta})|\leq C\,e^{-\pi\delta x^2},\quad x>1,\ |\theta|\leq\theta_0. \tag{1.16}$$

Proof. With some $A>0$ to be chosen later and an arbitrary $\gamma>1$, set

$$\Phi(z)=\exp\left(2\pi(\delta+iA)e^{\frac{i\pi}{2\gamma}}\,z^2\right)f(z\,e^{\frac{i\pi}{4\gamma}}), \tag{1.17}$$

a function considered in the sector $|\mathrm{Arg}\ z|\leq\frac{\pi}{4\gamma}$ and satisfying the estimate $\log_+\,|\Phi(z)|\leq C\,|z|^2$ for z in this sector with $|z|$ large. When $z=|z|\,e^{-\frac{i\pi}{4\gamma}}$, one has

$$|\Phi(z)|\leq e^{2\pi\delta|z|^2}\,|f(|z|)|\leq C;$$

when $z=|z|\,e^{\frac{i\pi}{4\gamma}}$, one has

$$|\Phi(z)|\leq C\,\exp\left(2\pi|z|^2\left(\delta\,\cos\frac{\pi}{\gamma}-A\,\sin\frac{\pi}{\gamma}+\frac{R}{2}\right)\right),$$

a bounded expression if A is chosen large enough. Then, by the Phragmén–Lindelöf lemma, Φ is bounded in the whole sector and, for $0 < \mathrm{Arg}\, z < \frac{\pi}{2\gamma}$, one has

$$|f(z)| \leq C \, \exp\left(-2\pi \, \mathrm{Re}\, ((\delta + iA)z^2)\right) \; : \tag{1.18}$$

when $z = |z| e^{i\theta}, \; 0 \leq \theta \leq \frac{\pi}{2\gamma}$, one has

$$\begin{aligned} \mathrm{Re}\, ((\delta + iA)z^2) &= |z|^2 \, \mathrm{Re}\, ((\delta + iA)e^{2i\theta}) \\ &= |z|^2 \, (\delta \cos 2\theta - A \sin 2\theta) \\ &\geq \frac{\delta}{2} \, |z|^2 \end{aligned} \tag{1.19}$$

if θ is small enough. The same holds if $-\frac{\pi}{2\gamma} \leq \theta \leq 0$, considering instead the function $z \mapsto \overline{f(\bar{z})}$. □

In a similar way, one can prove the following:

Lemma 1.5. *Let g be a function defined and holomorphic in some angular sector around the positive half-line, satisfying for some pair of positive constants C, R and every $z \in \mathbb{C}$ the estimate*

$$|g(z)| \leq C \, e^{2\pi R |z|}. \tag{1.20}$$

Assume that, for some $\delta > 0$, one has the inequality

$$|g(x)| \leq C \, e^{-2\pi\delta x}, \qquad x > 0 \; : \tag{1.21}$$

then, there exists $\theta_0 > 0$ such that

$$|g(x \, e^{i\theta})| \leq C \, e^{-\pi\delta x}, \qquad x > 1, \; \|\theta| \leq \theta_0. \tag{1.22}$$

Lemma 1.6. *Let f be an entire function such that, for some pair of positive constants C, R,*

$$|f(z)| \leq C \, e^{\pi R |z|^2}, \quad z \in \mathbb{C}. \tag{1.23}$$

If there exists $\delta > 0$, such that

$$|f(x)| + |f(ix)| \leq C \, e^{-\pi\delta x^2}, \qquad x \in \mathbb{R}, \tag{1.24}$$

the function f is identically zero.

Proof. By Lemma 1.4,

$$|f(x \, e^{i\theta})| \leq C \, e^{-2\pi\delta x^2}, \qquad x > 1, \; 0 \leq \theta \leq \theta_0 \; : \tag{1.25}$$

now the half-width of the sector $\theta_0 \leq \mathrm{Arg}\, z \leq \frac{\pi}{2}$ is $< \frac{\pi}{4}$, so that the Phragmén–Lindelöf lemma applies and shows that $f(z)$ goes to zero, as $|z| \to \infty$, in a uniform way in the first quadrant. The same goes with the three other quadrants, so that the lemma is a consequence of Liouville's theorem. □

Corollary 1.7. *Let $u \in \mathfrak{A}$, the space of entire functions introduced in Definition* 1.1*. Then the pair of nice functions f_0, f_1 the existence of which is asserted there is unique.*

Proof. Taking the difference of any two such pairs, one remarks that if f_0 is nice, *odd* and if the function $z \mapsto f_0(iz) + if_0(-iz) = (1-i)\, f_0(iz)$ is nice too, then $f_0 = 0$ according to the lemma that precedes; something similar goes with f_1. $\square$

We now show how the vector $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ can be rebuilt from the knowledge of $u \in \mathfrak{A}$. We shall postpone to the next section the proof that, given an entire function u satisfying some global estimate $|u(z)| \leq C\, e^{\pi R |z|^2}$, the additional properties expressed below in terms of the pair (w_0, w_1) whose definition follows characterize the fact that u lies in $\mathfrak{A}$.

Theorem 1.8. *Let $u \in \mathfrak{A}$. Set, for σ real and large enough,*

$$w_0(\sigma) = \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, u(x\, e^{-\frac{i\pi}{4}})\, dx,$$

$$w_1(\sigma) = \frac{1-i}{2} \int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\, x\, u(x\, e^{-\frac{i\pi}{4}})\, dx. \tag{1.26}$$

On the one hand, each of these two functions extends as a holomorphic function, still denoted as w_0 (resp. w_1), in some strip $|\mathrm{Im}\ \sigma| < \varepsilon$. On the other hand, for $|\sigma|$ large enough, $w_0(\sigma)$ and $w_1(\sigma)$ admit the convergent expansions

$$w_0(\sigma) = \sum_{n \geq 0} a_n\, \sigma^{-n}\, |\sigma|^{-\frac{1}{2}}, \qquad w_1(\sigma) = \sum_{n \geq 0} b_n\, \sigma^{-n-1}\, |\sigma|^{-\frac{1}{2}} \tag{1.27}$$

so that, for R large enough, w_0 (resp. w_1) extends as a holomorphic function, denoted as $\tilde{w}_0$ (resp. $\tilde{w}_1$), in the part of the Riemann surface of the square root function lying above the set $|z| > R$: the two continuations of the two functions under consideration are related by the equations, valid for σ real and large,

$$\tilde{w}_0(\sigma\, e^{i\pi}) = -i\, w_0(-\sigma), \qquad \tilde{w}_1(\sigma\, e^{i\pi}) = -i\, w_1(-\sigma). \tag{1.28}$$

Finally, the $\mathbb{C}^4$-realization of u can be obtained, in terms of w_0 and w_1, by the formulas (involving semi-convergent only integrals in the first two cases), valid for $x > 0$ only,

$$f_0(x) = 2^{-\frac{1}{2}}\, x \int_{-\infty}^{\infty} w_0(\sigma)\, e^{i\pi\sigma x^2}\, d\sigma,$$

$$f_{i,0}(x) = 2^{-\frac{1}{2}}\, x \int_{-\infty}^{\infty} w_0(\sigma)\, e^{-i\pi\sigma x^2}\, d\sigma,$$

$$f_1(x) = \int_{-\infty}^{\infty} w_1(\sigma)\, e^{i\pi\sigma x^2}\, d\sigma,$$

$$f_{i,1}(x) = \int_{-\infty}^{\infty} w_1(\sigma)\, e^{-i\pi\sigma x^2}\, d\sigma. \tag{1.29}$$

Proof. Let $(f_0,\, f_1,\, f_{i,0},\, f_{i,1})$ be the $\mathbb{C}^4$-realization of u. Since the even part of u coincides with that of f_0 and the odd part of u coincides with that of f_1, one can substitute f_0 (*resp.* f_1) for u in the integral defining w_0 (*resp.* w_1). Using Lemma 1.4 together with the global estimate of u, one sees that the integral

$$w_0^+(\sigma) = \int_0^\infty e^{-\pi\sigma x^2}\, f_0(x\, e^{-\frac{i\pi}{4}})\, dx \tag{1.30}$$

can also be written, for σ real and large, as

$$w_0^+(\sigma) = \frac{1+i}{2^{\frac{1}{2}}} \int_0^\infty e^{-i\pi\sigma x^2}\, f_0(x)\, dx\ : \tag{1.31}$$

this makes it possible to write

$$\begin{aligned} w_0(\sigma) - 2^{\frac{1}{2}} \int_0^\infty e^{-i\pi\sigma x^2}\, f_0(x)\, dx &= \int_0^\infty e^{-\pi\sigma x^2}\, [f_0(x\, e^{\frac{3i\pi}{4}}) + i\, f_0(x\, e^{-\frac{i\pi}{4}})]\, dx \\ &= (1+i) \int_0^\infty e^{-\pi\sigma x^2}\, f_{i,0}(x\, e^{\frac{i\pi}{4}})\, dx. \end{aligned} \tag{1.32}$$

With a new deformation of contour, made possible by a new application of Lemma 1.4, this time to the function $f_{i,0}$, one finds that, for large σ,

$$w_0(\sigma) = 2^{\frac{1}{2}} \int_0^\infty [e^{-i\pi\sigma x^2}\, f_0(x) + e^{i\pi\sigma x^2}\, f_{i,0}(x)]\, dx. \tag{1.33}$$

The same method, starting from the identity

$$\frac{1-i}{2} \int_0^\infty e^{-\pi\sigma x^2}\, x\, f_1(x\, e^{-\frac{i\pi}{4}})\, dx = -\frac{1+i}{2} \int_0^\infty e^{i\pi\sigma x^2}\, x\, f_1(-ix)\, dx, \tag{1.34}$$

shows that

$$w_1(\sigma) = \int_0^\infty x\, [e^{-i\pi\sigma x^2}\, f_1(x) + e^{i\pi\sigma x^2}\, f_{i,1}(x)]\, dx. \tag{1.35}$$

Since the four components of $\boldsymbol{f}$ are nice in the sense of Definition 1.1, the equations (1.33) and (1.35) show that w_0 and w_1 indeed extend as holomorphic functions in some open strip containing the real line.

The expansion of $w_0(\sigma)$ for large σ can be derived directly from (1.26): indeed, the estimate $|u(z)| \leq C\, \exp(\pi R |z|^2)$ and Cauchy's inequalities make it possible to write

$$u_{\text{even}}(x\, e^{-\frac{i\pi}{4}}) = \sum_{k\geq 0} c_k\, x^{2k} \tag{1.36}$$

with $|c_k| \leq C\, \frac{(2\pi R)^k}{k\,!}$: since $\int_{-\infty}^\infty e^{-\pi\sigma x^2}\, x^{2k}\, dx = \Gamma(k+\frac{1}{2})\, (\pi\sigma)^{-k-\frac{1}{2}}$, the expansion (1.36) can be integrated term-by-term against $e^{-\pi\sigma x^2}\, dx$, leading to the first series expansion (1.27) as soon as $\sigma > 2R$; the same goes with $w_1(\sigma)$ for large σ. We now prove (1.28), which will also imply the validity of these series expansions for $-\sigma$ large.

To do this, we go back to (1.26) and, for large σ, accompany, up to $\theta=\pi$, the change $\sigma\mapsto\sigma\,e^{i\theta}$ by the change of contour $x\mapsto x\,e^{-\frac{i\theta}{2}}$, ending up, with $u_i(x)=u(ix)$, with the pair of equations

$$\begin{aligned}\tilde{w}_0(\sigma\,e^{i\pi}) &= -i\int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\,u_i(x\,e^{-\frac{i\pi}{4}})\,dx,\\ \tilde{w}_1(\sigma\,e^{i\pi}) &= \frac{1-i}{2}\int_{-\infty}^{\infty} e^{-\pi\sigma x^2}\,x\,u_i(x\,e^{-\frac{i\pi}{4}})\,dx.\end{aligned} \tag{1.37}$$

Now, if $u\in\mathfrak{A}$ is associated to the vector $(f_0,\,f_1,\,f_{i,0},\,f_{i,1})$, it is immediate to check (more about it in Proposition 1.13, which does not depend on any previous result) that u_i is associated to the vector $(f_{i,0},\,-i\,f_{i,1},\,f_0,\,-i\,f_1)$: using (1.33) and (1.35) and comparing the results obtained if one utilizes the $\mathbb{C}^4$-realization of u or that of u_i, one obtains the relation (1.28).

The inversion formulas (1.29) are obtained from (1.33) and (1.35), using the change of variable $y=\frac{x^2}{2}$ followed by the Fourier inversion formula. $\square$

Examples. (i) Take for some non-negative integer n, and $x\in\mathbb{R}$,

$$u(x)=|x|^{2n+\frac{1}{2}}\,I_{n-\frac{1}{4}}(\pi x^2), \tag{1.38}$$

so that u extends as an entire even function. One finds if $\sigma>0$, using [17, p. 66, 91],

$$\begin{aligned}w_0(\sigma) &= 2\,(-1)^n\int_0^{\infty} e^{-\pi\sigma x^2}\,x^{2n+\frac{1}{2}}\,J_{n-\frac{1}{4}}(\pi x^2)\,dx\\ &= (-1)^n\,2^{n-\frac{1}{4}}\,\pi^{-n-\frac{5}{4}}\,\Gamma\left(n+\frac{1}{4}\right)\,(1+\sigma^2)^{-n-\frac{1}{4}}.\end{aligned} \tag{1.39}$$

Clearly, for $\sigma>1$,

$$\tilde{w}_0(\sigma\,e^{i\pi})=-i\,w_0(\sigma)=-i\,w_0(-\sigma), \tag{1.40}$$

which confirms (1.28). Using (1.29), one finds for $x>0$

$$f_0(x)=(-1)^n\,2^{\frac{1}{2}}\,\pi^{-1}\,x^{2n+\frac{1}{2}}\,K_{n-\frac{1}{4}}(\pi x^2), \tag{1.41}$$

and it is indeed immediate to check that the even part of the continuation of f_0 as an entire function coincides with u, and that the conditions of Definition 1.1 are satisfied. Thus $u\in\mathfrak{A}$: the particular case when $n=0$ is the function $\pi^{-\frac{1}{2}}\phi$, where ϕ is the function introduced in Proposition 1.2.

(ii) More generally, with $n=0,1,\dots$ and $j\in\mathbb{Z}$, one defines two (disjoint) classes in $\mathfrak{A}$ by the consideration of the functions

$$|x|^{2(j+n)+\frac{1}{2}}\,I_{n-j-\frac{1}{4}}(\pi\,x^2)\qquad\text{and}\qquad|x|^{2(j+n)+\frac{1}{2}}\,I_{n-j+\frac{3}{4}}(\pi\,x^2). \tag{1.42}$$

Indeed, setting $u(x) = |x|^\lambda I_\rho(\pi x^2)$, assuming that $\rho + \frac{\lambda}{2} = 2n$ or $2n+1$ with $n = 0, 1, \dots$ so that u should be analytic and even, one finds [17, p. 91]

$$\begin{aligned} w_0(\sigma) &= 2\,(-i)^{\rho+\frac{\lambda}{2}} \int_0^\infty x^\lambda\, e^{-\pi\sigma x^2}\, J_\rho(\pi x^2)\, dx \\ &= (-i)^{\rho+\frac{\lambda}{2}}\, \Gamma\left(\rho + \frac{\lambda+1}{2}\right) \pi^{\frac{-1-\lambda}{2}}\, (1+\sigma^2)^{\frac{-1-\lambda}{4}}\, P^{-\rho}_{\frac{\lambda-1}{2}} \left(\frac{\sigma}{\sqrt{1+\sigma^2}}\right), \end{aligned} \tag{1.43}$$

where the Legendre function involved is even (*resp.* odd) in the case when $-\rho + \frac{\lambda-1}{2}$ is an even (*resp.* odd) integer [17, p. 170]. On the other hand, the continuation $\tilde{w}_0$ can be found from the expression [17, p. 47]

$$\begin{aligned} w_0(\sigma) = 2^{-\rho}\,(-i)^{\rho+\frac{\lambda}{2}}\, \pi^{\frac{-1-\lambda}{2}} \frac{\Gamma(\rho + \frac{\lambda+1}{2})}{\Gamma(1+\rho)}\, \sigma^{-\rho-\frac{1+\lambda}{2}} \left(1 + \frac{1}{\sigma^2}\right)^{-\frac{\rho}{2}-\frac{1+\lambda}{4}} \\ \times {}_2F_1\left(\frac{\lambda+1}{4} + \frac{\rho}{2}, \frac{1-\lambda}{4} + \frac{\rho}{2}; \rho+1; \frac{1}{1+\sigma^2}\right) \end{aligned} \tag{1.44}$$

since, if $\sigma > 1$, $\frac{1}{1+(\sigma\, e^{i\theta})^2}$ can never be a real number > 1 so that, in the continuation process, the argument of the hypergeometric function remains in a domain where this function is uniform: this makes it possible to conclude.

(iii) On the other hand, the function $u(x) = e^{-\pi x^2}$ does not belong to $\mathfrak{A}$. For w_0, as obtained by an application of (1.26), is given for $\sigma > 1$ as $w_0(\sigma) = (\sigma - i)^{-\frac{1}{2}}$: indeed, this function extends as an analytic function in the strip $|\mathrm{Im}\ \sigma| < 1$. However, following the determinations of the square root, one notices that, for $\sigma > 1$, one has the relation $\tilde{w}_0(\sigma\, e^{i\pi}) = -w_0(-\sigma)$ rather than the relation (1.28). A large class of entire functions not in $\mathfrak{A}$ is the class $\mathfrak{M}$ of multipliers of $\mathfrak{A}$ introduced in Proposition 1.15 below which, as proven there, only intersects $\mathfrak{A}$ trivially.

(iv) If the four components of the $\mathbb{C}^4$-realization of some function in $\mathfrak{A}$ are all less, on the positive half-line, than a multiple of $\exp(-\pi\varepsilon x^2)$ for some specific ε, it is clear that they can all be multiplied without harm by any even entire function of z globally less than a multiple of $\exp(\pi a|z|^2)$ for some $a < \varepsilon$: the same goes, as a consequence, for the function in $\mathfrak{A}$ we started out with.

As an explicit example of function in $\mathfrak{A}$ obtained in this way, take for some $\theta \in]0, \frac{\pi}{4}[$ the function defined for $x \in \mathbb{R}$ as

$$u(x) = \pi^{\frac{1}{2}}\, |x|\, I_{-\frac{1}{4}}(\pi x^2 \cos\theta)\, I_{-\frac{1}{4}}(\pi x^2 \sin\theta). \tag{1.45}$$

Note that this is the product of two factors, each of which is a rescaled version of the function ϕ from Proposition 1.2: also note that we explicitly discard the case when the two rescaling factors would be the same. Applying (1.26), one finds with the help of [10, p. 95] the equation

$$w_0(\sigma) = 2^{\frac{1}{2}}\, \pi^{-\frac{3}{2}}\, (\sin\, 2\theta)^{-\frac{1}{2}}\, \mathfrak{Q}_{-\frac{3}{4}} \left(\frac{\sigma^2+1}{\sin\, 2\theta}\right), \tag{1.46}$$

where the function $\mathfrak{Q}_{-\frac{3}{4}}$ is the Legendre function of the second species defined for $t > 1$ as

$$\mathfrak{Q}_{-\frac{3}{4}}(t) = \frac{\Gamma(\frac{1}{4})\Gamma(\frac{1}{2})}{2^{\frac{1}{4}}\,\Gamma(\frac{3}{4})}\ t^{-\frac{1}{4}}\ {}_2F_1\left(\frac{5}{8}, \frac{1}{8}; \frac{3}{4}; t^{-2}\right)\ : \tag{1.47}$$

it is analytic for $t > 1$. It is immediate that the equation (1.28) linking the two continuations of w_0 is satisfied. Using the equation (1.29) again, we find for $x > 0$, using [17, p. 194], that

$$\begin{aligned} f_0(x) &= 2\pi^{-\frac{3}{2}}\,(\sin 2\theta)^{-\frac{1}{2}} \int_0^\infty \mathfrak{Q}_{-\frac{3}{4}}\left(\frac{\sigma^2+1}{\sin 2\theta}\right) \cos(\pi\sigma x^2)\, d\sigma \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} x\ K_{\frac{1}{4}}(\pi x^2 \cos\theta)\, I_{-\frac{1}{4}}(\pi x^2 \sin\theta). \end{aligned} \tag{1.48}$$

Of course, this is the result we expected: but the proof above also shows that u is no longer in $\mathfrak{A}$ in the case when $\theta = \frac{\pi}{4}$, since then the function w_0 in (1.46) ceases to be analytic at $\sigma = 0$.

As a matter of fact, this example may be connected to the family in the example (i), since one has the so-called Neumann series [17, p. 125]

$$u(x) = \sum_{n\geq 0} \frac{(-1)^n}{n\,!\,\Gamma(\frac{3}{4}+n)}\ \left(\frac{\sin 2\theta}{2}\right)^{2n-\frac{1}{4}}\ (\pi x^2)^{2n+\frac{1}{4}}\ I_{2n-\frac{1}{4}}(\pi x^2). \tag{1.49}$$

(v) Other examples of functions lying in $\mathfrak{A}$, or not lying in that space, will be given in Remark 1.2, at the end of this section.

One last pair of lemmas in the Phragmén–Lindelöf spirit will be useful later.

Lemma 1.9. *Let g be an entire function of one variable such that, for some pair of positive constants C, R, the estimate*

$$|g(z)| \leq C\, e^{2\pi R|z|}, \quad z \in \mathbb{C}, \tag{1.50}$$

holds. If $|g(x)|$ is less than $C\, e^{-\pi\delta|x|}$ for some $\delta > 0$ or if $|g(x)| + |g(ix)|$ goes to zero, as x is real and goes to $\pm\infty$, g is identically zero.

Proof. In the first case, we argue just as in the proof of Lemma 1.6, starting from Lemma 1.5 in place of Lemma 1.4, thus ending up with an application of the Phragmén–Lindelöf lemma in some angle of half-width $< \frac{\pi}{2}$. The second case is easier. □

Lemma 1.10. *Let f be an entire function satisfying some estimate*

$$|f(z)| \leq C\, e^{\pi R|z|^2}, \quad z \in \mathbb{C}, \tag{1.51}$$

together with some estimate

$$|f(x)| \leq C\, e^{-2\pi\delta x^2}, \quad x > 0. \tag{1.52}$$

Assume, moreover, that for every $\varepsilon > 0$*, there exists* $C > 0$ *such that*

$$|f(\pm ix)| \leq C\, e^{\pi\varepsilon x^2}, \quad x > 0. \tag{1.53}$$

Then, for every $\beta \in [0, \frac{\pi}{2}[$*,* $f(z)$ *goes to zero, as* $|z| \to \infty$ *and* $|\mathrm{Arg}\, z| \leq \beta$*, in a uniform way.*

Proof. Since the function $z \mapsto \overline{f(\bar{z})}$ satisfies the same assumptions as f, one may interest oneself in the sector $0 \leq \mathrm{Arg}\, z \leq \beta$ only. Set

$$\Phi_\varepsilon(z) = f(z)\, \exp\left(\pi\varepsilon\, z^2\, e^{-i\alpha}\right) \tag{1.54}$$

for some $\varepsilon \in]0, \delta[$ and some $\alpha \in [0, \frac{\pi}{2}[$ to be determined later. From Lemma 1.4, one gets

$$|\Phi_\varepsilon(x\, e^{i\theta})| \leq C\, e^{-\pi(\delta-\varepsilon)x^2}, \qquad x > 0, \qquad 0 \leq \theta \leq \theta_0. \tag{1.55}$$

On the other hand,

$$|\Phi_\varepsilon(ix)| = |f(ix)|\, e^{-\pi\varepsilon\, x^2\ \cos\alpha} \tag{1.56}$$

goes to zero as $x \to \infty$ so that, as an application of the Phragmén–Lindelöf lemma, $\Phi_\varepsilon(x\, e^{i\theta})$ goes to zero, as $x \to \infty$, uniformly for $\theta_0 \leq \theta \leq \frac{\pi}{2}$. Now, with $z = x + iy$,

$$\mathrm{Re}\ \left(z^2\, e^{-i\alpha}\right) = (x^2 - y^2)\, \cos\alpha + 2xy\, \sin\alpha \tag{1.57}$$

is ≥ 0 provided that $\frac{x}{y} \geq \frac{1-\sin\alpha}{\cos\alpha}$, an expression that is less than $\frac{\cos\beta}{\sin\beta}$ if α is chosen close enough from $\frac{\pi}{2}$. □

Proposition 1.11. *For any* complex y, η*, the transformation*

$$\pi(y, \eta) = e^{2i\pi\, (\eta\, Q - y\, P)}$$

defined by the equation (1.1) *preserves the space* $\mathfrak{A}$*.*

Proof. Abbreviating $\pi(y, 0) = e^{-2i\pi y P}$ as τ_y, one may verify that $\tau_y u$ is given, in the $\mathbb{C}^2$-realization, as

$$(h_0, h_1) = \left(\frac{1}{2}(\tau_y\, f_0 + \tau_{-y}\, f_0 + \tau_y\, f_1 - \tau_{-y}\, f_1),\, \frac{1}{2}(\tau_y\, f_0 - \tau_{-y}\, f_0 + \tau_y\, f_1 + \tau_{-y}\, f_1)\right) : \tag{1.58}$$

then, the other two components of the $\mathbb{C}^4$-realization of the same function are

$$\begin{aligned}(h_{i,0}, h_{i,1}) = \Bigg(&\frac{1}{2}(\tau_{iy}\, f_{i,0} + \tau_{-iy}\, f_{i,0} + i\, \tau_{iy}\, f_{i,1} - i\, \tau_{-iy}\, f_{i,1}),\\ &\frac{1}{2}(-i\, \tau_{iy}\, f_{i,0} + i\, \tau_{-iy}\, f_{i,0} + \tau_{iy}\, f_{i,1} + \tau_{-iy}\, f_{i,1})\Bigg)\ :\end{aligned} \tag{1.59}$$

as a consequence of Lemma 1.4, all the components are nice in the sense of Definition 1.1. We also need the explicit formulas relative to $\pi(0,\eta) = e^{2i\pi\eta Q}$ abbreviated as τ^η: the $\mathbb{C}^4$-realization $\boldsymbol{g}$ of $\tau^\eta\, u$ is given as

$$(g_0,g_1)=\left(\frac{1}{2}(\tau^\eta f_0+\tau^{-\eta} f_0+\tau^\eta f_1-\tau^{-\eta} f_1),\,\frac{1}{2}(\tau^\eta f_0-\tau^{-\eta} f_0+\tau^\eta f_1+\tau^{-\eta} f_1)\right) \tag{1.60}$$

and

$$\begin{aligned}(g_{i,0},g_{i,1}) =\Bigg(&\frac{1}{2}(\tau^{i\eta}\, f_{i,0}+\tau^{-i\eta}\, f_{i,0}-i\,\tau^{i\eta}\, f_{i,1}+i\,\tau^{-i\eta}\, f_{i,1}), \\ &\frac{1}{2}(i\,\tau^{i\eta}\, f_{i,0}-i\,\tau^{-i\eta}\, f_{i,0}+\tau^{i\eta}\, f_{i,1}+\tau^{-i\eta}\, f_{i,1})\Bigg).\end{aligned} \tag{1.61}$$

□

Proposition 1.12. *On analytic functions of x on the real line, define the operator Q as the operator of multiplication by x, and the operator P as $\frac{1}{2i\pi}\frac{d}{dx}$. The space $\mathfrak{A}$ is preserved under the action of the algebra generated by Q and P.*

Proof. In the $\mathbb{C}^4$-realization, the operator Q or P expresses itself as $\boldsymbol{f} \mapsto \boldsymbol{h}$ with

$$\boldsymbol{h}(z) = (z\, f_1(z),\ z\, f_0(z),\, z\, f_{i,1}(z),\, -z\, f_{i,0}(z)) \tag{1.62}$$

in the first case, and

$$\boldsymbol{h} = \frac{1}{2i\pi}\,(f_1',\ f_0',\, -f_{i,1}',\, f_{i,0}') \tag{1.63}$$

in the second one. □

Obviously, the multiplication by z preserves the space of nice functions introduced in Definition 1.1, and the same holds for the operation of taking the derivative by virtue of Lemma 1.4 (together with Cauchy's integral formula for the derivative).

It is immediate to check how some basic symmetries on $\mathfrak{A}$ transfer to the $\mathbb{C}^4$-realization: the formulas below thus constitute a proof that the symmetries under examination do preserve $\mathfrak{A}$.

Proposition 1.13. *Define the linear operators $\mathcal{R}$ (for* rotation*) and $\mathcal{R}^2$ by the equations*

$$(\mathcal{R}^2\, u)(z) = u(-z), \qquad (\mathcal{R}\, u)(z) = u(iz)\,; \tag{1.64}$$

define the antilinear operator C (for conjugation*) by the equation*

$$(Cu)(x) = \bar{u}(x) \ \text{ if } x\in\mathbb{R} \qquad \text{or} \ \ (Cu)(z) = \overline{u(\bar{z})},\ z\in\mathbb{C}. \tag{1.65}$$

The operations $\mathcal{R}^2$, $\mathcal{R}$, C transfer respectively, in the $\mathbb{C}^4$-realization, to the operations

$$\boldsymbol{f} = (f_0,\, f_1,\, f_{i,0},\, f_{i,1}) \mapsto \boldsymbol{h} = (h_0,\, h_1,\, h_{i,0},\, h_{i,1}) \tag{1.66}$$

with

$$\begin{pmatrix} h_0 \\ h_1 \\ h_{i,0} \\ h_{i,1} \end{pmatrix} = \begin{pmatrix} f_0 \\ -f_1 \\ f_{i,0} \\ -f_{i,1} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} f_{i,0} \\ -i\, f_{i,1} \\ f_0 \\ -i\, f_1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} Cf_0 \\ Cf_1 \\ Cf_{i,0} \\ Cf_{i,1} \end{pmatrix}. \tag{1.67}$$

We can now define the scalar product on $\mathfrak{A}$, at the same time proving that it is non-degenerate.

Proposition 1.14. *Let* $(\ |\ )$ *be the scalar product on* $\mathfrak{A}$ *defined, in the* $\mathbb{C}^4$*-realization, as*

$$(\boldsymbol{h}\,|\,\boldsymbol{f}) \tag{1.68}$$
$$= 2^{\frac{1}{2}} \int_0^\infty \left(\bar{h}_0(x) f_0(x) + \bar{h}_1(x) f_1(x) + \bar{h}_{i,0}(x) f_{i,0}(x) - \bar{h}_{i,1}(x) f_{i,1}(x)\right) dx.$$

This scalar product is non-degenerate.

Proof. Obviously, the subspaces of $\mathfrak{A}$ consisting of all even (*resp.* odd) functions are orthogonal with respect to $(\ |\ )$. On $\mathfrak{A}_{\text{even}}$, the scalar product is positive-definite. On the other hand, it follows from (1.62) and (1.68) that the operator Q is self-adjoint with respect to $(\ |\ )$: the non-degeneracy of the scalar product on the odd part of $\mathfrak{A}$ is then a consequence of its non-degeneracy on the even part together with the equation $(Q^2u|u) = (Qu|Qu)$. □

Proposition 1.15. *Let* $\mathfrak{M}$ *denote the space of all entire functions* m *satisfying for some pair* (R, C) *of positive numbers the estimate*

$$|m(z)| \le C\, e^{\pi R|z|^2}, \qquad z \in \mathbb{C} \tag{1.69}$$

and the property that, for every $\varepsilon > 0$, *one has for some* $C > 0$ *the estimate*

$$|m(x)| + |m(ix)| \le C\, e^{\pi \varepsilon x^2}, \qquad x \in \mathbb{R}. \tag{1.70}$$

Then, for every $u \in \mathfrak{A}$, *the product* mu *belongs to* $\mathfrak{A}$ *as well. The intersection* $\mathfrak{M} \cap \mathfrak{A}$ *reduces to zero.*

Proof. If $u \in \mathfrak{A}$ is associated to the vector $\boldsymbol{f}$ as before, the function mu is then associated to the vector $\boldsymbol{h}$, with

$$\begin{aligned} h_0(z) &= m_{\text{even}}(z)\, f_0(z) + m_{\text{odd}}(z)\, f_1(z), \\ h_1(z) &= m_{\text{odd}}(z)\, f_0(z) + m_{\text{even}}(z)\, f_1(z) \end{aligned} \tag{1.71}$$

and

$$\begin{aligned} h_{i,0}(z) &= m_{\text{even}}(iz)\, f_{i,0}(z) - i\, m_{\text{odd}}(iz)\, f_{i,1}(z), \\ h_{i,1}(z) &= i\, m_{\text{odd}}(iz)\, f_{i,0}(z) + m_{\text{even}}(iz)\, f_{i,1}(z), \end{aligned} \tag{1.72}$$

which proves the first part.

Since the assumptions relative to m are invariant under the symmetry $\mathcal{R}^2$ introduced in Proposition 1.13, one may, in the proof of the second part, deal separately with the even and odd parts of m; in view of Proposition 1.12, one may even consider only the case when m is even. Thus, assuming this to be the case, and that $m \in \mathfrak{M} \cap \mathfrak{A}$, let $\boldsymbol{f}$, reducing in this case to $(f_0, 0, f_{i,0}, 0)$, be the vector associated with m. Since m is the even part of f_0, (1.8) yields the equations

$$\begin{aligned} (1-i)\, f_0(ix) &= (1+i)\, f_{i,0}(x) - 2i\, m(ix), \\ (1-i)\, f_0(-ix) &= 2\, m(ix) - (1+i)\, f_{i,0}(x), \end{aligned} \tag{1.73}$$

which show, since $m \in \mathfrak{M}$, that $|f_0(ix)|$ is, for every $\varepsilon > 0$, a $\mathrm{O}(e^{\pi\varepsilon x^2})$ as $x \to \pm\infty$. Lemma 1.10 thus shows that $f_0(z)$ goes to zero, as $|z| \to \infty$, in any closed sector contained in the half-plane $\mathrm{Re}\, z > 0$. Exchanging the roles of f_0 and $f_{i,0}$, and using the result already obtained for f_0, one finds that $f_0(z)$ also goes to zero, as $|z| \to \infty$, in any closed sector contained in the quadrant defined by $-\pi < \mathrm{Arg}\, z < -\frac{\pi}{2}$ or $\frac{\pi}{2} < \mathrm{Arg}\, z < \pi$. One concludes with the help of the Phragmén–Lindelöf lemma together with Liouville's theorem. □

We now define a substitute for the notion of integral, to wit a translation-invariant linear form on $\mathfrak{A}$.

Proposition 1.16. *If $\boldsymbol{f} = (f_0, f_1, f_{i,0}, f_{i,1})$ is the $\mathbb{C}^4$-realization of some function $u \in \mathfrak{A}$, set*

$$\mathrm{Int}\,[u] = 2^{\frac{1}{2}} \int_0^\infty (f_0(x) + f_{i,0}(x))\, dx. \tag{1.74}$$

For every $y \in \mathbb{C}$, with $(e^{-2i\pi\, yP}\, u)(z) = u(z-y)$, one has

$$\mathrm{Int}\,[e^{-2i\pi\, yP}\, u] = \mathrm{Int}\,[u]. \tag{1.75}$$

Proof. Set $v = e^{-2i\pi yP}\, u$. From the proof of Proposition 1.11, one has

$$\begin{aligned} 2^{\frac{1}{2}}\, \mathrm{Int}\,[v] = &\int_0^\infty [f_0(x-y) + f_0(x+y) + f_{i,0}(x-iy) + f_{i,0}(x+iy)]\, dx \\ &+ \int_0^\infty [f_1(x-y) - f_1(x+y) + i\, f_{i,1}(x-iy) - i\, f_{i,1}(x+iy)]\, dx. \end{aligned} \tag{1.76}$$

The second line is

$$\begin{aligned} \int_{-y}^{y} f_1(z)\, dz + i \int_{-iy}^{iy} f_{i,1}(z)\, dz &= \int_{-y}^{y} f_1(z)\, dz - \int_{-y}^{y} f_{i,1}(iz)\, dz \\ &= \int_{-y}^{y} \frac{1+i}{2}\, (f_1(z) - f_1(-z))\, dz = 0. \end{aligned} \tag{1.77}$$