Operator Theory: Advances and Applications
Vol. 164

Pseudo-Differential Operators and Related Topics

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Birkhäuser Verlag
Basel · Boston · Berlin

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2000 Mathematics Subject Classification: Primary 35A17, 35L15, 35L40, 35L45, 35L60, 35L80, 35P05, 35S05, 35S30, 42C15, 43A80, 46F05, 47G30, 53D12, 58J40, 60G12, 60G15, 81R30; Secondary 17B45, 35B65, 35C20, 35J50, 35L30, 42C40, 44A05, 45C05, 46G10, 53D05, 53D50, 58B15, 58D30, 58J30, 60G57

A CIP catalogue record for this book is available from the Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 3-7643-7513-2 Birkhäuser Verlag, Basel – Boston – Berlin

Member of the BertelsmannSpringer Publishing Group
Printed on acid-free paper produced from chlorine-free pulp. TCF∞
Cover design: Heinz Hiltbrunner, Basel
Printed in Germany
ISBN-10: 3-7643-7513-2
ISBN-13: 978-3-7643-7513-3
e-ISBN: 3-7643-7514-0

9 8 7 6 5 4 3 2 1

www.birkhauser.ch

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Preface

As a satellite conference to the Fourth Congress of European Mathematics held at Stockholm University in 2004, the International Conference on Pseudo-Differential Operators and Related Topics was held at Växjö University in Sweden from June 22 to June 25, 2004. The conference was supported by Växjö University, the FIRB Research Group on Microlocal Analysis of Università di Torino, and the International Society for Analysis, its Applications and Computation (ISAAC). The conference was well attended by about 50 mathematicians from Bulgaria, Canada, Denmark, England, Finland, Germany, Italy, Japan, Mexico, Serbia and Montenegro, Russia and Sweden.

The conference covered a broad spectrum of topics related to pseudo-differential operators such as partial differential equations, quantization, Wigner transforms and Weyl transforms on Lie groups, mathematical physics, time-frequency analysis and stochastic processes. The speakers were enthusiastic about the prospect of publishing articles based on their presentations in a volume to be published in Professor Israel Gohberg's prestigious series entitled "Operator Theory: Advances and Applications".

All contributions from speakers have been carefully refereed and the articles collected in this volume give a representative flavour of the mathematics presented at the conference. This volume is a permanent record of the conference and a valuable complement to the volume "Advances in Pseudo-Differential Operators" published in the same series in 2004, which is devoted to the Special Session on Pseudo-Differential Operators at the Fourth ISAAC Congress held at York University in August 2003.

Operator Theory:
Advances and Applications, Vol. 164, 1–21

Strongly Elliptic Second Order Systems with Spectral Parameter in Transmission Conditions on a Nonclosed Surface

M.S. Agranovich

Abstract. We consider a class of second order strongly elliptic systems in $\mathbb{R}^n$, $n \geq 3$, outside a bounded nonclosed surface S with transmission conditions on S containing a spectral parameter. Assuming that S and its boundary γ are Lipschitz, we reduce the problems to spectral equations on S for operators of potential type. We prove the invertibility of these operators in suitable Sobolev type spaces and indicate spectral consequences. Simultaneously, we prove the unique solvability of the Dirichlet and Neumann problems with boundary data on S.

Mathematics Subject Classification (2000). Primary 35P05; Secondary 35J50, 45C05.

Keywords. Strong ellipticity, transmission condition, spectral equation, Lipschitz surface, surface potential, variational approach, Wiener–Hopf method.

1. Introduction

1.1. Statement of the Problems

We consider the second order system of partial differential equations

$$L_\omega(\partial)u(x) := L_0(\partial)u(x) + \omega^2 u(x) = 0 \tag{1.1}$$

in $\mathbb{R}^n \setminus S$, where S is an $(n-1)$-dimensional surface with $(n-2)$-dimensional boundary γ. More precisely, we assume that S is a part of a closed surface[1] Γ; the latter consists of two open parts $S = \Gamma_+$ and Γ_- without common points and of their common boundary γ. By ∂ we denote the "vector" of partial derivatives $\partial_j = \partial/\partial_j$, $j = 1, \ldots, n$. The operator L_0 is homogeneous with respect to differentiations and has constant coefficients. The numerical parameter $\omega = \omega_1 + i\omega_2$ belongs to

The work was supported by the grant of RFFI No. 04-01-00914.

[1] A closed surface is a compact surface without boundary.

the closed upper half-plane ($\omega_2 \geq 0$) and is nonnegative if it is real. At infinity, the solutions are subjected to natural radiation or decay conditions depending on ω. The assumptions are formulated more precisely below in Subsection 1.2. We assume that $n \geq 3$ to avoid the consideration of logarithmic potentials. The surface Γ divides the complement of itself into a bounded simply connected domain Ω^+ and an unbounded domain Ω^-. The superscripts $^+$ and $^-$ will also be used to denote the boundary values of functions on the inner and outer sides of Γ, respectively. By $\nu = \nu(x)$ we denote the unit outward normal at points $x \in \Gamma$.

Our main goal is to consider two spectral problems for (1.1) with transmission conditions on S containing a spectral parameter λ. The corresponding spaces will be specified later.

Problem I.

$$u^+ = u^-, \quad Tu^- - Tu^+ = \lambda u^\pm \quad \text{on } S. \tag{1.2}$$

Problem II.

$$Tu^+ = Tu^-, \quad Tu^\pm = \lambda[u^- - u^+] \quad \text{on } S. \tag{1.3}$$

Here Tu is the conormal derivative, see (1.8) below. In the simplest case of the Helmholtz equation, Tu is the normal derivative $\partial_\nu u$. We wish to describe some spectral properties of these problems. We will see shortly that they are closely connected with the non-spectral Dirichlet and Neumann problems for system (1.1):

The Dirichlet problem.

$$u^\pm = f \quad \text{on } S. \tag{1.4}$$

The Neumann problem.

$$Tu^\pm = g \quad \text{on } S. \tag{1.5}$$

The surfaces Γ and γ are assumed to be either C^∞ or Lipschitz; Γ is connected, while γ can consist of several components. The normal $\nu(x)$ is defined almost everywhere in the Lipschitz case.

In the case of a closed surface $S = \Gamma$, Problems I and II and some other problems for the Helmholtz equation were posed by the physicists Katsenelenbaum and his collaborators Sivov and Voitovich in the 60s. See the book [1] and its revised English edition [2]. In [1], a mathematical supplement [3] is contained, written by the author of the present paper, with the analysis of these and similar problems in acoustics and electrodynamics by tools of the theories of surface potentials and pseudo-differential operators. The surface was assumed to be smooth. The initial results were obtained by the author in collaboration with his graduate student Golubeva; see also her note [4]. Conditions (1.2) and (1.3) can be interpreted as related to a half-transparent screen.

Similar problems with boundary and transmission conditions on a closed Lipschitz surface for the Helmholtz equation were considered in [5] and for the Lamé system in elasticity theory (and $n = 3$) in [6]. The general case of systems (1.1) was considered in [7] and [8]. In [8], systems with variable coefficients were included into consideration. More precisely, in the last three papers the surface is assumed to be either smooth or Lipschitz. Of course, no theory of elliptic pseudodifferential operators exists in the case of a Lipschitz surface, but there is an extensive theory of classical surface potentials and non-spectral problems; see [7, 8] and numerous references therein.

In Section 2, we recall some definitions and technical tools from [7, 8] related to the case of a closed surface. We also add some supplementary material. In particular, we introduce the hypersingular operator and Calderón projections for general systems (1.1) following, e.g., the paper [9] on the Laplace and Helmholtz equations.

Non-spectral problems (Dirichlet, Neumann, and more general) with data on a nonclosed surface for the Helmholtz equation and the Lamé system were first considered by Stephan [10, 11] and then by Costabel and Stephan [12]. In elasticity theory, a non-closed surface has the meaning of a crack. The Lamé system models an isotropic medium. Cracks in anisotropic elastic media were considered by Duduchava, Natroshvili and Shargorodski [13] and by some other authors. Moreover, non-spectral problems in elasticity were considered with much more general conditions on S and in much more general spaces than in the present paper, see also, e.g., [14] and references therein.

These authors followed Vishik and Eskin (e.g., see [15], [16]) and Eskin [17] and used the Wiener–Hopf method assuming that Γ and γ are sufficiently smooth.

In Section 3, the main in the present paper, we will show that it is possible to consider Lipschitz surfaces Γ and γ using the simplest Sobolev type spaces $H^{\pm 1/2}(S)$ and $\widetilde{H}^{\pm 1/2}(S)$. (The spaces are defined in Subsection 2.5.) Instead of the Wiener–Hopf method, a modification of the classical variational approach is used (see our Propositions 3.2 and 3.4 for the case of pure imaginary ω). We prove the unique solvability of the Dirichlet and Neumann problems for general systems (1.1) and the invertibility of the potential type operators on S corresponding to these problems. It seems to us that these non-spectral results are of interest even for the Helmholtz and Lamé equations.

Exactly the same invertible operators occur in the spectral equations on S corresponding to our spectral problems I and II. Thus the simplest spectral results become available in the Lipschitz case; see Subsection 3.4.

In Section 4, we briefly mention some further results; they will be published elsewhere.

1.2. Exact Statement of the Assumptions

(See [7] for details.) The operator $L_0(\partial)$ is an $m \times m$ matrix. Replacing ∂ by $\xi = (\xi_1, \dots, \xi_n)$, we obtain the principal symbol of the operator $-L_0(\partial)$:

$$L_0(\xi) = \sum \sum A_{jk} \xi_j \xi_k, \quad A_{jk} = (a_{jk}^{rs}). \tag{1.6}$$

Here the A_{jk} are real matrices satisfying the symmetry condition $a_{jk}^{rs} = a_{kj}^{sr}$. The matrix $L_0(\xi)$ is assumed to be positive definite for $\xi \neq 0$, which is the strong ellipticity condition for the operator $-L(\partial)$. Besides, as in [7, 8], we impose the additional condition

$$\sum a_{jk}^{rs}\xi_j^r\xi_k^s \geq C_1 \sum |\xi_j^r + \xi_r^j|^2 \tag{1.7}$$

for all real ξ_j^r; here and below, the C_j are positive constants, and summation over all indices is implied. If $m \neq n$, then the ξ_j^r with $j > n$ or $r > m$ are assumed to be zero. Condition (1.7) is satisfied, in particular, for the Helmholtz and Lamé equations.

The conormal derivative at a point x of the boundary is the matrix operator

$$T_x = \sum \nu_j(x) A_{jk}\partial_k. \tag{1.8}$$

If Γ is smooth, then it follows from (1.7) that the Neumann problems, interior and exterior, are elliptic. As to the interior and exterior Dirichlet problems, their ellipticity is the well-known consequence of the strong ellipticity. However, these problems can be considered in Lipschitz domains as well. Here it is essential that, under Condition (1.7), the expression

$$\left\{ \int_{\Omega^\pm} \sum A_{jk}\partial_k u \cdot \partial_j \overline{u}\, dx + \|u\|_{0,\Omega^\pm}^2 \right\}^{1/2}$$

is equivalent to the usual norm $\|u\|_{1,\Omega^\pm}$ in the Sobolev space $H^1(\Omega^\pm)$. This follows from the well-known Korn inequalities (e.g., see [18]).

The conditions at infinity are connected with the choice of a fundamental solution $\mathcal{E}_\omega(x)$. In all cases, this is a matrix function analytic outside the origin.

If $\omega = 0$, then the fundamental solution is defined by the formula

$$\mathcal{E}_0(x) = -F^{-1}L_0(\xi), \tag{1.9}$$

where F^{-1} is the inverse Fourier transform in the sense of distributions. This matrix function is positively homogeneous of degree $2-n$. Accordingly, the conditions at infinity have the form

$$u(x) = O(|x|^{-n+2}), \quad \partial_k u(x) = O(|x|^{-n+1}) \tag{1.10}$$

for all k.

If $\omega_2 > 0$, then the complete symbol $\omega^2 E - L_0(\xi)$ of system (1.1) is a nondegenerate matrix, and the fundamental solution is defined by the formula

$$\mathcal{E}_\omega(x) = F^{-1}[\omega^2 E - L_0(\xi)]. \tag{1.11}$$

It decays exponentially, and accordingly the conditions at infinity have the form

$$u(x) = O(\exp(-\delta|x|)), \quad \partial_k u(x) = O(\exp(-\delta|x|)) \tag{1.12}$$

for all k with some $\delta > 0$.

If $\omega > 0$, an additional condition is imposed on the matrix $L_0(\xi)$ for $\xi \neq 0$. Let $d_l(\xi)$ $(l = 1, \ldots, q)$ be all its pairwise distinct eigenvalues. They are positive.

Condition for $\omega > 0$. The eigenvalues $d_l(\xi)$ have constant multiplicities. The surfaces S_l in $\mathbb{R}^n$ defined by the equations $d_l(\xi) = \omega^2$ are, roughly speaking, convex.

More precisely, they are star-shaped with respect to the origin, the principal curvatures are positive at each point $x \in S_l$, and the radius vector drawn from the origin to $x \in S_l$ forms an acute angle with the outward normal to S_l at x.

Under this condition, the fundamental solution $\mathcal{E}_\omega(x)$ is defined as the limit of $\mathcal{E}_{\omega+i\varepsilon}(x)$ as $\varepsilon \downarrow 0$. Its analysis leads to the following radiation conditions:

$$u(x) = u_1(x) + \cdots + u_q(x), \tag{1.13}$$

where

$$u_l(x) = O(|x|^{-(n-1)/2}), \quad \partial_k u_l(x) - i\xi_k^l(\alpha)u_l(x) = O(|x|^{-(n+1)/2}). \tag{1.14}$$

Here $\alpha = x/|x|$ and $\xi^l = \xi^l(\alpha)$ is the radius vector drawn from the origin to the point on S_l, at which the unit exterior normal vector coincides with α.

In all three cases, $\mathcal{E}_\omega(x)$ satisfies the corresponding conditions at infinity. In the second and third cases, the difference $\mathcal{E}_\omega(x) - \mathcal{E}_0(x)$ has smaller order of singularity at the origin than $\mathcal{E}_0(x)$ and $\mathcal{E}_\omega(x)$, the difference of the orders being equal to 1.

In what follows, the solutions of (1.1) in Ω^- are subjected to conditions at infinity just indicated.

2. Preliminary Material

Here we recall some facts and technical tools related to the case of a closed surface $S = \Gamma$. The omitted details and proofs or references can be found mainly in [7] and [8]. Subsections 2.2–2.4 contain the supplementary material used later in the present paper. In Subsection 2.5, we recall the definitions and some properties of the spaces $H^t(S)$ and $\widetilde{H}^t(S)$ in the case of a nonclosed S, cf., e.g., [17], Section 4, and [19], Section 3.

2.1. Surface Potentials and Integral Formulas in the Case of a Smooth Closed Boundary

In the first three subsections, we assume that Γ and all functions are infinitely smooth. System (1.1) and boundary problems are considered in the classical setting.

First, we note that the exterior Dirichlet and Neumann problems are uniquely solvable. The corresponding classes of functions will be indicated in Subsection 2.3.

Using the fundamental solution described above, we introduce the single layer potential ($x \in \mathbb{R}^n$) and the double layer potential ($x \notin \Gamma$):

$$\mathcal{A}_\omega\varphi(x) = \int_\Gamma \mathcal{E}_\omega(x-y)\varphi(y)\,dS_y, \quad \mathcal{B}_\omega\psi(x) = \int_\Gamma [T_y\mathcal{E}_\omega(x-y)]'\psi(x)\,dS_y, \tag{2.1}$$

where $'$ stands for transposition. These functions are solutions of (1.1) in $\Omega^\pm$ and satisfy our conditions at infinity.

Denote by $A_\omega\varphi(x)$ the restriction of the single layer potential to Γ and by $B_\omega\psi(x)$ the direct value of the double layer potential on Γ. The first of these operators is an integral operator with a weak singularity; it is an elliptic pseudo-differential operator of order -1. The second operator is a pseudo-differential operator of order not greater than 0; it can be a singular integral operator.

For the boundary values of these potentials and their conormal derivatives, we have

$$\begin{gathered}(\mathcal{A}_\omega\varphi)^\pm = A_\omega\varphi, \quad (\mathcal{B}_\omega\psi)^\pm = (B_\omega \pm \tfrac{1}{2}I)\psi,\\ (T\mathcal{A}_\omega\varphi)^\pm = (B'_\omega \mp \tfrac{1}{2}I)\varphi, \quad (T\mathcal{B}_\omega\psi)^+ = (T\mathcal{B}_\omega\psi)^-.\end{gathered} \tag{2.2}$$

Here the operator B'_ω is the transpose of B_ω:

$$B'_\omega\psi = \int_\Gamma T_x E_\omega(x-y)\,dS_y. \tag{2.3}$$

It is again a pseudo-differential operator of order not greater than 0.

The following formulas for solutions of system (1.1) in $\Omega^\pm$ are true:

$$\begin{aligned}\mathcal{B}_\omega u^+ - \mathcal{A}_\omega(Tu^+) &= \begin{cases} u \text{ in } \Omega^+,\\ 0 \text{ in } \Omega^-,\end{cases}\\ \mathcal{A}_\omega(Tu^-) - \mathcal{B}_\omega u^- &= \begin{cases} u \text{ in } \Omega^-,\\ 0 \text{ in } \Omega^+.\end{cases}\end{aligned} \tag{2.4}$$

Passing to Γ in the upper formulas, we obtain the following relations between the Cauchy data:

$$(\tfrac{1}{2}I - B_\omega)u^+ = -A_\omega(Tu^+), \quad (\tfrac{1}{2}I + B_\omega)u^- = A_\omega(Tu^-). \tag{2.5}$$

Under condition (1.7), the zero order pseudo-differential operators $\frac{1}{2}I \pm B_\omega$ are elliptic.

2.2. The Hypersingular Operator and Calderón Projections

Definition 2.1. We introduce *the hypersingular operator*

$$D_\omega\psi = -(T\mathcal{B}_\omega\psi)^\pm \tag{2.6}$$

on Γ. It is a pseudo-differential operator of order 1.

Applying the operator T to both sides of the upper formulas in (2.4), passing to the boundary Γ, and using (2.2) and (2.5), we obtain

$$Tu^+ = -Du^+ + (\tfrac{1}{2}I - B'_\omega)Tu^+, \quad Tu^- = Du^- + (\tfrac{1}{2}I + B'_\omega)Tu^-. \tag{2.7}$$

The left-hand sides are replaced by zero if we interchange $^+$ and $^-$ in the right-hand sides.

Definition 2.2. We introduce the matrix operators

$$P^+ = \begin{pmatrix} \frac{1}{2}I + B_\omega & -A\\ -D & \frac{1}{2}I - B'_\omega\end{pmatrix}, \quad P^- = \begin{pmatrix} \frac{1}{2}I - B_\omega & A\\ D & \frac{1}{2}I + B'_\omega\end{pmatrix}. \tag{2.8}$$

They possess the following properties.

1. They are bounded operators in the space of column vectors $(\varphi, \psi)'$ with components $\varphi \in H^t(\Gamma)$, $\psi \in H^{t-1}(\Gamma)$ for all t. (Here we mean that φ and ψ are, in turn, column vectors of dimension m.) This space will be denoted by $\mathcal{H}^t(\Gamma)$.

2. If the vector $w^\pm = (u^\pm, Tu^\pm)'$ consists of the Cauchy data of system (1.1) in $\Omega^\pm$, then

$$P^+ w^+ = w^+, \quad P^+ w^- = 0, \quad P^- w^- = w^-, \quad P^- w^+ = 0. \tag{2.9}$$

This follows from (2.5) and (2.7). Here we can assume that $w^\pm \in \mathcal{H}^t(\Gamma)$, with $t \geq 3/2$ in the classical setting of the Dirichlet and Neumann problems and even with $t > 1$ if we additionally use the approach in [20]. The first and third equalities in (2.9) are not only necessary but also sufficient for vectors $w^\pm \in \mathcal{H}^t(\Gamma)$ to consist of Cauchy data for (1.1) in Ω^+ and Ω^-. Indeed, if these relations are satisfied, then the corresponding solutions are reconstructed by upper formulas in (2.4).

3. The relations

$$P^+ + P^- = I_2, \quad (P^+)^2 = P^+, \quad (P^-)^2 = P^-, \quad P^+P^- = P^-P^+ = 0 \tag{2.10}$$

are satisfied, where I_2 is the unit operator in the space of vector functions in question. These relations follow from (2.8) and (2.9).

Thus $P^\pm$ *are the Calderón projections for system* (1.1). (Concerning this notion for general elliptic equations, e.g., see [21].) They define the decomposition of the space $\mathcal{H}^t(\Gamma)$ into the direct sum of Cauchy data subspaces for system (1.1) in Ω^+ and Ω^-.

It follows, say, from the relation $(P^+)^2 = P^+$ that

$$(\tfrac{1}{2}I + B_\omega)(\tfrac{1}{2}I - B_\omega) = A_\omega D_\omega, \quad A_\omega B'_\omega = B_\omega A_\omega, \quad D_\omega B_\omega = B'_\omega D_\omega. \tag{2.11}$$

The second of these relations was obtained in [7] in a different way. From the first relation we see that *the operator D_ω is elliptic. Its principal symbol is expressed by the formula*

$$\sigma_{D_\omega} = \sigma_{A_\omega}^{-1} \sigma_{\frac{1}{2}I + B_\omega} \sigma_{\frac{1}{2}I - B_\omega}. \tag{2.12}$$

The right-hand side is known, see formulas for the principal symbols σ_{A_ω} and $\sigma_{\frac{1}{2}I \pm B'_\omega}$ in [7] or [8]. If A_ω is invertible (has the inverse of order 1), then

$$D_\omega = A_\omega^{-1}(\tfrac{1}{2}I + B_\omega)(\tfrac{1}{2}I - B_\omega) \tag{2.13}$$

(cf. (4.6) and (4.7) in [7]). However, in definition (2.6) we did not assume that A_ω is invertible.

2.3. The Dirichlet and Neumann Problems and Problems of the Form I and II in the Case of a Smooth Closed Boundary

If u is a solution of system (1.1) outside S with $u^+ = u^-$, then it follows from (2.4) that $u = \mathcal{A}_\omega(Tu^- - Tu^+)$. Therefore, *the Dirichlet problems* (1.1), (1.4) (interior and exterior simultaneously) *are reduced to the equation*

$$A_\omega \varphi = f \tag{2.14}$$

on Γ *by the substitution*

$$\varphi = [Tu] := Tu^- - Tu^+. \tag{2.15}$$

Here $f \in H^t(\Gamma)$, $\varphi \in H^{t-1}(\Gamma)$, $u \in H^{t+1/2}(\Omega^+)$, and $u \in H^{t+1/2}_{\mathrm{loc}}(\Omega^-)$,[2] $t > 1$. Besides, the solution u is analytic inside $\Omega^\pm$. Once equation (2.14) is solved, the corresponding solution of the Dirichlet problem is given by the formula

$$u = \mathcal{A}_\omega \varphi. \tag{2.16}$$

The interior Dirichlet problem is uniquely solvable for all ω except for some positive values ω_j tending to $+\infty$ (for which ω^2 is an eigenvalue of the operator $-L_0(\partial)$ in Ω^+ under the homogeneous Dirichlet condition $u^+ = 0$).

The operator A_ω has the inverse (a pseudo-differential operator of order 1) precisely for non-exceptional ω. This inverse can also be defined as follows. We find the solution u of the interior and exterior Dirichlet problems with $u^\pm = f$ and set $A_\omega^{-1} f = [Tu]$.

Problem I *for eigenfunctions is reduced to the equation*

$$\varphi = \lambda A_\omega \varphi \tag{2.17}$$

by the same substitution (2.15). In [7], the spectral properties of A_ω are described.

If u is the solution of (1.1) outside S with $Tu^+ = Tu^-$, then it follows from (2.4) that $u = -\mathcal{B}_\omega(u^- - u^+)$. Taking (2.6) into account, we see that *the Neumann problems* (1.1), (1.5) (interior and exterior) *are reduced to the equation*

$$D_\omega \psi = g \tag{2.18}$$

by the substitution

$$\psi = [u] := u^- - u^+. \tag{2.19}$$

Once this equation is solved, the corresponding solution of the Neumann problem is given by the formula

$$u = -\mathcal{B}_\omega \psi. \tag{2.20}$$

Here $g \in H^{t-1}(\Gamma)$, $\psi \in H^t(\Gamma)$, $u \in H^{t+1/2}(\Omega^+)$, and $u \in H^{t+1/2}_{\mathrm{loc}}(\Omega^-)$ for $t > 1$. Besides, the solution is analytic inside $\Omega^\pm$.

The interior Neumann problem is uniquely solvable for all ω except for some nonnegative values $\omega = \omega_j'$ tending to $+\infty$ (such that $(\omega_j')^2$ is an eigenvalue of $-L_0(\partial)$ in Ω^+ under the homogeneous Neumann condition $Tu^+ = 0$).

The unique solvability of the interior Dirichlet problem is equivalent to the invertibility of the operators A_ω, $\frac{1}{2}I + B_\omega$, and $\frac{1}{2}I + B_\omega'$, and the unique solvability of the interior Neumann problem is equivalent to the invertibility of the operators $\frac{1}{2}I - B_\omega$ and $\frac{1}{2}I - B_\omega'$. In this case, the following formulas are true for the Neumann-to-Dirichlet operators (see (2.5)):

$$u^+ = -(\tfrac{1}{2}I - B_\omega)^{-1} A_\omega Tu^+, \quad u^- = (\tfrac{1}{2}I + B_\omega)^{-1} A_\omega Tu^-. \tag{2.21}$$

[2]Here and below, the subscript $_{\mathrm{loc}}$ may be omitted if $\omega_2 > 0$. We will not repeat this remark.

However, as we already mentioned, the exterior Dirichlet and Neumann problems are always uniquely solvable. The second formula in (2.21) can be modified if ω is exceptional with respect to the interior Dirichlet problem. See [7].

The operator D_ω has the inverse (a pseudo-differential operator of order -1) for all ω except for $\omega = \omega_j'$. It can also be defined as follows. If u is the solution of the interior and exterior Neumann problems with $Tu^\pm = g$ on Γ, then $\psi = D_\omega^{-1} g = [u]$.

Problem II *for eigenfunctions is reduced to the equation*

$$D_\omega \psi = \lambda \psi \tag{2.22}$$

on Γ *by the substitution* (2.19). Essentially, the spectral properties of D_ω were investigated in [7]. More precisely, the operator D_ω^{-1} was there considered. Namely, assuming that the interior Dirichlet and Neumann problems are uniquely solvable for given ω and using (2.21), we considered the operator

$$u^- - u^+ = (\tfrac{1}{2} I + B_\omega)^{-1} (\tfrac{1}{2} I - B_\omega)^{-1} A_\omega T u^\pm. \tag{2.23}$$

If ω is exceptional only for the interior Dirichlet problem, then the corresponding generalization of the second formula in (2.21) can be used. However, it is easier to consider D_ω.

In conclusion, we note that A_ω and D_ω are analogs of the Neumann-to-Dirichlet and Dirichlet-to-Neumann operators for the problems in question with transmission conditions on Γ.

2.4. The Case of a Lipschitz Boundary

Here we again follow [7] and [8], see also references therein. We now assume that the surface Γ is Lipschitz. (In particular, it can have edges and conical points.) Then the Sobolev spaces $H^\tau(\Gamma)$ are defined in general only for $|\tau| \le 1$. The Dirichlet and Neumann problems remain meaningful in a generalized (weak) setting. The solution belongs to $H^{t+1}(\Omega^+)$ and $H^{t+1}_{\mathrm{loc}}(\Omega^-)$, $|t| < 1/2$ (but is analytic in $\Omega^\pm$). The boundary values $u^\pm$ obviously belong to $H^{t+1/2}(\Gamma)$, while the values of $Tu^\pm$ are defined by Green's formulas and belong to $H^{t-1/2}(\Gamma)$. Namely, the following Green formula in Ω^+ is well known (cf., e.g., [22]):

$$-\omega^2 \int_{\Omega^+} u \cdot \overline{v}\, dx = (Tu^+, v^+)_{0,\Gamma} - \int_{\Omega^+} E(u, \overline{v})\, dx, \tag{2.24}$$

where

$$E(u, v) = \sum A_{jk} \partial_k u \cdot \partial_j v \tag{2.25}$$

and $(\cdot,\cdot)_{0,\Gamma}$ is the extension of the standard inner product in $H^0(\Gamma)$ to $H^{t-1/2}(\Gamma) \times H^{-t+1/2}(\Gamma)$ (the latter two spaces are dual with respect to this extension). Assuming that u is a solution to (1.1) in $H^{t+1}(\Omega)$ and v is an arbitrary function in $H^{-t+1}(\Omega^+)$, we define $Tu^+ \in H^{t-1/2}(\Gamma)$ as an anti-linear continuous functional on the space $H^{-t+1/2}(\Gamma)$ of functions v^+ on Γ. To define Tu^-, we write out a similar formula for $\Omega_R^- = \Omega^- \cap O_R$, where O_R is a large ball containing Γ and centered at the origin:

$$-\omega^2 \int_{\Omega_R^-} u \cdot \overline{v}\, dx = \int_{S_R} Tu \cdot \overline{v}\, dS - (Tu^-, v^-)_{0,\Gamma} - \int_{\Omega_R^-} E(u, \overline{v})\, dx. \tag{2.26}$$

If $\omega = 0$ or $\omega_2 > 0$, then we can pass to the limit as $R \to \infty$ and use the formula

$$-\omega^2 \int_{\Omega^-} u \cdot \overline{v}\, dx = -(Tu^-, v^-)_{0,\Gamma} - \int_{\Omega^-} E(u, \overline{v})\, dx. \tag{2.27}$$

Variational arguments are used for $t = 0$, and the unique solvability of the interior Dirichlet and Neumann problems is proved for all ω except for some positive values $\omega_j \to +\infty$ and nonnegative values $\omega_j' \to +\infty$, respectively. These results admit partial extensions to t with $|t| \le 1/2$ and complete extensions to t with $|t| < 1/2$. Namely, in the case of the Dirichlet problem for all systems and in the case of the Neumann problem for some systems, the value $t = 1/2$ can be considered by means of the Rellich identities, which gives the results for $|t| \le 1/2$; see, e.g., [19]. In the case of the Neumann problem for all systems, the extensions to t with $|t| < 1/2$ are obtained with the use of the Savaré theorem on the smoothness of solutions to variational problems in Lipschitz domains. See [7, 8] and references therein, starting from [23].

This approach should be compared with that based on the use of potentials (cf. [22]). Formulas (2.2) remain true for $\varphi \in H^{-1/2+t}(\Gamma)$ and $\psi \in H^{1/2+t}(\Gamma)$ with $|t| < 1/2$. For solutions in $H^{1+t}(\Omega^+)$ and $H^{1+t}_{\mathrm{loc}}(\Omega^-)$, $|t| < 1/2$, the integral representations (2.4) and relations (2.5) on Γ remain true. The formulas for the Calderón projections also remain true. Of course, we have no calculus of pseudo-differential operators. However, A_ω is a bounded operator from $H^{t-1/2}(\Gamma)$ to $H^{t+1/2}(\Gamma)$ and is invertible for all ω except for ω_j. Similarly, D_ω is a bounded operator from $H^{t+1/2}(\Gamma)$ to $H^{t-1/2}(\Gamma)$ and is invertible for all ω except for ω_j'. All this is always true for $|t| < 1/2$; the values $t = \pm 1/2$ will not be used in the present paper.

Now assume that $t = 0$ and $\omega = i\tau$, $\tau > 0$. For the solution of the Dirichlet problems, we have the formulas

$$u = \mathcal{A}_\omega A_\omega^{-1} u^\pm \tag{2.28}$$

and the two-sided estimates

$$\|u^\pm\|_{1/2,\Gamma} \le C_1 \|u\|_{1,\Omega^\pm} \le C_2 \|u^\pm\|_{1/2,\Gamma}. \tag{2.29}$$

Here the first inequality is well known for all $u \in H^1(\Omega^\pm)$ and the second follows from (2.28).

Similarly, for the solution of the Neumann problems we have the formulas

$$u = -\mathcal{B}_\omega D_\omega^{-1} Tu^\pm \tag{2.30}$$

and the two-sided estimates

$$\|Tu^\pm\|_{-1/2,\Gamma} \le C_3 \|u\|_{1,\Omega^\pm} \le C_4 \|Tu^\pm\|_{-1/2,\Gamma}. \tag{2.31}$$

Here the second inequality follows from (2.30). Let us explain the first inequality in (2.31). We have

$$\|Tu^\pm\|_{-1/2,\Gamma} \le C_5 \sup_{v:\|v^\pm\|_{1/2,\Gamma}=1} |(Tu^\pm, v^\pm)_{0,\Gamma}|.$$

Define v as the solution to the Dirichlet problems for system (1.1) with given $v^\pm$. Then the right-hand side is not greater than

$$C_6\|u\|_{1,\Omega^\pm} \sup \|v\|_{1,\Omega^\pm} \le C_7\|u\|_{1,\Omega^\pm}$$

(in view of (2.29) for v), which yields the desired inequality.

The operators A_ω and D_ω with $\omega_1 = 0$ are self-adjoint in the following sense:

$$(A_\omega\varphi_1, \varphi_2)_{0,\Gamma} = (\varphi_1, A_\omega\varphi_2)_{0,\Gamma} \quad (\varphi_j \in H^{-1/2}(\Gamma)), \tag{2.32}$$

$$(D_\omega\psi_1, \psi_2)_{0,\Gamma} = (\psi_1, D_\omega\psi_2)_{0,\Gamma} \quad (\psi_j \in H^{1/2}(\Gamma)). \tag{2.33}$$

Indeed, (2.32) is true for these ω and $\varphi_j \in H^0(\Gamma)$, since the matrix $\mathcal{E}_\omega(x)$ is real symmetric, and it is carried over $\varphi_j \in H^{-1/2}(\Gamma)$ by a passage to the limit. Formula (2.33) follows from the self-adjointness of the operators (2.21) for pure imaginary ω.

2.5. The Spaces $H^\tau(S)$ and $\widetilde{H}^\tau(S)$ on a Nonclosed Surface S

The *space* $H^\tau(S)$ consists, by definition, of the restrictions to S of functions[3] belonging to $H^\tau(\Gamma)$. Here S is considered as an open part of Γ, and for $\tau < 0$ the restriction is understood in the sense of distributions. The norm in $H^\tau(S)$ is defined by the formula

$$\|\psi\|_{\tau,S} = \inf\{\|\phi\|_{\tau,\Gamma} : \ \phi \in H^\tau(\Gamma), \ \phi|_S = \psi\}. \tag{2.34}$$

The *space* $\widetilde{H}^\tau(S)$ is defined as the subspace of $H^\tau(\Gamma)$ consisting of functions supported in $\overline{S}$. The norm in this space is defined by the formula

$$\|\psi\|_{\widetilde{H}^\tau(S)} = \|\psi\|_{\tau,\Gamma}. \tag{2.35}$$

The spaces $H^\tau(\Gamma_-)$ and $\widetilde{H}^\tau(\Gamma_-)$ are defined similarly.

The space $H^\tau(S)$ is the factor space $H^\tau(\Gamma)/\widetilde{H}^\tau(\Gamma_-)$.

If Γ and γ are Lipschitz, then these spaces are defined only for $|\tau| \le 1$. However, even in the case of smooth Γ and γ we need these spaces only for $|\tau| < 1$.

If $\tau_1 < \tau_2$, then $H^{\tau_1}(S) \supset H^{\tau_2}(S)$ and $\widetilde{H}^{\tau_1}(S) \supset \widetilde{H}^{\tau_2}(S)$, the corresponding embeddings are compact, and the spaces with index 2 are dense in the corresponding spaces with index 1.

Obviously, for $\tau \ge 0$ we may view the spaces $H^\tau(S)$ and $\widetilde{H}^\tau(S)$ as spaces of functions defined on S. For $\tau < 0$, the space $H^\tau(S)$ consists of distributions defined in S, and $\widetilde{H}^\tau(S)$ consists of distributions on Γ with supports lying in $\overline{S}$. We allow ourself to say that these distributions are functions defined on S.

For $|\tau| < 1/2$, the spaces $H^\tau(S)$ and $\widetilde{H}^\tau(S)$ can be identified. For $0 \le \tau < 1/2$, this follows from the fact that the continuation by zero on Γ_- is a

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[3]Usual functions or generalized functions, i.e. distributions.

bounded operator from $H^\tau(S)$ to $H^\tau(\Gamma)$ giving a function in $\widetilde{H}^\tau(\Gamma)$ with norm equivalent to the norm of the original function in $H^\tau(S)$. For $-1/2 < \tau < 0$, the possibility of identifying these spaces follows from duality arguments (see below). In [17], it was shown that for $|\tau| < 1/2$ the space $H^\tau(\Gamma)$ is the direct sum of the subspaces $\widetilde{H}^\tau(\Gamma_+)$ and $\widetilde{H}^\tau(\Gamma_-)$. This result for the smooth case is carried over to the Lipschitz case by means of a Lipschitz diffeomorphism.

If the surfaces Γ and γ are smooth, then the linear manifolds $C_0^\infty(S)$ and $C^\infty(\overline{S})$ are dense in $\widetilde{H}^\tau(S)$ and $H^\tau(S)$, respectively. Besides, $C_0^\infty(S)$ is dense in $H^\tau(S)$ for $\tau < 1/2$. (Indeed, the space $H^{\tau_2}(S)$ is dense in $H^{\tau_1}(S)$ for $\tau_1 < \tau_2$, while $C_0^\infty(S)$ is dense in $H^{\tau_2}(S) = \widetilde{H}^{\tau_2}(S)$ for $\tau_2 \in (-1/2, 1/2)$.) In the case of Lipschitz Γ and γ, the first assertion is true for linear manifolds of functions satisfying the Lipschitz condition on S and $\overline{S}$, respectively, with supports in S in the first case.

The spaces $H^\tau(S)$ and $\widetilde{H}^{-\tau}(S)$ are dual to each other with respect to the continuation of the inner product $(\varphi, \psi)_{0,S}$ in $H^0(S)$ from functions belonging to the dense linear manifolds just indicated to the direct product of these spaces. See, e.g., [19].

The spaces $H^\tau(S)$ and $\widetilde{H}^\tau(S)$ form two interpolation scales with respect to the real and complex interpolation methods. This means that if we take two points τ_1 and τ_2, then in each of these scales the space X^τ with intermediate index τ is obtained from the spaces with indices τ_1 and τ_2 by the interpolation rules: for $0 \le \theta \le 1$,

$$[X^{\tau_1}, X^{\tau_2}]_\theta = [X^{\tau_1}, X^{\tau_2}]_{\theta,2} = X^{(1-\theta)\tau_1 + \theta\tau_2}. \tag{2.36}$$

See, e.g., [24].

However, there are two more interpolation scales obtained by pasting together: 1) The scale consisting of $\widetilde{H}^\tau(S)$ for $\tau \le \theta$ and $H^\tau(S)$ for $\tau \ge \theta$; 2) the scale consisting of $H^\tau(S)$ for $t \le \theta$ and $\widetilde{H}^\tau(S)$ for $\tau \ge \theta$. Here the point θ of pasting together is an arbitrary point in $(-1/2, 1/2)$. The possibility of pasting together follows from Wolff's theorem (see [25])[4] and the reiteration theorem [24].

Similar spaces are defined in the case of the space and the half-space instead of Γ and S, respectively.

3. Problems with Transmission Conditions on a Lipschitz Nonclosed Surface in the Simplest Spaces

3.1. The Contents of This Section

In this section, assuming that Γ and γ are Lipschitz, we reduce the Dirichlet and Neumann problems to equations on S and prove the invertibility of the corresponding operators $A_{\omega,S}$ and $D_{\omega,S}$. We use only the spaces $H^{\pm 1/2}(S)$ and $\widetilde{H}^{\pm 1/2}(S)$ and apply a version of the variational approach (cf., e.g., [19] and see references

[4] I obtained this information from V.I. Ovchinnikov and use here a possibility to express him my sincere gratitude.

therein). At the end of the section, we discuss the spectral properties of these operators. They are analogs of the Neumann-to-Dirichlet and Dirichlet-to-Neumann operators for problems in question.

Note that if system (1.1) is satisfied outside S, then

$$[u] = 0 \text{ and } [Tu] = 0 \text{ on } \Gamma_-. \tag{3.1}$$

We denote by p_+ the operator of restriction of functions defined on Γ to S and by E_0 the operator of continuation of functions defined on S to Γ by zero outside S.

3.2. The Dirichlet Problem and Problem I

We introduce the operator

$$A_{\omega,S} := p_+ A_\omega. \tag{3.2}$$

It will be applied to functions defined on Γ and equal to zero on Γ_-. We also can consider it as acting on functions defined on S and extended by zero on Γ_-. Then the right-hand side of (3.2) should be rewritten in the form $p_+ A_\omega E_0$.

If u is the solution to the Dirichlet problem (1.1), (1.4) in $H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus S)$, then we set

$$\varphi = [Tu] \tag{3.3}$$

as in (2.15), but now this is a function in $\widetilde{H}^{-1/2}(S)$, for which we obtain the equation

$$A_{\omega,S}\varphi = f \quad \text{on } S, \tag{3.4}$$

similar to (2.14). The operator $A_{\omega,S}$ acts boundedly from $\widetilde{H}^{-1/2}(S)$ to $H^{1/2}(S)$. For u we have the formula

$$u = \mathcal{A}_\omega \varphi. \tag{3.5}$$

All this follows from the discussion in Subsection 2.4. Conversely, if $f \in H^{1/2}(S)$ and φ is the solution to equation (3.4) in $\widetilde{H}^{-1/2}(S)$, then (3.5) is the solution of the Dirichlet problem (1.1), (1.4) in $H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus S)$. Thus, *we have equivalently reduced the Dirichlet problem* (1.1), (1.4) *to equation* (3.4) *on* S.

Passing to the spectral problem I, we see that if u is an eigenfunction of this problem with eigenvalue λ, then φ satisfies the equation

$$\varphi = \lambda A_{\omega,S}\varphi \quad \text{on } S, \tag{3.6}$$

similar to (2.17). Here we mean that φ in the left-hand side is restricted to S. Conversely, if φ is a solution in $\widetilde{H}^{-1/2}(S)$ to this equation, then (3.5) is a solution of Problem I. We add that taking our identifications into account, we have

$$H^{1/2}(S) \subset H^0(S) = \widetilde{H}^0(S) \subset \widetilde{H}^{-1/2}(S). \tag{3.7}$$

Thus we can treat $A_{\omega,S}$ as a bounded operator in $\widetilde{H}^{-1/2}(S)$ with range contained in $H^{1/2}(S)$ (we will see that actually the range coincides with this space) and hence lying in $\widetilde{H}^{-1/2}(S)$. Therefore, the spectral equation (3.6) make sense. (Another point of view is also possible: in (3.6), we can replace the operator $A_{\omega,S} = p_+ A_\omega$ by $\theta_+ A_\omega$, where $\theta_+(x)$ is the characteristic function of S on Γ. Then φ can be treated as equal to zero in Γ_- on both sides in (3.6).)

Like (2.17), equation (3.6) relates only to eigenfunctions; if $A_{\omega,D}$ has associated functions, their relation to the corresponding solutions of Problem I is somewhat more complicated. We do not discuss it and further will discuss the spectral properties of the operator $A_{\omega,S}$. Thus *Problem I for eigenfunctions is equivalently reduced to equation* (3.6) *on* Γ.

The following theorem is known in the case of smooth S and γ at least for the Helmholtz equation, the Lamé system, and the system of anisotropic elasticity for $n=3$. See additional references in Section 4.

Theorem 3.1. *The Dirichlet problem* (1.1), (1.4) *with* $f \in H^{1/2}(S)$ *has a unique solution in* $H^1_{\mathrm{loc}}(\mathbb{R}^n \setminus S)$, *and the operator* $A_{\omega,S} : \widetilde{H}^{-1/2}(S) \mapsto H^{1/2}(S)$ *is invertible.*

Proof. First, we check the uniqueness in the Dirichlet problem. For $\omega = 0$ or $\operatorname{Im}\omega > 0$, we use the standard arguments. We set $v = u$ in (2.24) and (2.27) and assume that this function belongs to $H^1(\mathbb{R}^n \setminus S)$ and has zero values $u^{\pm}$ on S. Adding left- and right-hand sides and taking (3.1) into account, we obtain

$$-\omega^2 \int_{\mathbb{R}^n} |u|^2 dx = -\int_{\mathbb{R}^n} E(u,\overline{u})\, dx. \tag{3.8}$$

If $\operatorname{Im}\omega > 0$, then $\int |u|^2 dx = 0$ and hence $u = 0$. If $\omega = 0$, then $\int E(u,\overline{u})\, dx = 0$ and hence (cf. [18], Section 3) $u = \mathrm{const} = 0$. The case $\omega > 0$ is somewhat more complicated. In this case, from (2.24) and (2.26) we obtain

$$-\omega^2 \int_{O_R} |u|^2 dx = \int_{S_R} Tu \cdot \overline{u}\, dS - \int_{O_R} E(u,\overline{u})\, dx,$$

and hence

$$\operatorname{Im} \int_{S_R} Tu \cdot \overline{u}\, dx = 0. \tag{3.9}$$

Now the arguments from [7] can be applied, and we find that $u = 0$ in Ω^-. Since u is an analytic function outside S, it vanishes identically.

It follows from the uniqueness in the Dirichlet problem that $A_{\omega,S}$ annihilates only the zero function.

The form

$$\langle \varphi_1, \varphi_2 \rangle_{-1/2,S} = -(A_{\omega,S}\varphi_1, \varphi_2)_{0,S} = -(A_\omega \varphi_1, \varphi_2)_{0,\Gamma} \tag{3.10}$$

is defined and bounded on the space $\widetilde{H}^{-1/2}(S)$. To check the latter equality, it suffices to approximate φ_2 by functions with supports lying inside S. We now need the following assertion (cf. Proposition 7.10 in [7]):

Proposition 3.2. *For* $\omega = i\tau$, $\tau > 0$, *the form* (3.10) *is an inner product in* $\widetilde{H}^{-1/2}(S)$, *the corresponding norm being equivalent to* $\|\varphi\|_{\widetilde{H}^{-1/2}(S)}$.

Proof. From (3.10) and (2.32), we have

$$\langle \varphi_1, \varphi_2 \rangle_{-1/2,S} = -(A_{i\tau}\varphi_1, \varphi_2)_{0,\Gamma} = -(\varphi_1, A_{i\tau}\varphi_2)_{0,\Gamma} = \overline{\langle \varphi_2, \varphi_1 \rangle}_{-1/2,S}$$

for $\varphi_j \in \widetilde{H}^{-1/2}(S)$.

It remains to estimate $\langle\varphi,\varphi\rangle_{-1/2,S}$ by $\|\varphi\|^2_{\widetilde{H}^{-1/2}(S)}$ from below. We define the function u by (3.5). This is a solution to (1.1) outside S belonging to $H^1(\mathbb{R}^n\setminus S)$, and we have

$$[Tu]=\varphi \quad\text{and}\quad u^{\pm}=A_{i\tau}\varphi \quad \text{on } \Gamma. \tag{3.11}$$

From (2.24), (2.27) with $v=u$ and (3.10), we obtain

$$\langle\varphi,\varphi\rangle_{-1/2,S}=\int_{\mathbb{R}^n}[\tau^2|u|^2+E(u,\overline{u})]\,dx \geq C_1\|u\|^2_{1,\mathbb{R}^n\setminus\Gamma}. \tag{3.12}$$

On the other hand, from the same formulas (2.24), (2.27) with $v^+=v^-$ on Γ, we have

$$\tau^2\int_{\mathbb{R}^n}u\cdot\overline{v}\,dx=-([Tu],v)_{0,\Gamma}-\int_{\mathbb{R}^n}E(u,\overline{v})\,dx. \tag{3.13}$$

Assuming that

$$\|v\|_{1,\mathbb{R}^n}\leq C_2\|v^{\pm}\|_{1/2,\Gamma}\leq C_2'\|v\|_{1,\mathbb{R}^n}$$

(we construct v as the solution of the Dirichlet problems in $\Omega^{\pm}$ with given $v^{\pm}$, see (2.29)) and using the left inequality in (2.31), we obtain

$$\|\varphi\|_{-1/2,\Gamma}=\sup_{v:\|v^{\pm}\|_{1/2,\Gamma}=1}|([Tu],v^{\pm})_{0,\Gamma}|\leq C_3\|u\|_{1,\mathbb{R}^n\setminus\Gamma}\sup\|v\|_{1,\mathbb{R}^n}\leq C_4\|u\|_{1,\mathbb{R}^n\setminus\Gamma}.$$

Combining this with (3.12), we obtain the desired estimate. □

Now we return to the proof of the theorem.

If we take f in $H^{1/2}(S)$, then the relation

$$\langle\varphi_1,\varphi_2\rangle_{-1/2,S}=-(f,\varphi_2)_{0,S}\quad(\varphi_2\in\widetilde{H}^{-1/2}(S))$$

uniquely determines $\varphi_1\in\widetilde{H}^{-1/2}(S)$ by the Riesz theorem. Thus we have shown that the equation $A_{i\tau,S}\varphi_1=f$ has a solution in $\widetilde{H}^{-1/2}(S)$. We see that the operator $A_{\omega,S}$ is invertible for pure imaginary ω.

For other ω, this operator is a weak perturbation of the operator just investigated, since the difference of their kernels is by order one less singular and hence is the kernel of a compact operator from $\widetilde{H}^{-1/2}(S)$ to $H^{1/2}(S)$. (Moreover, its range lies even in $H^1(S)$.) Therefore, $A_{\omega,S}$ is a Fredholm operator for each ω and its index is zero. Since $\operatorname{Ker} A_{\omega,S}$ is trivial, we conclude that $A_{\omega,S}$ is invertible. Simultaneously, we have proved the unique solvability of the Dirichlet problem. □

In addition, we note that for $\omega=i\tau$, $\tau>0$, if we make the substitution $\varphi_j=A^{-1}_{\omega,S}\psi_j$ $(j=1,2)$ in (3.10), then we obtain an inner product

$$\langle\psi_1,\psi_2\rangle_{1/2,S}=-(\psi_1,A^{-1}_{\omega,S}\psi_2)_{0,S} \tag{3.14}$$

in $H^{1/2}(S)$. The corresponding norm is equivalent to the usual norm in this space.

3.3. The Neumann Problem and Problem II

Here our considerations are similar to those in the previous subsection (but the results do not follow from the results obtained there).

We introduce the operator

$$D_{\omega,S}\psi = p_+ D_\omega \psi. \tag{3.15}$$

Here ψ is a function on Γ equal to zero on Γ_-. We also can apply this operator to functions defined on S and extended by zero to Γ_-. Then we rewrite the right-hand side in (3.15) in the form $p_+ D_\omega E_0 \psi$.

If u is a solution to the Neumann problem (1.1), (1.5) in $H^1_{\rm loc}(\mathbb{R}^n \setminus S)$, then we set

$$\psi = [u] \tag{3.16}$$

as in (2.19), but now this function belongs to $\widetilde{H}^{1/2}(S)$. By (2.4) and (2.7), we have

$$u = -\mathcal{B}_\omega \psi. \tag{3.17}$$

According to definitions (2.6) and (3.15), we obtain the equation

$$D_{\omega,S}\psi = g \quad \text{on} \ \ S \tag{3.18}$$

for ψ. The operator $D_{\omega,S}$ acts boundedly from $\widetilde{H}^{1/2}(S)$ to $H^{-1/2}(S)$.

Conversely, if $g \in H^{-1/2}(S)$ and ψ is a solution to equation (3.18) in $\widetilde{H}^{1/2}(S)$, then the function (3.17) is a solution to the Neumann problem (1.1), (1.5) in $H^1_{\rm loc}(\mathbb{R}^n \setminus S)$. Thus, *the Neumann problem* (1.1), (1.5) *is equivalently reduced to equation* (3.18) *on* S, similar to equation (2.18) in the case of a closed surface.

Problem II *for eigenfunctions is equivalently reduced to the equation*

$$D_{\omega,S}\psi = \lambda\psi \quad \text{on} \ \ S, \tag{3.19}$$

similar to equation (2.22) in the case of a closed surface. Here we mean that ψ in the right-hand side is restricted to S. The operator $D_{\omega,S}$ can be considered as acting in the space $H^{-1/2}(S)$ and having the domain $\widetilde{H}^{1/2}(S)$. It lies in $H^{-1/2}(S)$, since with our identifications we have

$$\widetilde{H}^{1/2}(S) \subset \widetilde{H}^0(S) = H^0(S) \subset H^{-1/2}(S). \tag{3.20}$$

Therefore, the spectral equation (3.19) make sense. (Another possible point of view is that we can replace the operator $D_{\omega,S} = p_+ D_\omega$ by $\theta_+ D_\omega$. Then ψ on both sides in (3.19) will be equal to zero in Γ_-.) The following theorem is also known in the case of smooth surfaces for the Helmholtz equation, the Lamé system and the system of anisotropic elasticity (for $n = 3$). See also Section 4 for additional references.

Theorem 3.3. *The Neumann problem* (1.1), (1.5) *with* $g \in H^{-1/2}(S)$ *has a unique solution in* $H^1_{\rm loc}(\mathbb{R}^n \setminus S)$, *and the operator* $D_{\omega,S} : \widetilde{H}^{1/2}(S) \mapsto H^{-1/2}(S)$ *is invertible.*

Proof. The uniqueness in the Neumann problem is checked literally in the same way as in the Dirichlet problem. It follows that the operator $D_{\omega,S}$ annihilates only the zero function.

The form

$$< \psi_1, \psi_2 >_{1/2,S} = -(D_{\omega,S}\psi_1, \psi_2)_{0,S} = -(D_\omega\psi_1, \psi_2)_{0,\Gamma} \tag{3.21}$$

is defined and bounded on the space $\widetilde{H}^{1/2}(S)$. The latter equality is true because of possibility to approximate ψ_2 by functions with supports lying inside S.

Proposition 3.4. *For $\omega = i\tau$, $\tau > 0$, the form* (3.21) *is an inner product in $\widetilde{H}^{1/2}(S)$. The corresponding norm is equivalent to the original norm in $\widetilde{H}^{1/2}(S)$.*

Proof. From (3.21) and (2.33), we have

$$< \psi_1, \psi_2 >_{1/2,S} = -(D_\omega\psi_1, \psi_2)_{0,\Gamma} = -(\psi_1, D_\omega\psi_2)_{0,\Gamma} = \overline{< \psi_2, \psi_1 >}_{1/2,S}.$$

Now we estimate $< \psi, \psi >_{1/2,S}$ from below by $\|\psi\|^2_{\widetilde{H}^{1/2}(S)}$. We define the function u by (3.17). Then, according to the second formula in (2.2) and definitions (2.6) and (3.15), we have

$$[u] = \psi \ \text{ and } \ Tu^\pm = D_\omega\psi \quad \text{on } \Gamma. \tag{3.22}$$

From (2.24), (2.27) with $v = u$, and (3.21) we have

$$< \psi, \psi >_{1/2,S} = \int_{\mathbb{R}^n} \left[\tau^2|u|^2 + E(u, \overline{u})\right] dx \geq C_5\|u\|^2_{1,\mathbb{R}^n\setminus\Gamma}. \tag{3.23}$$

On the other hand, from (2.24) and (2.27), by interchanging the letters u and v, we obtain

$$\tau^2 \int_{\mathbb{R}^n} v \cdot \overline{u}\, dx = -(Tv^\pm, [u])_{0,S} - \int_{\mathbb{R}^n} E(v, \overline{u})\, dx. \tag{3.24}$$

Here we assume that v belongs to $H^1(\mathbb{R}^n \setminus S)$ and satisfies the conditions

$$Tv^+ = Tv^- \ \text{ and } \ \|v\|_{1,\Omega^\pm} \leq C_6\|Tv^\pm\|_{-1/2,\Gamma} \leq C_7\|v\|_{1,\Omega^\pm}$$

on Γ (v is constructed as the solution to the Neumann problems in $\Omega^\pm$ with given $Tv^\pm$; see (2.31)). Hence

$$\begin{aligned} \|\psi\|_{1/2,\Gamma} &= \sup\nolimits_{v:\|Tv^\pm\|_{-1/2,\Gamma}=1} |(Tv^\pm, [u])_{0,\Gamma}| \leq C_8\|u\|_{1,\mathbb{R}^n\setminus\Gamma} \sup \|v\|_{1,\mathbb{R}^n\setminus\Gamma} \\ &\leq C_9\|u\|_{1,\mathbb{R}^n\setminus\Gamma}, \end{aligned}$$

and, using (3.23), we obtain the desired estimate

$$< \psi, \psi >_{1/2,S} \geq C_{10}\|\psi\|^2_{\widetilde{H}^{1/2}(S)}.$$

□

Now we finish the proof of Theorem 3.3 in the same way as the proof of Theorem 3.1. For $\omega = i\tau$, $\tau > 0$, taking $g \in H^{-1/2}(S)$, we define a function $\psi_1 \in \widetilde{H}^{1/2}(S)$ by the relation

$$< \psi_1, \psi_2 >_{1/2,S} = -(g, \psi_2)_{0,S} \quad (\psi_2 \in \widetilde{H}^{1/2}(S)). \tag{3.25}$$

It is a solution of the equation $D_{\omega,S}\psi_1 = g$. Thus the operator $D_{\omega,S}$ is invertible for $\omega = i\tau$, $\tau > 0$. For other ω, we obtain the invertibility treating $D_{\omega,S}$ as a weak perturbation of $D_{i\tau,S}$. Simultaneously, we conclude that the Neumann problem is uniquely solvable. □

The substitution $\psi_j = D^{-1}_{\omega,S}\varphi_j$ transforms the form (3.21) into the inner product

$$< \varphi_1, \varphi_2 >_{-1/2,S} = -(\varphi_1, D^{-1}_{\omega,S}\varphi_2)_{0,S} \tag{3.26}$$

in $H^{-1/2}(S)$. The corresponding norm is equivalent to the original norm in this space.

3.4. The Spectral Properties of the Operators $A_{\omega,S}$ and $D_{\omega,S}$

We list and comment these properties without trying to formulate a cumbersome theorem. In 6 and 7, we mention some expected results.

1. For $\omega = i\tau$, $\tau > 0$, the operator $A_{\omega,S}$ is a self-adjoint compact operator in the space $\widetilde{H}^{-1/2}(S)$ with inner product (3.10). Hence there exists an orthonormal basis $\{e_j\}$ in this space consisting of eigenfunctions. Since this operator maps this space continuously onto $H^{1/2}(S)$ and has a bounded inverse, it follows that the functions e_j belong to $H^{1/2}(S)$ and form an orthogonal basis there with respect to the inner product (3.14). More precisely, if $A_{\omega,S}e_j = \lambda_j e_j$, then $\{\lambda_j e_j\}$ is a basis in $H^{1/2}(S)$ orthonormal with respect to (3.14).

The power $(-A_{\omega,S})^\theta$, $0 < \theta < 1$, defines a continuous invertible mapping of $\widetilde{H}^{-1/2}(S)$ onto $H^{\theta-1/2}(S) = \widetilde{H}^{\theta-1/2}(S)$. Therefore, the same eigenfunctions form a basis in these spaces too. It is orthogonal with respect to the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\theta-1/2,S} = ((-A_{\omega,S})^{1-2\theta}\varphi_1, \varphi_2)_{0,S}.$$

In particular, in $L_2(S)$ it is the usual inner product. Here $|\lambda_j|^{1/2}e_j$ form an orthonormal basis.

2. Passing to other ω, we can treat $A_{\omega,S}$ in $\widetilde{H}^{-1/2}(S)$ as a weak perturbation of the operator $A_{i\tau,S}$ just considered. Unfortunately, we cannot go beyond the spaces with indices from $-1/2$ to $1/2$ and therefore can only state that $A_{\omega,S}$ is a relatively compact perturbation of the operator $A_{i\tau,S}$. This property is inherited in all spaces $H^t(S)$, $-1/2 < t \leq 1/2$.

3. The s-numbers of these operators admit the estimate

$$s_j \leq Cj^{-1/(n-1)} \tag{3.27}$$

(see, e.g., [26]).

4. It follows from Assertions 2 and 3 that it is possible to form a basis for the Abel–Lidskii method of summability with brackets of order $n-1+\varepsilon$ in $\widetilde{H}^{-1/2}(S)$ with arbitrarily small $\varepsilon > 0$ consisting of root functions (see formulations and references in [27]). This property is inherited by the same system of root functions in all spaces $H^t(S)$, $-1/2 < t \leq 1/2$.

5. It also follows from Assertion 2 that the characteristic numbers (i.e. the inverses of the eigenvalues) of the non-self-adjoint operator $A_{\omega,S}$ lie in the union of an arbitrarily narrow sectorial neighborhood of the ray $\mathbb{R}_-$ and a neighborhood