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Introduction to Complex Analysis in Several Variables

Birkhäuser Verlag
Basel · Boston · Berlin

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2000 Mathematics Subject Classification 32–01

A CIP catalogue record for this book is available from the
Library of Congress, Washington D.C., USA

Bibliografische Information Der Deutschen Bibliothek
Die Deutsche Bibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie;
detaillierte bibliografische Daten sind im Internet über <http://dnb.ddb.de> abrufbar.

ISBN 3-7643-7490-X Birkhäuser Verlag, Basel – Boston – Berlin

Part of Springer Science+Business Media
Cover design: Micha Lotrovsky, CH-4106 Therwil, Switzerland
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany
ISBN-10: 3-7643-7490-X e-ISBN: 3-7643-7491-8
ISBN-13: 978-3-7643-7490-7

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

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Preface

The idea for this book came when I was an assistant at the Department of Mathematics and Computer Science at the Philipps-University Marburg, Germany. Several times I faced the task of supporting lectures and seminars on complex analysis of several variables and found out that there are very few books on the subject, compared to the vast amount of literature on function theory of one variable, let alone on real variables or basic algebra. Even fewer books, to my understanding, were written primarily with the student in mind. So it was quite hard to find supporting examples and exercises that helped the student to become familiar with the fascinating theory of several complex variables.

Of course, there are notable exceptions, like the books of R.M. Range [9] or B. and L. Kaup [6], however, even those excellent books have a drawback: they are quite thick and thus quite expensive for a student's budget. So an additional motivation to write this book was to give a comprehensive introduction to the theory of several complex variables, illustrate it with as many examples as I could find and help the student to get deeper insight by giving lots of exercises, reaching from almost trivial to rather challenging.

There are not many illustrations in this book, in fact, there is exactly one, because in the theory of several complex variables I find most of them either trivial or misleading. The readers are of course free to have a different opinion on these matters.

Exercises are spread throughout the text and their results will often be referred to, so it is highly recommended to work through them.

Above all, I wanted to keep the book short and affordable, recognizing that this results in certain restrictions in the choice of contents. Critics may say that I left out important topics like pseudoconvexity, complex spaces, analytic sheaves or methods of cohomology theory. All of this is true, but inclusion of all that would have resulted in another frighteningly thick book. So I chose topics that assume only a minimum of prerequisites, i.e., holomorphic functions of one complex variable, calculus of several real variables and basic algebra (vector spaces, groups, rings etc.). Everything else is developed from scratch. I also tried to point out some of the relations of complex analysis with other parts of mathematics. For example, the Convergence Theorem of Weierstrass, that a compactly convergent sequence of holomorphic functions has a holomorphic limit is formulated in the language of functional analysis: the algebra of holomorphic functions is a closed subalgebra of the algebra of continuous functions in the compact-open topology.

Also the exercises do not restrict themselves only to topics of complex analysis of several variables in order to show the student that learning the theory of several complex variables is not working in an isolated ivory tower. Putting the knowledge of different fields of mathematics together, I think, is one of the major joys of the subject. Enjoy !

I would like to thank Dr. Thomas Hempfling of Birkhäuser Publishing for his friendly cooperation and his encouragement. Also, my thanks go to my wife Claudia for her love and constant support. This book is for you!

Chapter 1

Elementary theory of several complex variables

In this chapter we study the n-dimensional complex vector space $\mathbb{C}^n$ and introduce some notation used throughout this book. After recalling geometric and topological notions such as connectedness or convexity we will introduce holomorphic functions and mapping of several complex variables and prove the n-dimensional analogues of several theorems well-known from the one-dimensional case. Throughout this book n, m denote natural numbers (including zero). The set of strictly positive naturals will be denoted by $\mathbb{N}_+$, the set of strictly positive reals by $\mathbb{R}_+$.

1.1 Geometry of $\mathbb{C}^n$

The set $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$ is the n-dimensional complex vector space consisting of all vectors $z = x+iy$,where $x, y \in \mathbb{R}^n$and i is the imaginary unit satisfying $i^2 = -1$.By $\overline{z} = x - iy$ we denote the complex conjugate. $\mathbb{C}^n$ is endowed with the Euclidian inner product

$$(z|w) := \sum_{j=1}^{n} z_j \overline{w_j} \tag{1.1}$$

and the Euclidian norm

$$\|z\|_2 := \sqrt{(z|z)}. \tag{1.2}$$

$\mathbb{C}^n$ endowed with the inner product (1.1) is a complex Hilbert space and the mapping

$$\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}^n, (x, y) \mapsto x + iy$$

is an isometry. Due to the isometry between $\mathbb{C}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$ all metric and topological notions of these spaces coincide.

Remark 1.1.1. Let $p \in \mathbb{N}$ be a natural number ≥ 1. For $z \in \mathbb{C}^n$ the following settings define norms on $\mathbb{C}^n$:

$$\|z\|_\infty := \max_{j=1}^{n} |z_j|$$

and

$$\|z\|_p := \left(\sum_{j=1}^{n} |z_j|^p \right)^{\frac{1}{p}}.$$

$\|.\|_\infty$ is called the *maximum norm*, $\|.\|_p$ is called the p- *norm*. All norms define the same topology on $\mathbb{C}^n$. This is a consequence of the fact that, as we will show now, in finite dimensional space all norms are equivalent.

Definition 1.1.2. Two norms N_1, N_2 on a vector space V are called *equivalent*, if there are constants $c, c' > 0$ such that

$$cN_1(x) \leq N_2(x) \leq c' N_1(x) \text{ for all } x \in V.$$

Proposition 1.1.3. On a finite-dimensional vector space V (over $\mathbb{R}$ or $\mathbb{C}$) all norms are equivalent.

Proof. It suffices to show that an arbitrary norm $\|.\|$ on V is equivalent to the Euclidian norm (1.2), because one shows easily that equivalence of norms is an equivalence relation (Exercise !). Let $\{b_1, \ldots, b_n\}$ be a basis of V and put

$$M := \max \{\|b_1\|, \ldots, \|b_n\|\}.$$

Let $x \in V$, $x = \sum_{j=1}^{n} \alpha_j b_j$ with coefficients $\alpha_j \in \mathbb{C}$. The triangle inequality and Hölder's inequality yield

$$\begin{aligned} \|x\| &\leq \sum_{j=1}^{n} |\alpha_j| \|b_j\| \\ &\leq \left(\sum_{j=1}^{n} |\alpha_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \|b_j\|^2 \right)^{\frac{1}{2}} \\ &\leq \|x\|_2 \sqrt{n} M. \end{aligned}$$

Every norm is a continuous mapping, because $|\|x\| - \|y\|| \leq \|x - y\|$, hence, $\|.\|$ attains a minimum $s \geq 0$ on the compact unit sphere

$$S := \{x \in V \mid \|x\|_2 = 1\}.$$

S is compact by the Heine–Borel Theorem, because $\dim V < \infty$. Since $0 \notin S$ the identity property of a norm, i.e. that $\|x\| = 0$ if and only if $x = 0$, implies that $s > 0$. For every $x \neq 0$ we have

$$\frac{x}{\|x\|_2} \in S,$$

which implies

$$\left\| \frac{x}{\|x\|_2} \right\| \geq s > 0.$$

This is equivalent to $\|x\| \geq s \|x\|_2$. Putting both estimates together gives

$$s \|x\|_2 \leq \|x\| \leq \sqrt{n} M \|x\|_2 ,$$

which shows the equivalence of $\|.\|$ and $\|.\|_2$. □

Exercise 1.1.4. Give an alternative proof of Proposition 1.1.3 using the 1-norm.

Exercise 1.1.5. Show that $\lim_{p\to\infty} \|z\|_p = \|z\|_\infty$ for all $z \in \mathbb{C}^n$.

If we do not refer to a special norm, we will use the notation $\|.\|$ for any norm (not only p-norms).

Example 1.1.6. On infinite-dimensional vector spaces *not* all norms are equivalent. Consider the infinite-dimensional real vector space $\mathcal{C}^1[0,1]$ of all real differentiable functions on the interval $[0,1]$. Then we can define two norms by

$$\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$$

and

$$\|f\|_{\mathcal{C}^1} := \|f\|_\infty + \|f'\|_\infty .$$

The function $f(x) := x^n$, $n \in \mathbb{N}$, satisfies

$$\|f\|_\infty = 1, \ \|f\|_{\mathcal{C}^1} = 1 + n.$$

Since n can be arbitrarily large, there is no constant $c > 0$ such that

$$\|f\|_{\mathcal{C}^1} \leq c \|f\|_\infty$$

for all $f \in \mathcal{C}^1[0,1]$.

Exercise 1.1.7. Show that $\mathcal{C}^1[0,1]$ is a Banach space with respect to $\|.\|_{\mathcal{C}^1}$, but not with respect to $\|.\|_\infty$.

Let us recall some definitions.

Definition 1.1.8. Let E be a real vector space and $x, y \in E$.

1. The *closed segment* $[x,y]$ is the set

$$[x,y] := \{tx + (1-t)y \mid 0 \leq t \leq 1\}.$$

2. The *open segment* $]x,y[$ is the set

$$]x,y[:= \{tx + (1-t)y \mid 0 < t < 1\}.$$

3. A subset $C \subset E$ is called *convex* if $[x, y] \subset C$ for all $x, y \in C$.
4. Let $M \subset V$ be an arbitrary subset. The *convex hull* $\operatorname{conv}(M)$ of M is the intersection of all convex sets containing M.
5. An element x of a compact and convex set C is called an *extremal point* of C if the condition $x \in \,]y, z[$ for some $y, z \in C$ implies that $x = y = z$. The subset of extremal points of C is denoted by $\partial_{ex} C$.

Example 1.1.9. Let $r > 0$ and $a \in \mathbb{C}^n$. The set

$$B_r^n(a) := \{ z \in \mathbb{C}^n \mid \|z - a\| < r \} \tag{1.3}$$

is called the n-dimensional *open ball* with center a and radius r with respect to the norm $\|.\|$. It is a convex set, since for all $z, w \in B_r(a)$ and $t \in [0, 1]$ it follows from the triangle inequality that

$$\|tz + (1 - t) w\| \leq t \|z\| + (1 - t) \|w\| < tr + (1 - t) r = r.$$

The *closed ball* is defined by replacing the $<$ by $\leq$ in (1.3).

Exercise 1.1.10. Show that the closed ball with respect to the p-norm coincides with the topological closure of the open ball. Show that the closed ball is compact and determine all its extremal points.

The open (closed) ball in $\mathbb{C}^n$ is a natural generalization of the open (closed) disc in $\mathbb{C}$. It is, however, not the only one.

Definition 1.1.11. We denote by $\mathbb{R}_+^n$ the set of real vectors of strictly positive components. Let $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ and $a \in \mathbb{C}^n$.

1. The set

$$P_r^n(a) := \{ z \in \mathbb{C}^n | \; |z_j - a_j| < r_j \text{ for all } j = 1, \ldots, n\}$$

is called the open *polycylinder* with center a and polyradius r.

2. The set

$$T_r^n(a) := \{ z \in \mathbb{C}^n | \; |z_j - a_j| = r_j \text{ for all } j = 1, \ldots, n\}$$

is called the *polytorus* with center a and polyradius r. If $r_j = 1$ for all j and $a = 0$ it is called the unit polytorus and denoted $\mathbb{T}^n$.

Remark 1.1.12. The open polycylinder is another generalization of the one- dimensional open disc, since it is the Cartesian product of n open discs in $\mathbb{C}$.Therefore we also use the expression *polydisc*. For $n = 1$, open polycylinder and open ball coincide. $P_r^n(a)$ is also convex.

Lemma 1.1.13. Let C be a convex subset of $\mathbb{C}^n$. Then C is simply connected.

Proof. Let $\gamma : [0,1] \to C$ be a closed curve. Then

$$H : [0,1] \times [0,1] \to \mathbb{C}^n, (s,t) \mapsto s\gamma(0) + (1-s)\gamma(t)$$

defines a homotopy from γ to $\gamma(0)$. Since C is convex we have

$$H(s,t) \in C$$

for all $s,t \in [0,1]$. □

As in the one-dimensional case, the notion of connectedness and of a domain is important in several complex variables. We recall the definition for a general topological space.

Definition 1.1.14. Let X be a topological space.

1. The space X is called *connected*, if X cannot be represented as the disjoint union of two nonempty open subsets of X, i.e., if A, B are open subsets of X, $A \neq \emptyset, A \cap B = \emptyset$ and $X = A \cup B$, then $B = \emptyset$.
2. An open and connected subset $D \subset X$ is called a *domain*.

There are different equivalent characterizations of connected sets stated in the following lemma.

Lemma 1.1.15. Let X be a topological space and $D \subset X$ an open subset. The following statements are equivalent:

1. The set D is a domain.
2. If $A \neq \emptyset$ is a subset of D which is both open and closed, then $A = D$.
3. Every locally constant function $f : D \to \mathbb{C}$ is constant.

Proof. 1. $\Rightarrow$ 2. Let A be a nonempty subset of D which is both open and closed in D. Put $B := D \setminus A$. Then B is open in D, for A is closed, $A \cap B = \emptyset$ and $D = A \cup B$. Since D is connected and $A \neq \emptyset$ we conclude $B = \emptyset$, hence, $A = D$.

2. $\Rightarrow$ 3. Let $c \in D$ and $A := f^{-1}(\{f(c)\})$. In $\mathbb{C}$, sets consisting of a single point are closed (this holds for any Hausdorff space). f is continuous, because f is locally constant, so A is closed in D.Since $c \in A$, the set A is nonempty. Let $p \in A$. Then there is an open neighbourhood U of p, such that $f(x) = f(p) = f(c)$ for all $x \in U$, i.e., $U \subset A$. Thus, A is open. We conclude that $A = D$, so f is constant.

3. $\Rightarrow$ 1. If D can be decomposed into disjoint open nonempty subsets A, B, then

$$f : D \to \mathbb{C}, z \mapsto \begin{cases} 1, & z \in A \\ 0, & z \in B \end{cases}$$

defines a locally constant, yet not constant function □

Remark 1.1.16. In the one-variable case the celebrated Riemann Mapping Theorem states that all connected, simply connected domains in $\mathbb{C}$ are biholomorphically equivalent to either $\mathbb{C}$ or to the unit disc. This theorem is false in the multivariable case. We will later show that even the two natural generalizations of the unit disc, i.e., the unit ball and the unit polycylinder, are not biholomorphically equivalent. This is one example of the far-reaching differences between complex analysis in one and in more than one variable.

Exercise 1.1.17. Let X be a topological space.

1. If $A, B \subset X$, such that $A \subset B \subset \overline{A}$ and A is connected, then B is connected.
2. If X is connected and $f : X \to Y$ is a continuous mapping into some other topological space Y, then $f(X)$ is also connected.
3. The space X is called *pathwise connected*, if to every pair $x, y \in X$ there exists a continuous curve
$$\gamma_{x,y} : [0,1] \to X$$
with $\gamma_{x,y}(0) = x, \gamma_{x,y}(1) = y$. Show that a subset D of $\mathbb{C}^n$ is a domain if and only if D is open and pathwise connected. (Hint: You can use the fact that real intervals are connected.)
4. If $(U_j)_{j \in J}$ is a family of (pathwise) connected sets which satisfies
$$\bigcap_{j \in J} U_j \neq \emptyset,$$
then $\bigcup_{j \in J} U_j$ is (pathwise) connected.
5. Show that for every $R > 0$ and every $n \geq 1$ the set $\mathbb{C}^n \setminus B_R^n(0)$ is pathwise connected.
6. Check the set
$$M := \left\{ z \in \mathbb{C} \;\middle|\; 0 < \operatorname{Re} z \leq 1, \operatorname{Im} z = \sin \frac{1}{\operatorname{Re} z} \right\} \cup [-i, i]$$
for connectedness and pathwise connectedness.

Exercise 1.1.18. We identify the space $M(n, n; \mathbb{C})$ of complex $n \times n$ matrices as a topological space with $\mathbb{C}^{n^2}$ with the usual (metric) topology

1. Show that the set $GL_n(\mathbb{C})$ of invertible matrices is a domain in $M(n, n; \mathbb{C})$.
2. Show that the set $U_n(\mathbb{C})$ of unitary matrices is compact and pathwise connected.
3. Show that the set $P_n(\mathbb{C})$ of self-adjoint positive definite matrices is convex.

Exercise 1.1.19. Let C be a compact convex set.

1. Show that
$$\partial_{ex} C \subset \partial C.$$

2. Let $\overline{P_r^n(a)}$ be a compact polydisc in $\mathbb{C}^n$ and $T_r(a)$ the corresponding polytorus. Show that
$$\partial_{ex} P_r^n(a) = T_r^n(a).$$

Remark 1.1.20. By the celebrated Krein–Milman Theorem (see, e.g.,[11] Theorem VIII.4.4) every compact convex subset C of a locally convex vector space possesses extremal points. Moreover, C can be reconstructed as the closed convex hull of its subset of extremal points:
$$C = \overline{\operatorname{conv}(\partial_{ex} C)}$$

Notation 1.1.21. In the following we will use the expression that some proposition holds *near* a point a or near a set X if there is an open neighbourhood of a resp. X on which it holds.

1.2 Holomorphic functions in several complex variables

1.2.1 Definition of a holomorphic function

Definition 1.2.1. Let $U \subset \mathbb{C}^n$ be an open subset, $f : U \to \mathbb{C}^m, a \in U$ and $\|.\|$ an arbitrary norm in $\mathbb{C}^n$.

1. The function f is called *complex differentiable* at a, if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon, a) > 0$ and a $\mathbb{C}$-linear mapping
$$Df(a) : \mathbb{C}^n \to \mathbb{C}^m,$$
such that for all $z \in U$ with $\|z - a\| < \delta$ the inequality
$$\|f(z) - f(a) - Df(a)(z - a)\| \leq \varepsilon \|z - a\|$$
holds. If $Df(a)$ exists, it is called the *complex derivative* of f in a.

2. The function f is called *holomorphic* on U, if f is complex differentiable at all $a \in U$.

3. The set
$$\mathcal{O}(U, \mathbb{C}^n) := \{ f : U \to \mathbb{C}^m | \text{ f holomorphic}\}$$
is called the set of holomorphic mappings on U. If $m = 1$ we write
$$\mathcal{O}(U) := \mathcal{O}(U, \mathbb{C})$$
and call this set the set of *holomorphic functions* on U.

This definition is independent of the choice of a norm, since all norms on $\mathbb{C}^n$ are equivalent. The proofs of the following propositions are analogous to the real variable case, so we can leave them out.

Proposition 1.2.2.

1. If f is $\mathbb{C}$-differentiable in a, then f is continuous in a.
2. The derivative $Df(a)$ is unique.
3. The set $\mathcal{O}(U, \mathbb{C}^m)$ is a $\mathbb{C}-$ vector space and
$$D(\lambda f + \mu g)(a) = \lambda Df(a) + \mu Dg(a)$$
for all $f, g \in \mathcal{O}(U, \mathbb{C}^m)$ and all $\lambda, \mu \in \mathbb{C}$.
4. (Chain Rule) Let $U \subset \mathbb{C}^n, V \subset \mathbb{C}^m$ be open sets, $a \in U$ and
$$f \in \mathcal{O}(U, V) := \{\varphi : U \to V \mid \varphi \text{ holomorphic}\},$$
$g \in \mathcal{O}\left(V, \mathbb{C}^k\right)$. Then $g \circ f \in \mathcal{O}\left(U, \mathbb{C}^k\right)$ and
$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a).$$
5. Let $U \subset \mathbb{C}^n$ be an open set. A mapping
$$f = (f_1, \dots, f_m) : U \to \mathbb{C}^m$$
is holomorphic if and only if all components $f_1, \dots, f_m$ are holomorphic functions on U.
6. $\mathcal{O}(U)$ is a $\mathbb{C}-$ algebra. If $f, g \in \mathcal{O}(U)$ and $g(z) \neq 0$ for all $z \in U$, then $\frac{f}{g} \in \mathcal{O}(U)$.

Example 1.2.3. Let $U \subset \mathbb{C}^n$ be an open subset and $f : U \to \mathbb{C}$ be a locally constant function. Then f is holomorphic and $Df(a) = 0$ for all $a \in U$.

Proof. Let $a \in U$ and $\varepsilon > 0$. Since f is locally constant there is some $\delta > 0$, such that $f(z) = f(a)$ for all $z \in U$ with $\|z - a\| < \delta$. Therefore
$$\|f(z) - f(a)\| = 0 \leq \varepsilon \|z - a\|$$
for all $z \in U$ with $\|z - a\| < \delta$, i.e., f is holomorphic with $Df(a) = 0$ for all $a \in U$. □

Example 1.2.4. For every $k = 1, \dots, n$ the projection
$$\mathrm{pr}_k : \mathbb{C}^n \to \mathbb{C}, (z_1, \dots, z_n) \mapsto z_k$$
is holomorphic and $D\,\mathrm{pr}_k(a) = e_k$ (the k-th canonical basis vector) for all $a \in \mathbb{C}^n$.

Proof. Let $\varepsilon > 0$ and $a \in \mathbb{C}^n$. Then

$$|\mathrm{pr}_k(z) - \mathrm{pr}_k(a) - (z - a|e_k)| = 0 \leq \varepsilon \|z - a\|$$

for all $z \in \mathbb{C}^n$. □

Example 1.2.5. The complex subalgebra $\mathbb{C}[z_1, \ldots, z_n]$ of $\mathcal{O}(\mathbb{C}^n)$ generated by the constants and the projections is called the *algebra of polynomials.* Its elements are sums of the form

$$\sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

with $c_\alpha \neq 0$ only for finitely many $c_\alpha \in \mathbb{C}$, where for $z \in \mathbb{C}^n$ and $\alpha \in \mathbb{N}^n$ we use the notation

$$z^\alpha := z_1^{\alpha_1} \ldots z_n^{\alpha_n}.$$

The *degree* of a polynomial

$$p(z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ c_\alpha = 0 \text{ for almost all } \alpha}} c_\alpha z^\alpha$$

is defined as

$$\deg p := \max \{\alpha_1 + \cdots + \alpha_n \mid \alpha \in \mathbb{N}^n,\ c_\alpha \neq 0\}.$$

For example, the polynomial $p(z_1, z_2) := z_1^5 + z_1^3 z_2^3$ has degree 6. By convention the zero polynomial has degree $-\infty$.The following formulas for the degree are easily verified:

$$\begin{aligned} \deg(pq) &= \deg p + \deg q, \\ \deg(p+q) &\leq \max\{\deg p, \deg q\}. \end{aligned}$$

Exercise 1.2.6. Show that for all $z, w \in \mathbb{C}^n$ and all $\alpha \in \mathbb{N}^n$ there exists a polynomial $q \in \mathbb{C}[z, w]$ of degree $|\alpha| := \|\alpha\|_1$ such that

$$(z + w)^\alpha = z^\alpha + q(z, w).$$

Exercise 1.2.7. Show that the polynomial algebra $\mathbb{C}[z_1, \ldots, z_n]$ has no zero divisors.

Exercise 1.2.8. Show that the zero set of a complex polynomial in $n \geq 2$ variables is not compact in $\mathbb{C}^n$. (*Hint*: Use the Fundamental Theorem of Algebra). Compare this to the case $n = 1$.

Exercise 1.2.9. Show that every (affine) linear mapping $L : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic. Compute $DL(a)$ for all $a \in \mathbb{C}^n$.

Exercise 1.2.10. Let $U_1, \ldots, U_n$ be open sets in $\mathbb{C}$ and let $f_j : U_j \to \mathbb{C}$ be holomorphic functions, $j = 1, \ldots, n$.

1. Show that $U := U_1 \times \cdots \times U_n$ is open in $\mathbb{C}^n$.
2. Show that the functions

$$f : U \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto \prod_{j=1}^{n} f_j(z_j)$$

and

$$g : U \to \mathbb{C}, (z_1, \ldots, z_n) \mapsto \sum_{j=1}^{n} f_j(z_j)$$

are holomorphic on U.

1.2.2 Basic properties of holomorphic functions

We turn to the multidimensional analogues of some important theorems from the one variable case. The basic tool to this end is the following observation.

Lemma 1.2.11. Let $U \subset \mathbb{C}^n$ be open, $a \in U, f \in \mathcal{O}(U), b \in \mathbb{C}^n$ and $V := V_{a,b;U} := \{t \in \mathbb{C} \mid a + tb \in U\}$. Then V is open in $\mathbb{C}$, $0 \in V$ and the function

$$g_{a,b} : V \to \mathbb{C}, t \mapsto f(a + tb)$$

is holomorphic.

Proof. From $a \in U$ follows that $0 \in V$. If $b = 0$ then $V = \mathbb{C}$. Let $b \neq 0$. If $t_0 \in V$ then $z_0 := a + t_0 b \in U$. Since U is open, there is some $\varepsilon > 0$, such that $B_\varepsilon(z_0) \in U$. Put $z_t := a + tb$. Then

$$\|z_0 - z_t\| = \|b\| \, |t_0 - t| < \varepsilon$$

for all t with $|t_0 - t| < \frac{\varepsilon}{\|b\|}$, i.e., $B_{\frac{\varepsilon}{\|b\|}}(t_0) \subset V$. Since $g_{a,b}$ is the composition of the affine linear mapping $t \mapsto a + tb$ and the holomorphic function f, holomorphy of $g_{a,b}$ follows from the chain rule. □

Conclusion 1.2.12. *We have analogues of the following results from the one-dimensional theory.*

1. **Liouville's Theorem**: *Every bounded holomorphic function*

$$f : \mathbb{C}^n \to \mathbb{C}$$

is constant.

2. **Identity Theorem**: *Let $D \subset \mathbb{C}^n$ be a domain, $a \in D, f \in \mathcal{O}(D)$, such that $f = 0$ near a. Then f is the zero function.*
3. **Open Mapping Theorem**: *Let $D \subset \mathbb{C}^n$ be a domain, $U \subset D$ an open subset and $f \in \mathcal{O}(D)$ a non-constant function. Then $f(U)$ is open, i.e., every holomorphic function is an open mapping. In particular, $f(D)$ is a domain in $\mathbb{C}$.*

4. **Maximum Modulus Theorem**: *If $D \subset \mathbb{C}^n$ is a domain, $a \in D$ and $f \in \mathcal{O}(D)$, such that $|f|$ has a local maximum at a, then f is constant.*

Proof. 1. Let $a, b \in \mathbb{C}^n$. The function $g_{a,b-a}$ from Lemma 1.2.11 is holomorphic on $\mathbb{C}$, satisfies

$$g_{a,b-a}(0) = f(a), g_{a,b-a}(1) = f(b)$$

and

$$g_{a,b-a}(\mathbb{C}) \subset f(\mathbb{C}^n).$$

Since f is bounded, $g_{a,b-a}$ is bounded. By the one-dimensional version of Liouville's Theorem $g_{a,b-a}$ is constant, hence, $f(a) = f(b)$ for all $a, b \in \mathbb{C}^n$.

2. Let

$$U := \{z \in D \mid f = 0 \text{ near } z\}.$$

By prerequisite $a \in U$. U is closed in D, because either $U = D$ (if f is the zero function) or, by continuity of f, to every $z \in D \setminus U$ there exists a neighbourhood W, on which f does not vanish, i.e., $W \subset D \setminus U$. Let $c \in U \cap D$. There is a polyradius $r \in \mathbb{R}^n_+$, such that the polycylinder $P_r(c)$ is contained in D and such that $P_r(c) \cap U \neq \emptyset$. Choose some $z \in P_r(c)$ and $w \in P_r(c) \cap U$. From Lemma 1.2.11 we obtain that the set $V_{w,z-w;D}$ is open in $\mathbb{C}$ and because $P_r(c)$ is convex, we have $[0,1] \subset V_{w,z-w;D}$. Since f vanishes near w, there exists an open and connected neighbourhood $W \subset \mathbb{C}$ of $[0,1]$ on which $g_{w,z-w}$ vanishes. This implies that $P_r(c) \subset U$, so U is open in D. However, since D is connected, the only nonempty open and closed subset of D is D itself. Hence, $U = D$, i.e., $f = 0$ on D.

3. $f(D)$ is connected, because D is connected and f is continuous (cf. Exercise 1.1.17). We have to show that $f(U)$ is open. Let $b \in f(U)$. There is some $a \in U$ with $b = f(a)$. Since U is open, there is a polycylinder $P_r(a) \subset U$. By the Identity Theorem f is not constant on $P_r(c)$, since otherwise f would be constant on all of D, contradicting the prerequisites. This implies that there is some $w \in \mathbb{C}^n, w \neq 0$, such that $g_{a,w}$ from Lemma 1.2.11 is not constant on $V = V_{a,w;P_r(a)}$. From the one-dimensional theory we obtain that $g_{a,w}(V)$ is an open neighbourhood of b. Because

$$b \in g_{a,w}(V) \subset f(P_r(a)) \subset f(U),$$

$f(U)$ is a neighbourhood of b. Since b was arbitrary, $f(U)$ is open in $\mathbb{C}$.

4. $f(D)$ is open in $\mathbb{C}$. Since

$$|.| : \mathbb{C} \to [0, +\infty[$$

is an open mapping (Exercise !), the assertion follows. □

Corollary 1.2.13 (Maximal Modulus Principle for bounded domains). *Let $D \subset \mathbb{C}^n$ be a bounded domain and $f : \overline{D} \to \mathbb{C}$ be a continuous function, whose restriction to D is holomorphic. Then $|f|$ attains a maximum on the boundary ∂D.*

Proof. Since D is bounded, the closure $\overline{D}$ is compact by the Heine–Borel Theorem. Thus, the continuous real-valued function $|f|$ attains a maximum in a point $p \in \overline{D}$. If $p \in \partial D$ we are done. If $p \in D$ the Maximum Modulus Theorem says that $f|_D$ is constant. By continuity, f is constant on $\overline{D}$ and thus $|f|$ attains a maximum also on ∂D. □

In the one-dimensional version of the Identity Theorem it is sufficient to know the values of a holomorphic function on a subset of a domain, which has an accumulation point. This is no longer true in more than one dimension.

Example 1.2.14. The holomorphic function

$$f : \mathbb{C}^2 \to \mathbb{C}, (z, w) \mapsto zw$$

is not identically zero, yet it vanishes on the subsets $\mathbb{C}\times \{0\}$ and $\{0\} \times \mathbb{C}$ of $\mathbb{C}^2$, which clearly have accumulation points in $\mathbb{C}^2$.

Exercise 1.2.15. Let $U \subset \mathbb{C}^n$ be an open set. Show that U is a domain if and only if the ring $\mathcal{O}(U)$ is an integral domain, i.e., it has no zero divisors.

Exercise 1.2.16. Let $D \subset \mathbb{C}^n$ be a domain and $\mathcal{F} \subset \mathcal{O}(D)$ be a family of holomorphic functions. We denote by

$$N(\mathcal{F}) := \{z \in D \mid f(z) = 0 \text{ for all } f \in \mathcal{F}\}$$

the common zero set of the family $\mathcal{F}$.

1. Show that either $D \setminus N(\mathcal{F}) = \emptyset$ or $D \setminus N(\mathcal{F})$ is dense in D.
2. Show that $GL_n(\mathbb{C})$ is dense in $M(n, n; \mathbb{C})$.

Exercise 1.2.17. Consider the mapping

$$f : \mathbb{C}^2 \to \mathbb{C}^2, (z, w) \mapsto (z, zw)$$

Show that f is holomorphic, but is not an open mapping. Does this contradict the Open Mapping Theorem?

Exercise 1.2.18. Let

$$f : X \to E$$

be an open mapping from a topological space X to a normed space E. State and prove a Maximum Modulus Theorem for f.

Exercise 1.2.19. Let $D \subset \mathbb{C}^n$ be a domain, $B \subset D$ an open and bounded subset, such that also the closure $\overline{B}$ is contained in D. Let ∂B denote the topological boundary of B and $f \in \mathcal{O}(D)$. Show that

$$\partial(f(B)) \subset f(\partial B).$$

Does this also hold in general, if B is unbounded?