Operator Theory: Advances and Applications
Vol. 161

The State Space Method Generalizations and Applications

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Editors

Linear
Operators &
Linear
Systems

Birkhäuser Verlag
Basel · Boston · Berlin

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2000 Mathematics Subject Classification 47Axx, 93Bxx

A CIP catalogue record for this book is available from the
Library of Congress, Washington D.C., USA

Bibliographic information published by Die Deutsche Bibliothek
Die Deutsche Bibliothek lists this publication in the Deutsche Nationalbibliografie; detailed
bibliographic data is available in the Internet at <http://dnb.ddb.de>.

ISBN 3-7643-7370-9 Birkhäuser Verlag, Basel – Boston – Berlin

Part of Springer Science+Business Media
Printed on acid-free paper produced from chlorine-free pulp. TCF∞
Cover design: Heinz Hiltbrunner, Basel
Printed in Germany
ISBN-10: 3-7643-7370-9
ISBN-13: 978-3-7643-7370-2
e-ISBN: 3-7643-7431-4

9 8 7 6 5 4 3 2 1

www.birkhauser.ch

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Editorial Introduction

This volume of the *Operator Theory: Advances and Applications* series (OTAA) is the first volume of a new subseries. This subseries is dedicated to connections between the theory of linear operators and the mathematical theory of linear systems and is named *Linear Operators and Linear Systems* (LOLS). As the existing subseries *Advances in Partial Differential Equations* (ADPE), the new subseries will continue the traditions of the OTAA series and keep the high quality of the volumes. The editors of the new subseries are: Daniel Alpay (Beer–Sheva, Israel), Joseph Ball (Blacksburg, Virginia, USA) and André Ran (Amsterdam, The Netherlands).

In the last 25–30 years, Mathematical System Theory developed in an essential way. A large part of this development was connected with the use of the state space method. Let us mention for instance the "theory of H_∞ control". The state space method allowed to introduce in system theory the modern tools of matrix and operator theory. On the other hand the state space approach had an important impact on Algebra, Analysis and Operator Theory. In particular it allowed to solve explicitly some problems from interpolation theory, theory of convolution equations, inverse problems for canonical differential equations and their discrete analogs. All these directions are planned to be present in the subseries LOLS. The editors and the publisher are inviting authors to submit their manuscripts for publication in this subseries.

This volume contains five essays. The essay of D. Arov and O. Staffans, *State/signal linear time-invariant systems theory, part I: discrete time systems*, contains new results in classical system theory. The essays of D. Alpay and D.S. Kalyuzhnyĭ-Verbovetzkiĭ, *Matrix-J-unitary non-commutative rational formal power series*, and of J.A. Ball, G. Groenewald and T. Malakorn, *Conservative structured noncommutative multidimensional linear systems* are dedicated to a new branch in Mathematical system theory where discrete time is replaced by the free semigroup with N generators. The essay of I. Gohberg, I. Haimovici, M.A. Kaashoek and L. Lerer, *The Bezout integral operator: main property and underlying abstract scheme* contains new applications of the state space method to the theory of Bezoutiants and convolution equations. The essay of D. Alpay and I. Gohberg *Discrete analogs of canonical systems with pseudo-exponential potential. Definitions and formulas for the spectral matrix functions* is concerned with new results and formulas for the discrete analogs of canonical systems.

Daniel Alpay, Israel Gohberg

Operator Theory:
Advances and Applications, Vol. 161, 1–47

Discrete Analogs of Canonical Systems with Pseudo-exponential Potential. Definitions and Formulas for the Spectral Matrix Functions

Daniel Alpay and Israel Gohberg

Abstract. We first review the theory of canonical differential expressions in the rational case. Then, we define and study the discrete analogue of canonical differential expressions. We focus on the rational case. Two kinds of discrete systems are to be distinguished: one-sided and two-sided. In both cases the analogue of the potential is a sequence of numbers in the open unit disk (Schur coefficients). We define the characteristic spectral functions of the discrete systems and provide exact realization formulas for them when the Schur coefficients are of a special form called strictly pseudo-exponential.

Mathematics Subject Classification (2000). 34L25, 81U40, 47A56.

Contents

1. Introduction

Canonical differential expressions are differential equations of the form

$$-iJ\frac{\partial\Theta}{\partial x}(x,\lambda)=\lambda\Theta(x,\lambda)+v(x)\Theta(x,\lambda),\quad x\geq 0,\ \lambda\in\mathbb{C},\tag{1.1}$$

where

$$v(x)=\begin{pmatrix}0 & k(x)\\ k(x)^* & 0\end{pmatrix},\quad J=\begin{pmatrix}I_n & 0\\ 0 & -I_n\end{pmatrix},$$

and where $k\in\mathbf{L}_1^{n\times n}(\mathbb{R}_+)$ is called the potential. Such systems were introduced by M.G. Kreĭn (see, e.g., [37], [38]).

Associated to (1.1) are a number of functions of λ, which we called in [10] the *characteristic spectral functions* of the canonical system. These are:

1. The asymptotic equivalence matrix function $V(\lambda)$.
2. The scattering function $S(\lambda)$.
3. The spectral function $W(\lambda)$.
4. The Weyl function $N(\lambda)$.
5. The reflection coefficient function $R(\lambda)$.

Direct problems consist in computing these functions from the potential $k(x)$ while inverse problems consist in recovering the potential from one of these functions.

In the present paper we study discrete counterparts of canonical differential expressions. To present our approach, we first review various facts on the telegraphers' equations. By the term *telegraphers' equations*, one means a system of differential equations connecting the voltage and the current in a transmission line. The case of lossy lines can be found for instance in [45] and [18]. We here consider the case of lossless lines and follow the arguments and notations in [16, Section 2], [19, p. 110–111] and [46]. The telegraphers' equations which describe the evolution of the voltage $v(x,t)$ and current $i(x,t)$ in a lossless transmission line can be given as:

$$\begin{aligned}\frac{\partial v}{\partial x}(x,t)+Z(x)\frac{\partial i}{\partial t}(x,t)&=0\\ \frac{\partial i}{\partial x}(x,t)+Z(x)^{-1}\frac{\partial v}{\partial t}(x,t)&=0.\end{aligned}\tag{1.2}$$

In these equations, $Z(x)$ represents the local impedance at the point x. *A priori* there may be points where $Z(x)$ is not continuous, but it is important to bear in mind that voltage and current will be continuous at these points.

Let us assume that $Z(x)>0$ and is continuously differentiable on an interval (a,b), and introduce the new variables

$$\begin{aligned}V(x,t)&=Z(x)^{-1/2}v(x,t),\\ I(x,t)&=Z(x)^{1/2}i(x,t),\end{aligned}$$

and

$$W_R(x,t) = \frac{V(x,t) + I(x,t)}{2},$$
$$W_L(x,t) = \frac{V(x,t) - I(x,t)}{2}.$$

Then the function

$$W(x,t) = \begin{pmatrix} W_R(x,t) \\ W_L(x,t) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} Z(x)^{-1/2} & Z(x)^{1/2} \\ Z(x)^{-1/2} & -Z(x)^{1/2} \end{pmatrix} \begin{pmatrix} v(x,t) \\ i(x,t) \end{pmatrix} \tag{1.3}$$

satisfies the differential equation, also called *symmetric two components wave equation* (see [16, equation (2.6) p. 362], [46, p. 256], [19, equation (3.3) p. 111])

$$\frac{\partial W(x,t)}{\partial x} = -J\frac{\partial W(x,t)}{\partial t} + \begin{pmatrix} 0 & -\kappa(x) \\ -\kappa(x) & 0 \end{pmatrix} W(x,t),$$

where

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \kappa(x) = \frac{Z'(x)}{2Z(x)}. \tag{1.4}$$

We distinguish two cases:

(a) The case where $Z(x) > 0$ and is continuously differentiable on $\mathbb{R}_+$. Taking the (inverse) Fourier transform $f \mapsto \widehat{f}(\lambda) = \int_{\mathbb{R}} e^{i\lambda t} f(t)dt$ on both sides we get to a *canonical differential expressions* (also called *Dirac type system*), with $k(x) = i\kappa(x)$ and $\Theta(x,\lambda) = \widehat{W}(x,\lambda)$. The theory of *canonical differential expressions* is reviewed in the next section.

(b) The case where $Z(x)$ is constant on intervals $[nh,(n+1)h)$ for some preassigned $h > 0$. We are then lead to discrete systems.

The paper consists of three sections besides the introduction. In Section 2 we review the main features of the continuous case. The third section presents the discrete systems to be studied. These are of two kinds, one-sided and two-sided. Section 3 also contains a study of one-sided systems and of their associated characteristic spectral functions. In Section 4 we focus on two-sided systems and we also present an illustrative example.

In the parallel between the continuous and discrete cases a number of problems remains to be considered to obtain a complete picture. In the sequel to the present paper we study *inverse problems* associated to these first-order systems.

To conclude this introduction we set some definitions and notation. The open unit disk will be denoted by $\mathbb{D}$, the unit circle by $\mathbb{T}$, and the open upper half-plane by $\mathbb{C}_+$. The open lower half-plane is denoted by $\mathbb{C}_-$ and its closure by $\overline{\mathbb{C}_-}$. We will make use of the Wiener algebras of the real line and of the unit circle. These are defined as follows. The Wiener algebra of the real line $\mathcal{W}^{n\times n}(\mathbb{R}) = \mathcal{W}^{n\times n}$ consists of the functions of the form

$$f(\lambda) = D + \int_{-\infty}^{\infty} e^{i\lambda t} u(t)dt \tag{1.5}$$

where $D \in \mathbb{C}^{n\times n}$ and where $u \in \mathbf{L}_1^{n\times n}(\mathbb{R})$. Usually we will not stress the dependence on $\mathbb{R}$. The sub-algebra $\mathcal{W}_+^{n\times n}$ (resp. $\mathcal{W}_-^{n\times n}$) consists of the functions of the form (1.5) for which the support of u is in $\mathbb{R}_+$ (resp. in $\mathbb{R}_-$).

The Wiener algebra $\mathcal{W}(\mathbb{T})$ (we will usually write $\mathcal{W}$ rather than $\mathcal{W}(\mathbb{T})$) of the unit circle consists of complex-valued functions $f(z)$ of the form

$$f(z) = \sum_{\mathbb{Z}} f_\ell z^\ell$$

for which

$$\|f\|_{\mathcal{W}} \stackrel{\text{def.}}{=} \sum_{\mathbb{Z}} |f_\ell| < \infty.$$

2. Review of the continuous case

2.1. The asymptotic equivalence matrix function

We first review the continuous case, and in particular the definitions and main properties of the characteristic spectral functions. See, e.g., [7], [11], [10] for more information. We restrict ourselves to the case where the potential is of the form

$$k(x) = -2ce^{ixa}\left(I_p + \Omega\left(Y - e^{-2ixa^*}Ye^{2ixa}\right)\right)^{-1}(b + i\Omega c^*), \tag{2.1}$$

where $(a, b, c) \in \mathbb{C}^{p\times p} \times \mathbb{C}^{p\times n} \times \mathbb{C}^{n\times p}$ is a triple of matrices with the properties that

$$\cap_{\ell=0}^m \ker ca^\ell = \{0\} \quad \text{and} \quad \cup_{\ell=0}^m \operatorname{Im} a^\ell b = \mathbb{C}^p$$

for m large enough. In system theory, see for instance [30], the first condition means that the pair (c, a) is observable while the second means that the pair (a, b) is controllable. When both conditions are in force, the triple is called minimal. See also [14] for more information on these notions. We assume moreover that the spectra of a and of $a^\times = a - bc$ are in the open upper half-plane. Furthermore Ω and Y in (2.1) belong to $\mathbb{C}^{p\times p}$ and are the unique solutions of the Lyapunov equations

$$\begin{aligned} i(\Omega a^{\times *} - a^\times\Omega) &= bb^*, & (2.2)\\ -i(Ya - a^*Y) &= c^*c. & (2.3)\end{aligned}$$

This class of potentials was introduced in [7] and called in [26] *strictly pseudo-exponential potentials*. Note that both Ω and Y are strictly positive since the triple (a, b, c) is minimal, and that $I_p + \Omega Y$ and $I_p + Y\Omega$ are invertible since

$$\det(I_p + \Omega Y) = \det(I_p + Y\Omega) = \det(I_p + \sqrt{Y}\Omega\sqrt{Y}).$$

Note also that asymptotically,

$$k(x) \sim -2ce^{ixa}(I_p + \Omega Y)^{-1}(b + i\Omega c^*) \tag{2.4}$$

as $x \to +\infty$. Potentials of the form (2.1) can also be represented in a different form; see (2.22).

We first define the asymptotic equivalence matrix function. To that purpose (and here we follow closely our paper [12]) let F, G and T be the matrices given by

$$F = i\begin{pmatrix} -c & 0 \\ 0 & f_1 \end{pmatrix}, \quad T = \begin{pmatrix} ia & 0 \\ 0 & -ia^* \end{pmatrix}, \quad G = \begin{pmatrix} 0 & f_1^* \\ c^* & 0 \end{pmatrix}, \tag{2.5}$$

where $f_1 = (b^* - ic\Omega)(I_p + Y\Omega)^{-1}$.

Theorem 2.1. *Let $Q(x, y)$ be defined by*

$$Q(x, y) = Fe^{xT}(I_{2p} - e^{xT}Ze^{xT})^{-1}e^{yT}G$$

where (F, G, T) are defined by (2.5) *and where Z is the unique solution of the matrix equation*

$$TZ + ZT = -GF.$$

Then the matrix function

$$U(x, \lambda) = e^{i\lambda Jx} + \int_x^\infty Q(x, u)e^{i\lambda Ju}du$$

is the unique solution of (1.1) *with the potential as in* (2.1), *subject to the condition*

$$\lim_{x\to\infty} \begin{pmatrix} e^{-ix\lambda}I_n & 0 \\ 0 & e^{ix\lambda}I_n \end{pmatrix} U(x, \lambda) = I_{2n}, \quad \lambda \in \mathbb{R}. \tag{2.6}$$

Furthermore, the $\mathbb{C}^{n\times n}$-valued blocks in the decomposition of the matrix function $U(0, \lambda) = (U_{ij}(0, \lambda))$ are given by

$$\begin{aligned}
U_{11}(0, \lambda) &= I_n + ic\Omega(\lambda I_p - a^*)^{-1}c^*, \\
U_{21}(0, \lambda) &= (-b^* + ic\Omega)(\lambda I_p - a^*)^{-1}c^*, \\
U_{12}(0, \lambda) &= -c(I_p + \Omega Y)(\lambda I_p - a)^{-1}(I_p + \Omega Y)^{-1}(b + i\Omega c^*), \\
U_{22}(0, \lambda) &= I_n - (ib^*Y + c\Omega Y)(\lambda I_p - a)^{-1}(I_p + \Omega Y)^{-1}(b + i\Omega c^*).
\end{aligned}$$

See [9, Theorem 2.1].

Definition 2.2. *The function $V(\lambda) = U(0, \lambda)$ is called the asymptotic equivalence matrix function.*

The terminology *asymptotic equivalence matrix function* is explained in the following theorem:

Theorem 2.3. *The asymptotic equivalence matrix function has the following property: let $x \in \mathbb{R}$ and ξ_0 and ξ_1 in $\mathbb{C}^{2n}$. Let $f_0(x, \lambda) = e^{i\lambda xJ}\xi_0$ be the $\mathbb{C}^{2n}$-valued solution to* (1.1) *corresponding to $k(x) = 0$ and $f_0(0, \lambda) = \xi_0$ and let $f_1(x, \lambda)$ corresponding to an arbitrary potential k of the form* (2.1), *with $f_1(0, \lambda) = \xi_1$. The two solutions are asymptotic in the sense that*

$$\lim_{x\to\infty} \|f_1(x, \lambda) - f_0(x, \lambda)\| = 0$$

if and only if $\xi_1 = U(0, \lambda)\xi_0$.

For a proof, see [10, Section 2.2].

The asymptotic equivalence matrix function takes J-unitary values on the real line:

$$V(\lambda)JV(\lambda)^* = J, \quad \lambda \in \mathbb{R}.$$

We recall the following: if R be a $\mathbb{C}^{2n\times 2n}$-valued rational functions analytic at infinity, it can be written as $R(\lambda) = D + C(\lambda I_m - A)^{-1}B$, where A, B, C and D are matrices of appropriate sizes. Such a representation of R is called a *realization*. The realization is said to be minimal if the size of A is minimal (equivalently, the triple (A, B, C) is minimal, in the sense recalled above). The McMillan degree of R is the size of the matrix A in any minimal realization. Minimal realizations of rational matrix-valued functions taking J-unitary values on the real line were characterized in [5, Theorem 2.8 p. 192]: R takes J-unitary values on the real line if and only if there exists an Hermitian invertible matrix $H \in \mathbb{C}^{m\times m}$ solution of the system of equations

$$\begin{aligned} i(A^*H - HA) &= C^*JC && (2.7)\\ C &= iJB^*H. && (2.8) \end{aligned}$$

The matrix H is uniquely defined by the minimal realization of R and is called the *associated Hermitian matrix* to the minimal realization matrix function. The matrix function R is moreover J-inner, that is J-contractive in the open upper half-plane:

$$R(\lambda)JR(\lambda) \le J \quad \text{for all points of analyticity in the open upper half-plane,}$$

if and only if $H > 0$. The asymptotic equivalence matrix function $V(\lambda)$ has no pole on the real line, but an arbitrary rational function which takes J-unitary values on the real line may have poles on the real line. See [5] and [4] for examples.

The next theorem presents a minimal realization of the asymptotic equivalence matrix function and its associated Hermitian matrix.

Theorem 2.4. *Let $k(x)$ be given in the form* (2.1)*. Then, a minimal realization of the asymptotic equivalence matrix function associated to the corresponding canonical differential system is given by $V(\lambda) = I_{2n} + C(\lambda I_{2p} - A)^{-1}B$, where*

$$A = \begin{pmatrix} a^* & 0 \\ 0 & a \end{pmatrix}, \quad B = \begin{pmatrix} c^* & 0 \\ 0 & (I_p + \Omega Y)^{-1}(b + i\Omega c^*) \end{pmatrix}$$

and

$$C = \begin{pmatrix} ic\Omega & -c(I_p + \Omega Y) \\ -b^* + ic\Omega & -ib^*Y - c\Omega Y \end{pmatrix},$$

and the associated Hermitian matrix is given by

$$H = \begin{pmatrix} \Omega & i(I_p + \Omega Y) \\ -i(I_p + Y\Omega) & (I_p + Y\Omega)Y \end{pmatrix}.$$

We now prove a factorization result for $V(\lambda)$. We first recall the following: let as above R be a rational matrix-valued function analytic at infinity. The factorization

$R = R_1 R_2$ of R into two other rational matrix-valued functions analytic at infinity (all the functions are assumed to have the same size) is said to be minimal if

$$\deg R = \deg R_1 + \deg R_2.$$

Minimal factorizations of rational matrix-valued functions have been characterized in [14, Theorem 1.1 p. 7]. Assume now that R takes J-unitary values on the real line. Minimal factorizations of R into two factors which are J-unitary on the real line were characterized in [5]. Such factorizations are called *J-unitary factorizations*. To recall the result (see [5, Theorem 2.6 p. 187]), we introduce first some more notations and definitions: let $H \in \mathbb{C}^{p \times p}$ be an invertible Hermitian matrix. The formula

$$[x, y]_H = y^* H x, \qquad x, y \in \mathbb{C}^p$$

defines a non-degenerate and in general indefinite inner product. Two vectors are orthogonal with respect to this inner product if $[x, y]_H = 0$. The orthogonal complement of a subspace $\mathcal{M} \subset \mathbb{C}^p$ is:

$$\mathcal{M}^{[\perp]} = \{x \in \mathbb{C}^p \ ; [x, m]_H = 0 \ \forall m \in \mathcal{M}\}.$$

We refer to [29] for more information on finite-dimensional indefinite inner product spaces.

Theorem 2.5. *Let R be a rational matrix-valued function analytic at infinity and J-unitary on the real line, and let $R(\lambda) = D + C(zI_p - A)^{-1}B$ be a minimal realization of R, with associated matrix H. Let $\mathcal{M}$ be a A-invariant subspace of $\mathbb{C}^p$ non-degenerate with respect to the inner product $[\cdot, \cdot]_H$. Let π denote the orthogonal (with respect to $[\cdot, \cdot]_H$) projection such that*

$$\ker \pi = \mathcal{M}, \quad \mathrm{Im}\ \pi = \mathcal{M}^{[\perp]}$$

and let $D = D_1 D_2$ be a factorization of D into two J-unitary constants. Then $R = R_1 R_2$ with

$$\begin{aligned} R_1(z) &= D_1 + C(zI_p - A)^{-1}(I_p - \pi)BD_2^{-1} \\ R_2(z) &= D_2 + D_1^{-1}C\pi(zI_p - A)^{-1}BD_2 \end{aligned}$$

is a minimal J-unitary factorization of R. Conversely, every J-unitary factorization of R is obtained in such a way.

As a consequence we have:

Theorem 2.6. *Let $V(\lambda)$ be the asymptotic equivalence matrix function of a canonical differential expression (1.1) with potential of the form (2.1). Then it admits a minimal factorization*

$$V(\lambda) = V_1(\lambda) V_2(\lambda)^{-1}$$

where V_1 and V_2 are J-inner and of same degree.

To prove this result we consider the realization of $V(\lambda)$ given in Theorem 2.4 and note that the space $\binom{\mathbb{C}^p}{0}$ is A-invariant and H-non-degenerate (in fact, H-positive). The factorization follows from Theorem 2.5. The fact that V_2 is inner follows from

$$H = \begin{pmatrix} I_p & 0 \\ -i(I_p+Y\Omega)\Omega^{-1} & I_p \end{pmatrix} \begin{pmatrix} \Omega & 0 \\ 0 & -\Omega^{-1}-Y \end{pmatrix} \begin{pmatrix} I_p & 0 \\ -i(I_p+Y\Omega)\Omega^{-1} & I_p \end{pmatrix}^* .$$

To prove this last formula we have used the formula for Schur complements:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & 0 \\ A_{21}A_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{pmatrix} \begin{pmatrix} I & A_{11}^{-1}A_{12} \\ 0 & I \end{pmatrix}$$

for matrices of appropriate sizes and A_{11} being invertible. See [20, formula (0.3) p. 3].

One could have started with the space $\binom{0}{\mathbb{C}^p}$, which is also A-invariant and H-positive. In particular, the above factorization is not unique.

2.2. The other characteristic spectral functions

In this section we review the definitions and main properties of the characteristic spectral functions associated to a canonical differential expression.

It follows from Theorem 2.4 that $U(0,\lambda)$ has no pole on the real line and that, furthermore:

$$\begin{aligned} U_{11}(0,\lambda)U_{11}(0,\lambda)^* - U_{12}(0,\lambda)U_{12}(0,\lambda)^* &= I_n \\ U_{22}(0,\lambda)U_{22}(0,\lambda)^* - U_{21}(0,\lambda)U_{21}(0,\lambda)^* &= I_n \end{aligned}$$

and

$$U_{11}(0,\lambda)^*U_{12}(0,\lambda) = U_{21}(0,\lambda)^*U_{22}(0,\lambda)$$

for real λ.

In particular, $U_{11}(0,\lambda)$ is invertible on the real line and $U_{21}(0,\lambda)U_{11}(0,\lambda)^{-1}$ is well defined and takes contractive values on the real line.

Definition 2.7. *The function*

$$R(\lambda) = (U_{21}(0,\lambda)U_{11}(0,\lambda)^{-1})^* = U_{12}(0,\lambda)U_{22}(0,\lambda)^{-1}, \qquad \lambda\in\mathbb{R},$$

is called the reflection coefficient function.

To present an equivalent definition of the reflection coefficient function, we need some notation: if

$$\Theta = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{C}^{(p+q)\times(p+q)}, \quad A\in\mathbb{C}^{p\times p}, \quad \text{and} \quad X\in\mathbb{C}^{p\times q}$$

we set

$$T_\Theta(X) = (AX+B)(CX+D)^{-1}.$$

Note that

$$T_{\Theta_1\Theta_2}(X) = T_{\Theta_1}(T_{\Theta_2}(X)) \tag{2.9}$$

when all expressions are well defined.

Theorem 2.8. *Let $\Theta(x,\lambda) = U(x,\lambda)U(0,\lambda)^{-1}$. Then, $\Theta(x,\lambda)$ is also a solution of* (1.1). *It is an entire function of λ. It is J-expansive in $\mathbb{C}_+$,*

$$J - \Theta(x,\lambda)J\Theta(x,\lambda)^* \begin{cases} = 0, & \lambda \in \mathbb{R} \\ \leq 0, & \lambda \in \mathbb{C}_+, \end{cases}$$

and satisfies the initial condition $\Theta(0,\lambda) = I_{2n}$. Moreover

$$R(\lambda) = \lim_{x\to\infty} T_{\Theta(x,\lambda)^{-1}}(0), \quad \lambda \in \mathbb{R}. \tag{2.10}$$

The matrix function $\Theta(x,\lambda)$ is called the matrizant, or fundamental solution of the canonical differential expression. Its properties may be found in [22, p. 150]. For real λ the matrix function $U(0,\lambda)$ is J-unitary. Hence we have:

$$\Theta(x,\lambda)^{-1} = U(0,\lambda)U(x,\lambda)^{-1}.$$

The result follows using (2.9) and the asymptotic property (2.6).

In fact, the function R is analytic and takes contractive values in the closed lower half-plane. For a proof and references, see [10] and [13, Theorem 3.1 p 6].

Theorem 2.9. *A minimal realization of $R(\lambda)$ is given by*

$$R(\lambda) = -c(\lambda I_p - (a + i\Omega c^*c))^{-1}(b + i\Omega c^*). \tag{2.11}$$

See [10]. It follows in particular that the spectrum of the matrix $a + i\Omega c^*c$ is in the open upper half-plane. Note that Ω is not arbitrary but is related to a, b and c via the Lyapunov equation (2.2).

A direct proof that R is analytic and contractive in $\mathbb{C}_-$ can be given using the results in [33], as we now explain.

Definition 2.10. *A $\mathbb{C}^{n\times n}$-valued rational function R is called a proper contraction if it takes contractive values on the real line and if moreover it is analytic at infinity and such that*

$$R(\infty)R(\infty)^* < I_n.$$

The following results are respectively [33, Theorem 3.2 p. 231, Theorem 3.4 p. 235].

Theorem 2.11. *Let R be a $\mathbb{C}^{n\times n}$-valued rational function analytic at infinity and let $R(z) = D + C(zI - A)^{-1}B$ be a minimal realization of W. Let*

$$\mathcal{A} = \begin{pmatrix} \alpha & \beta \\ \gamma & \alpha^* \end{pmatrix} = \begin{pmatrix} A + BD^*(I_n - DD^*)^{-1}C & B(I_n - D^*D)^{-1}B^* \\ C^*(I_n - DD^*)^{-1}C & A^* + C^*(I_n - DD^*)^{-1}DB^* \end{pmatrix}.$$

Then the following are equivalent:

1) *The matrix function R is a proper contraction.*
2) *The real eigenvalues of $\mathcal{A}$ have even partial multiplicities.*
3) *The Riccati equation*

$$X\gamma X - iX\alpha^* + i\alpha X + \beta = 0. \tag{2.12}$$

has an Hermitian solution.

The matrix $\mathcal{A}$ is called the state characteristic matrix of W and the Riccati equation (2.12) is called its state characteristic equation.

Theorem 2.12. *Let R be a $\mathbb{C}^{n\times n}$-valued proper contraction, with minimal realization $R(z) = D + C(zI - A)^{-1}B$ and let* (2.12) *be its state characteristic equation. Then, any Hermitian solution of* (2.12) *is invertible and the number of negative eigenvalues of X is equal to the number of poles of R in $\mathbb{C}_-$.*

Consider now the minimal realization (2.11). The corresponding state characteristic equation is

$$Xc^*cX - iX(a^* - icc^*\Omega) + i(a + i\Omega cc^*)X + (b + i\Omega c^*)(b^* - ic\Omega) = 0.$$

To show that $X = \Omega$ is a solution of this equation is equivalent to prove that Ω solves the Lyapunov equation (2.3). Indeed,

$$\begin{aligned}
&0 = \Omega c^*c\Omega - i\Omega(a^* - icc^*\Omega) + i(a + i\Omega cc^*)\Omega + (b + i\Omega c^*)(b^* - ic\Omega)\\
&\quad\Longleftrightarrow\\
&0 = -i\Omega a^* + ia\Omega + bb^* - i\Omega(a - c^*b^*) + i(a - bc)\Omega + bb^*\\
&\quad\Longleftrightarrow\\
&0 = i(a^\times\Omega - \Omega a^{\times *}) + bb^*,
\end{aligned}$$

which is (2.3).

The *scattering matrix function* is defined as follows:

Theorem 2.13. *The differential equation* (1.1) *has a uniquely defined $\mathbb{C}^{2n\times n}$-valued solution such that for $\lambda \in \mathbb{R}$,*

$$\begin{aligned}
\begin{pmatrix} I_n & -I_n \end{pmatrix} X(0, \lambda) &= 0,\\
\lim_{x\to\infty} \begin{pmatrix} 0 & e^{ix\lambda}I_n \end{pmatrix} X(x, \lambda) &= I_n.
\end{aligned}$$

The limit

$$\lim_{x\to\infty} \begin{pmatrix} e^{-ix\lambda}I_n & 0 \end{pmatrix} X(x, \lambda) = S(\lambda)$$

exists for all real λ and is called the scattering matrix function of the canonical system. The scattering matrix function takes unitary values on the real line, belongs to the Wiener algebra $\mathcal{W}$ and admits a factorization $S = S_+S_-$ where S_+ and its inverse are analytic in the closed upper half-plane while S_- and its inverse are analytic in the closed lower half-plane.

We note that the general factorization of a function in the Wiener algebra and unitary on the real line involves in general a diagonal term taking into account quantities called partial indices; see [31], [32], [34], [17]. We also note that conversely, functions with the properties as in the theorem are scattering matrix functions of a more general class of differential equations; see [41] and the discussion in [7, Appendix].

Theorem 2.14. *The scattering matrix function of a canonical system* (1.1) *with potential* (2.1) *is given by:*

$$\begin{aligned} S(\lambda) &= (I_n + b^*(\lambda I_p - a^*)^{-1}c^*)^{-1} \\ &\quad \times (I_n - (ib^*Y - c)(\lambda I_p - a)^{-1}(I_p + \Omega Y)^{-1}(b + i\Omega c^*)). \end{aligned}$$

A minimal realization of the scattering matrix function is given by $S(\lambda) = I_n + C(\lambda I_{2p} - A)^{-1}B$, *where*

$$\begin{aligned} A &= \begin{pmatrix} a & b(ic\Omega - b^*) \\ 0 & a^{\times *} \end{pmatrix}, \\ B &= \begin{pmatrix} b \\ (I_p + Y\Omega)^{-1}(c^* + iYb) \end{pmatrix}, \\ C &= (c \quad ic\Omega - b^*). \end{aligned}$$

Set

$$G = \begin{pmatrix} -\Omega & iI_p \\ -iI_p & -Y(I_p + \Omega Y)^{-1} \end{pmatrix}.$$

Then it holds that

$$\begin{aligned} i(AG - GA^*) &= -BB^*, \\ CG &= iB^*, \end{aligned}$$

and thus S *takes unitary values on the real line.*

For a proof, see [8, p. 7]. The last statement follows from [5, Theorem 2.1 p. 179], that is from equations (2.7) and (2.8) with $H = X^{-1}$ and $J = I_p$. Since

$$X = \begin{pmatrix} I_p & 0 \\ i\Omega^{-1} & I_p \end{pmatrix} \begin{pmatrix} -\Omega & 0 \\ 0 & (\Omega + \Omega Y \Omega)^{-1} \end{pmatrix} \begin{pmatrix} I_p & 0 \\ i\Omega^{-1} & I_p \end{pmatrix}^*$$

the space $\begin{pmatrix} \mathbb{C}^p \\ 0 \end{pmatrix}$ is A invariant and H-negative. Thus Theorem 2.5 on factorizations leads to:

Theorem 2.15. *The scattering matrix function of a canonical system* (1.1) *with potential* (2.1) *admits a minimal factorization of the form*

$$S(z) = U_1(z)^{-1}U_2(z)$$

where both U_1 *and* U_2 *are inner* (*that is, are contractive in* $\mathbb{C}_+$ *and take unitary values on the real line*).

The fact that U_2 is inner (and not merely unitary) stems from the fact that the Schur complement of $-\Omega$ in H is equal to

$$-Y(I_p + \Omega Y)^{-1} - iI_p(-\Omega)^{-1}(-iI_p) = (\Omega + \Omega Y \Omega)^{-1}$$

and in particular is strictly positive.

Such a factorization result was also proved in [12, Theorem 7.1] using different methods. It is a particular case of a factorization result of M.G. Kreĭn and H. Langer for functions having a finite number of negative squares; see [39].

We now turn to the spectral function. We first recall that the operator

$$Hf(x) = -iJ\frac{\mathrm{d}\,f}{\mathrm{d}\,x}(x) - v(x)f(x)$$

restricted to the space of $\mathbb{C}^{2n}$-valued absolutely continuous functions with entries in $\mathbf{L}_2$ and such that

$$\begin{pmatrix} I_n & -I_n \end{pmatrix} f(0) = 0$$

is self-adjoint.

Definition 2.16. *A positive function $W : \mathbb{R} \to \mathbb{C}^{n\times n}$ is called a spectral function if there is a unitary map U from $\mathbf{L}_2^n$ onto $\mathbf{L}_2^n(W)$ mapping H onto the operator of multiplication by the variable in $\mathbf{L}_2^n(W)$.*

Theorem 2.17. *The function*

$$W(\lambda) = (V_{22}(\lambda) - V_{12}(\lambda))^{-*}(V_{22}(\lambda) - V_{12}(\lambda))^{-1}$$

is a spectral function, the map U being given by

$$F(\lambda) = \frac{1}{\sqrt{2\pi}}\int_0^\infty \begin{pmatrix} I_n & I_n \end{pmatrix} \Theta(x,\lambda)^* f(x)dx. \tag{2.13}$$

A direct proof in the rational case can be found in [26]. When $k(x) \equiv 0$, we have that $W(\lambda) = I_n d\lambda$, and the unitary map (2.13) is readily identified with the Fourier transform.

Definition 2.18. *The Weyl coefficient function $N(\lambda)$ is defined in the open upper half plane; it is the unique $\mathbb{C}^{n\times n}$-valued function such that*

$$\int_0^\infty \begin{pmatrix} iN(\lambda)^* & I_n \end{pmatrix} \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \Theta(x,\lambda)^*\Theta(x,\lambda) \begin{pmatrix} I_n & I_n \\ I_n & -I_n \end{pmatrix} \begin{pmatrix} -iN(\lambda) \\ I_n \end{pmatrix} dx$$

is finite for $-i(\lambda - \lambda^*) > 0$.

In the setting of differential expressions (1.1), the function N was introduced in [27]. The motivation comes from the theory of the Sturm-Liouville equation. The Weyl coefficient function is analytic in the open upper half-plane and has a nonnegative imaginary part there. Such functions are called Nevanlinna functions.

Theorem 2.19. *The Weyl coefficient function is given by the formula*

$$\begin{aligned} N(\lambda) &= i(U_{12}(0,\lambda) + U_{22}(0,\lambda))(U_{12}(0,\lambda) - U_{22}(0,\lambda))^{-1} \\ &= i(I_n - 2c(\lambda I_p - a^\times)^{-1}(b + i\Omega c^*)). \end{aligned} \tag{2.14}$$

Proof. We first look for a $\mathbb{C}^{n\times 2n}$-valued function $P(\lambda)$ such that $x \mapsto P(\lambda)\Theta(x,\lambda)^*$ has square summable entries for $\lambda \in \mathbb{C}^+$. Let $U(\lambda, x)$ be the solution of the differential system (1.1) subject to the asymptotic condition (2.6). Then, $U(x,\lambda) = \Theta(x,\lambda)U(0,\lambda)$. We thus require the entries of the function

$$x \mapsto P(\lambda)U(0,\lambda)^{-*}U(x,\lambda) \tag{2.15}$$

to be square summable. By definition of U, it is necessary for $P(\lambda)U(0,\lambda)^{-*}$ to be of the form $(0,\ \ p(\lambda))$ where $p(\lambda)$ is $\mathbb{C}^{n\times n}$-valued. It follows from the definition of $U(0,\lambda)$ that one can take

$$P(\lambda) = \begin{pmatrix} 0 & I_n \end{pmatrix} U(0,\lambda)^* = \begin{pmatrix} U_{12}(0,\lambda)^* & U_{22}(0,\lambda)^* \end{pmatrix}$$

and hence the necessity condition. Conversely, we have to show that the function (2.15) has indeed summable entries. But this is just doing the above argument backwards.

The realization formula follows then from the realization formulas for the block entries of the asymptotic equivalence matrix function. □

Any of the functions in the spectral domain determines all the others, as follows from the next theorem:

Theorem 2.20. *Assume given a differential system of the form* (1.1) *with potential* $k(x)$ *of the form* (2.1)*. Assume* $W(\lambda)$, $V(\lambda)$, $R(\lambda)$, $S(\lambda)$ *and* $N(\lambda)$ *are the characteristic spectral functions of* (1.1)*, and let* $S = S_-S_+$ *be the spectral factorization of the scattering matrix function* S*, where* S_- *and its inverse are invertible in the closed lower half-plane and* S_+ *and its inverse are invertible in the closed upper half-plane. Then, the connections between these functions are:*

$$\begin{aligned}
W(\lambda) &= S_-(\lambda)^{-1}S_-(\lambda)^{-*} = S_+(\lambda)S_+(\lambda)^*,\\
W(\lambda) &= \mathrm{Im}\ N(\lambda),\\
S(\lambda) &= S_-(\lambda)S_+(\lambda),\\
R(\lambda) &= (iN(\lambda)^* - I_n)(iN(\lambda)^* + I_n)^{-1},\\
N(\lambda) &= i(I_n + R(\lambda)^*)(I_n - R(\lambda)^*)^{-1},\\
V(\lambda) &= \frac{1}{2}\begin{pmatrix} (iN(\lambda)^* + I_n)S_-(\lambda)^* & (-iN(\lambda) - I_n)S_+(\lambda)^{-*} \\ (iN(\lambda)^* - I_n)S_-(\lambda)^* & (-iN(\lambda) + I_n)S_+(\lambda)^{-*} \end{pmatrix}
\end{aligned}$$

for $\lambda \in \mathbb{R}$.

See [10, Theorem 3.1].

We note that $R^* = T_V(0)$. We now wish to relate V to a unitary completion of the reflection coefficient function. It is easier to look at

$$\widetilde{V}(\lambda) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} V(\lambda) \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

We set

$$P = \frac{I_{2n}+J}{2} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Q = \frac{I_{2n}-J}{2} = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}.$$

Theorem 2.21. *Let* $\Theta \in \mathbb{C}^{2n\times 2n}$ *be such that* $\det(P+Q\Theta) \neq 0$*. Then* $\det(P-\Theta Q) \neq 0$ *and*

$$\Theta^\times \stackrel{\text{def.}}{=} (P\Theta + Q)(P+Q\Theta)^{-1} = (P-\Theta Q)^{-1}(\Theta P - Q) \tag{2.16}$$

Finally

$$\begin{aligned} I_{2n} - \Theta^\times \Theta^{\times *} &= (P - \Theta Q)^{-1} (J - \Theta J \Theta^*) (P - \Theta Q)^{-*} && (2.17)\\ I_{2n} - \Theta^{\times *} \Theta^\times &= (P + Q\Theta)^{-*} (J - \Theta^* J \Theta) (P + Q\Theta)^{-1}. && (2.18) \end{aligned}$$

Proof. We set $\Theta = \begin{pmatrix} A & B \\ C & C \end{pmatrix}$ where $A \in \mathbb{C}^{n \times n}$. We have:

$$P + Q\Theta = \begin{pmatrix} I_n & 0 \\ C & D \end{pmatrix}, \qquad P - \Theta Q = \begin{pmatrix} I_n & -B \\ 0 & -D \end{pmatrix}.$$

Thus either of these matrices is invertible if and only if D is invertible. Thus both equalities in (2.16) make sense. To prove that they define the same object is equivalent to prove that

$$(P - \Theta Q)(P\Theta + Q) = (\Theta P - Q)(P + Q\Theta),$$

i.e., since $PQ = QP = 0$,

$$P\Theta - \Theta Q = \Theta P - Q\Theta.$$

This in turn clearly holds since $P + Q = I_{2n}$.

We now prove (2.17). The proof of (2.18) is similar and will be omitted. We have

$$\begin{aligned} I_{2n} - \Theta^\times \Theta^{\times *} &= I_{2n} - (P - \Theta Q)^{-1} (\Theta P - Q)(\Theta P - Q)^* (P - \Theta Q)^{-*} \\ &= (P - \Theta Q)^{-1} \{ (P - \Theta Q)(P - \Theta Q)^* - (\Theta P - Q)(\Theta P - Q)^* \} \\ &\quad \times (P - \Theta Q)^{-*} \\ &= (P - \Theta Q)^{-1} \{ P - Q + \Theta Q \Theta^* - \Theta P \Theta^* \} (P - \Theta Q)^{-*} \end{aligned}$$

and hence the result since $J = P - Q$. □

The function defined by (2.16) is called the Potapov–Ginzburg transform of Θ. We have

$$\Theta^\times = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix}. \qquad (2.19)$$

Theorem 2.22. *The Potapov–Ginzburg transform of $\widetilde{V}$ is a unitary completion of the reflection coefficient function.*

Indeed, from (2.19) the 22 block of the Potapov–Ginzburg transform of $\widetilde{V}$ is exactly R. It is not a minimal completion (in particular it has n poles in $\mathbb{C}_-$). See [20] for more information on this transform. Minimal unitary completions of a proper contraction are studied in [33, Theorem 4.1 p. 236].

2.3. The continuous orthogonal polynomials

As already mentioned, for every $x \geq 0$ the function $\lambda \mapsto \Theta(x,\lambda) = U(x,\lambda)U(0,\lambda)^{-1}$ is entire. Albeit their name, the continuous orthogonal polynomials are entire functions, first introduced by M.G. Kreĭn (see [37]) and in terms of which one can compute the matrix function $\Theta(x,\lambda)$. To define these functions we start with a function W of the form

$$W(\lambda) = I_n - \int_{\mathbb{R}} e^{it\lambda}\omega(t)dt, \quad \lambda \in \mathbb{R}, \tag{2.20}$$

with $\omega \in \mathbf{L}_1^{n\times n}(\mathbb{R})$ and such that $W(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. This last condition insures that the integral equation

$$\Gamma_T(t,s) - \int_0^T \omega(t-u)\Gamma_T(u,s)du = \omega(t-s), \quad t,s \in [0,T]$$

has a unique solution for every $T > 0$.

Definition 2.23. *The continuous orthogonal polynomial is given by:*

$$P(t,\lambda) = e^{it\lambda}\left(I_n + \int_0^{2t} \Gamma_{2t}(u,0)e^{-i\lambda u}du\right).$$

Theorem 2.24. *It holds that*

$$\begin{pmatrix} I_n & I_n \end{pmatrix}\Theta(x,\lambda) = \begin{pmatrix} P(t,-\lambda) & R(t,\lambda) \end{pmatrix}$$

where $R(t,\lambda) = e^{it\lambda}\left(I_n + \int_0^{2t}\Gamma_{2t}(2t-u,2t)e^{-i\lambda u}du\right)$.

In view of Theorem 2.20, note that every rational function analytic at infinity, such that $W(\infty) = I_n$, with no poles and strictly positive on the real line, is the spectral function of a canonical differential expression of the form (1.1) with potential of the form (2.1). Furthermore, let $W(\lambda) = I_n + C(\lambda I_p - A)^{-1}B$ be a minimal realization of W. Then, W is of the form (2.20) with

$$\omega(u) = \begin{cases} iCe^{-iuA}(I_p - P)B, & u > 0, \\ -iCe^{-iuA}PB, & u < 0, \end{cases}$$

where P is the Riesz projection of A in $\mathbb{C}_+$. We recall that

$$P = \int_\gamma (\zeta I_p - A)^{-1}d\zeta$$

where γ is a positively oriented contour which encloses only the eigenvalues of A in $\mathbb{C}_+$.

Theorem 2.25. *Let W be a rational $\mathbb{C}^{n\times n}$-valued function analytic and invertible on $\mathbb{R}$ and at infinity. Assume moreover that $W(\lambda) > 0$ for real λ and that $W(\infty) = I_n$. Let $W(\lambda) = I_n + C(\lambda I_p - A)^{-1}B$ be a minimal realization of W. Let P (resp. $P^\times$) denote the Riesz projection corresponding to the eigenvalues of A (resp. of $A^\times = A - BC$) in $\mathbb{C}_+$. Then, the continuous orthogonal polynomials $P(t,\lambda)$ are given by the formula*

$$P(t,\lambda) = e^{i\lambda t}\left\{I_n + C(\lambda I_p + A^\times)^{-1}(I_p - e^{-2i\lambda t}e^{-2itA^\times})\pi_{2t}B\right\}$$

where

$$\pi_t = (I_p - P + Pe^{-itA^\times})^{-1}(I_p - P).$$

Furthermore,

$$\lim_{t\to\infty} e^{-it\lambda}P(t,\lambda) = S_-(-\lambda)^*. \tag{2.21}$$

See [7, Theorem 3.3 p 10]. The computations in [7] use exact formulas for the function $\Gamma_T(t,s)$ in terms of the realization of W which have been developed in [15].

We note that the potential $k(x)$ can be written as

$$k(x) = 2C\left(Pe^{-2ixA^\times}|_{\mathrm{Im}\ P}\right)^{-1} PB \tag{2.22}$$

in terms of the realization of the spectral function W.

2.4. Perturbations

In this subsection we address the following question: assume that $k(x)$ is a strictly pseudo-exponential potential. Is $-k(x)$ also such a potential? This is not quite clear from formulas (2.1) or (2.22). One could attack this problem using the results in [11], where we studied a trace formula for a pair of self-adjoint operators corresponding to the potentials $k(x)$ and $-k(x)$. Here we present a direct argument in the rational case. More precisely, if N is a Nevanlinna function so are the three functions

$$\begin{aligned} \lambda &\to -N^{-1}(\lambda),\\ \lambda &\to -N^{-1}(-\lambda^*)^*,\\ \lambda &\to N(-\lambda^*)^*, \end{aligned}$$

and we have three associated weight functions

$$\begin{aligned} W_-(\lambda) &= \mathrm{Im}\ -N(\lambda)^{-1},\\ W_1(\lambda) &= \mathrm{Im}\ -N(-\lambda^*)^{-*},\\ W_2(\lambda) &= \mathrm{Im}\ N(-\lambda^*)^*. \end{aligned}$$

The relationships between these three weight functions and the original weight function W and the associated potential have been reviewed in the thesis [36] and we recall the results in form of a table:

The potential	**The weight function**
$v(x) = \begin{pmatrix} 0 & k(x) \\ k(x)^* & 0 \end{pmatrix}$	$W(\lambda) = \mathrm{Im}\ N(\lambda)$
$-v(x) = -\begin{pmatrix} 0 & k(x) \\ k(x)^* & 0 \end{pmatrix}$	$W_-(\lambda) = \mathrm{Im} - N(\lambda)^{-1}$
$-\begin{pmatrix} 0 & k(x)^* \\ k(x) & 0 \end{pmatrix}$	$W_1(\lambda) = \mathrm{Im}\ N(-\lambda^*)^*$
$\begin{pmatrix} 0 & k(x)^* \\ k(x) & 0 \end{pmatrix}$	$W_2(\lambda) = \mathrm{Im} - N(-\lambda^*)^{-*}$

Let

$$N(\lambda) = i(I + c(\lambda I - a)^{-1}b)$$

be a minimal realization of N. Then,

$$W(\lambda) = I + C(\lambda I - A)^{-1}B$$

is a minimal realization of the weight function W, where

$$A = \begin{pmatrix} a & 0 \\ 0 & a^* \end{pmatrix}, \quad B = \begin{pmatrix} b \\ c^* \end{pmatrix}, \quad C = \frac{1}{2}\begin{pmatrix} c & b^* \end{pmatrix}, \tag{2.23}$$

and the Riesz projection corresponding to the spectrum of A in the open upper half-plane $\mathbb{C}_+$ is

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.24}$$

Furthermore, the potential associated to the weight function W is given by (2.22) where A, B, C and P are given by (2.23) and (2.24), and

$$A^\times = A - BC = \begin{pmatrix} a - \frac{bc}{2} & -\frac{bb^*}{2} \\ -\frac{c^*c}{2} & (a - \frac{bc}{2})^* \end{pmatrix}.$$

Consider now the weight function W_-. A minimal realization of $-N(\lambda)^{-1}$ is given by

$$-N(\lambda)^{-1} = i(I - c(\lambda I - a^\times)^{-1}b), \quad a^\times = a - bc,$$

and a minimal realization of W_- is given by

$$W_-(\lambda) = I + C_-(\lambda I - A_-)^{-1}B_-,$$

where

$$A_- = \begin{pmatrix} a^\times & 0 \\ 0 & a^{\times *} \end{pmatrix}, \quad B_- = B = \begin{pmatrix} b \\ c^* \end{pmatrix}, \quad C_- = -C = -\frac{1}{2}\begin{pmatrix} c & b^* \end{pmatrix},$$

and the Riesz projection corresponding to the spectrum of A_- in the open upper half-plane $\mathbb{C}_+$ is $P_- = P$ given by (2.24).

The potential associated to the weight function W_- is given by

$$k_-(x) = -2C\left(Pe^{-2itA_-^\times}|_{\mathrm{Im}\ P}\right)^{-1}PB,$$

where

$$A_-^\times = A_- - B_-C_- = \begin{pmatrix} a - \frac{bc}{2} & \frac{bb^*}{2} \\ \frac{c^*c}{2} & (a - \frac{bc}{2})^* \end{pmatrix}.$$

Setting

$$D = \begin{pmatrix} a - \frac{bc}{2} & 0 \\ 0 & (a - \frac{bc}{2})^* \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & \frac{b^*b}{2} \\ \frac{cc^*}{2} & 0 \end{pmatrix},$$

we have

$$A^\times = D - Z \quad \text{and} \quad A_-^\times = D + Z.$$

We are now in a position to prove the following result:

Theorem 2.26. *Let $k(x)$ be a strictly pseudo-exponential potential with associated Weyl function $N(\lambda)$. The potential associated to* $\mathrm{Im} - N^{-1}$ *is equal to $k_-(x) = -k(x)$.*

Proof. To prove that $k_-(x) = -k(x)$, it is enough to prove that

$$Pe^{-itA^\times}|_{\mathrm{Im}\ P} = Pe^{-it(A_- - B_-C_-)}|_{\mathrm{Im}\ P}.$$

To prove this equality, it is enough in turn to prove that for all positive integers ℓ, it holds that

$$PA^{\times\ell}|_{\mathrm{Im}\ P} = P(A_- - B_-C_-)^\ell|_{\mathrm{Im}\ P},$$

i.e., that

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} (D+Z)^\ell \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} (D-Z)^\ell \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

for all positive integers ℓ. Let $\epsilon = \pm 1$. The expression $(D + \epsilon Z)^\ell$ consists of a sum of terms of the form

$$D^{\alpha_1}(\epsilon Z)^{\beta_1} D^{\alpha_2}(\epsilon Z)^{\beta_2} \cdots ,$$

where the α_i and the β_i are equal to 1 or 0 and $\sum_i(\alpha_i + \beta_i) = \ell$. Each factor $D^{\alpha_i}Z^{\beta_i}$ for which $\beta_i \neq 0$ is anti block diagonal. We consider two cases, namely $\sum_i \beta_i$ being odd or even. When $\sum_i \beta_i$ is odd, we have the product of an odd number of anti block diagonal matrices, and the result is anti block diagonal, and so, premultiplying and postmultiplying this product by $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ we obtain the zero matrix. When $\sum_i \beta_i$ is even, the product is an even function of ϵ and have the same value at $\epsilon = 1$ and at $\epsilon = -1$.

The case of the other two weight functions is treated in much the same way. We focus on $W_1(\lambda) = \mathrm{Im}\ N(-\lambda^*)^*$. A minimal realization of $N(-\lambda^*)^*$ is given by

$$N(-\lambda^*)^* = i(I - b^*(\lambda I + a^*)^{-1}c^*),$$

and a minimal realization of the weight function W_1 is therefore given by

$$W_1(\lambda) = I + C_1(\lambda I - A_1)^{-1}B_1,$$

where

$$A_1 = \begin{pmatrix} -a^* & 0 \\ 0 & -a \end{pmatrix}, \quad B_1 = \begin{pmatrix} c^* \\ b \end{pmatrix}, \quad C_1 = -\frac{1}{2}\begin{pmatrix} b^* & c \end{pmatrix},$$

and the Riesz projection corresponding to the spectrum of A_1 in the open upper half-plane $\mathbb{C}_+$ is $P_1 = P$ given by (2.24). The potential associated to the weight function W_1 is given by

$$k_1(x) = 2C_1\left(P_1 e^{-2itA_1^\times}|_{\mathrm{Im}\ P_1}\right)^{-1} P_1 B_1.$$

We claim that $k_1(x) = -k(x)^*$. Indeed,

$$k_1(x)^* = 2B_1^* P_1^* \left(P_1 e^{2itA_1^{*\times}}|_{\mathrm{Im}\ P_1}\right)^{-1} P_1^* C_1^*.$$

But we have that

$$B_1^* P_1^* = 2CP = \begin{pmatrix} c & 0 \end{pmatrix}, \quad P_1 C_1^* = -PB = -\frac{1}{2}\begin{pmatrix} b \\ 0 \end{pmatrix}, \quad A_1^{*\times} = -A^\times,$$

which allows to conclude. □

3. The discrete case

3.1. First-order discrete system

In our previous work [6] we studied inverse problems for difference operators associated to Jacobi matrices. Such operators are the discrete counterparts of Sturm–Liouville differential operators, and one can associate to them a number of functions analytic in the open unit disk similar to the characteristic spectral functions of a canonical differential expression. In the present paper we chose a different avenue to define discrete systems, which has more analogy to the continuous case and is more natural. The analogies between the two cases are gathered in form of two tables at the end of the paper.

We note that another type of discrete systems has been considered by L. Sakhnovich in [42, Section 2 p. 389].

Our starting point is the telegraphers' equations (1.2). We now assume that the local impedance function $Z(x)$ defined in (1.2) is equal to a constant, say Z_n, on the interval $[nh,(n+1)h)$ for $n = 0, 1, \ldots$ In particular, $Z(x)$ may have discontinuities at the points nh. On the open interval $(nh,(n+1)h)$, we have $k(x) = 0$ and equation (1.3) becomes

$$\begin{pmatrix} (\frac{\partial}{\partial x} + \frac{\partial}{\partial t}) & 0 \\ 0 & (\frac{\partial}{\partial x} - \frac{\partial}{\partial t}) \end{pmatrix} W(x,t) = 0.$$

Hence one can write

$$W(x,t) = \begin{pmatrix} v_{1n}(x-t) \\ v_{2n}(x+t) \end{pmatrix}$$

on the interval $(nh,(n+1)h)$. Voltage and current are continuous at the points nh. Let us set

$$\alpha(n,t) = \lim_{\substack{x \to nh \\ x > nh}} W(x,t).$$

Taking into account (1.3) one gets to:

$$\alpha(n,t) = \frac{1}{2}\begin{pmatrix} Z_n^{-1/2} & Z_n^{1/2} \\ Z_n^{-1/2} & -Z_n^{1/2} \end{pmatrix}\begin{pmatrix} v(nh,t) \\ i(nh,t) \end{pmatrix}$$

$$\lim_{\substack{x \to nh \\ x < nh}} W(x,t) = \frac{1}{2}\begin{pmatrix} Z_{n-1}^{-1/2} & Z_{n-1}^{1/2} \\ Z_{n-1}^{-1/2} & -Z_{n-1}^{1/2} \end{pmatrix}\begin{pmatrix} v(nh,t) \\ i(nh,t) \end{pmatrix}.$$

We define the backward shift operator on functions of the variable t

$$\Delta f(t) = f(t-h).$$

We have:

$$\begin{aligned}\lim_{\substack{x\to nh\\ x<nh}} W(x,t) &= \begin{pmatrix} v_{1,n-1}(nh-t)\\ v_{2,n-1}(nh+t)\end{pmatrix}\\ &= \begin{pmatrix} v_{1,n-1}((n-1)h-(t-h))\\ v_{2,n-1}((n-1)h+t+h)\end{pmatrix}\\ &= \begin{pmatrix}\Delta & 0\\ 0 & \Delta^{-1}\end{pmatrix}\begin{pmatrix} v_{1,n-1}((n-1)h-t)\\ v_{2,n-1}((n-1)h+t)\end{pmatrix}\\ &= \begin{pmatrix}\Delta & 0\\ 0 & \Delta^{-1}\end{pmatrix}\alpha(n-1,t).\end{aligned}$$

Thus,

$$\begin{pmatrix}\Delta & 0\\ 0 & \Delta^{-1}\end{pmatrix}\alpha(n-1,t) = \frac{1}{2}\begin{pmatrix} Z_{n-1}^{-1/2} & Z_{n-1}^{1/2}\\ Z_{n-1}^{-1/2} & -Z_{n-1}^{1/2}\end{pmatrix}\begin{pmatrix} v(nh,t)\\ i(nh,t)\end{pmatrix},$$

and we have:

$$\begin{aligned}\alpha(n,t) &= \begin{pmatrix} Z_{n+1}^{-1/2} & Z_{n+1}^{1/2}\\ Z_{n+1}^{-1/2} & -Z_{n+1}^{1/2}\end{pmatrix}\begin{pmatrix} Z_{n}^{-1/2} & Z_{n}^{1/2}\\ Z_{n}^{-1/2} & -Z_{n}^{1/2}\end{pmatrix}^{-1}\begin{pmatrix}\Delta & 0\\ 0 & \Delta^{-1}\end{pmatrix}\alpha(n-1,t)\\ &= H(\rho_n)\begin{pmatrix}\Delta & 0\\ 0 & \Delta^{-1}\end{pmatrix}\alpha(n-1,t)\end{aligned}$$

where

$$\rho_n = \frac{Z_{n+1}-Z_n}{Z_{n+1}+Z_n} \quad\text{and}\quad H(\rho) = \frac{1}{\sqrt{1-|\rho|^2}}\begin{pmatrix} 1 & -\rho\\ -\rho^* & 1\end{pmatrix}$$

for $|\rho|<1$. See [19, p. 111].

Replacing Δ by the complex variable and removing the scalar constant factor $\frac{1}{\sqrt{1-|\rho|^2}}$ we see that the discretization of the telegraphers' equations leads to systems of the form

$$Y_{n+1}(z) = \begin{pmatrix} 1 & -\rho_n\\ -\rho_n^* & 1\end{pmatrix}\begin{pmatrix} z & 0\\ 0 & z^{-1}\end{pmatrix} Y_n(z), \tag{3.1}$$

which we will call *two-sided first-order discrete systems.*

The solution corresponding to $\rho_n \equiv 0$ is

$$Y_n(z) = \begin{pmatrix} z^n & 0\\ 0 & z^{-n}\end{pmatrix} Y_0(z),$$

that is, we are in a *two-sided* situation (the negative powers of z corresponding to signals coming from $-\infty$).

Recursions of the related forms

$$X_{n+1}(z) = \begin{pmatrix} 1 & -\rho_n\\ -\rho_n^* & 1\end{pmatrix}\begin{pmatrix} z & 0\\ 0 & 1\end{pmatrix} X_n(z) \tag{3.2}$$