# Progress in Nonlinear Differential Equations and Their Applications

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# Contributions to Nonlinear Analysis

## A Tribute to D.G. de Figueiredo on the Occasion of his 70th Birthday

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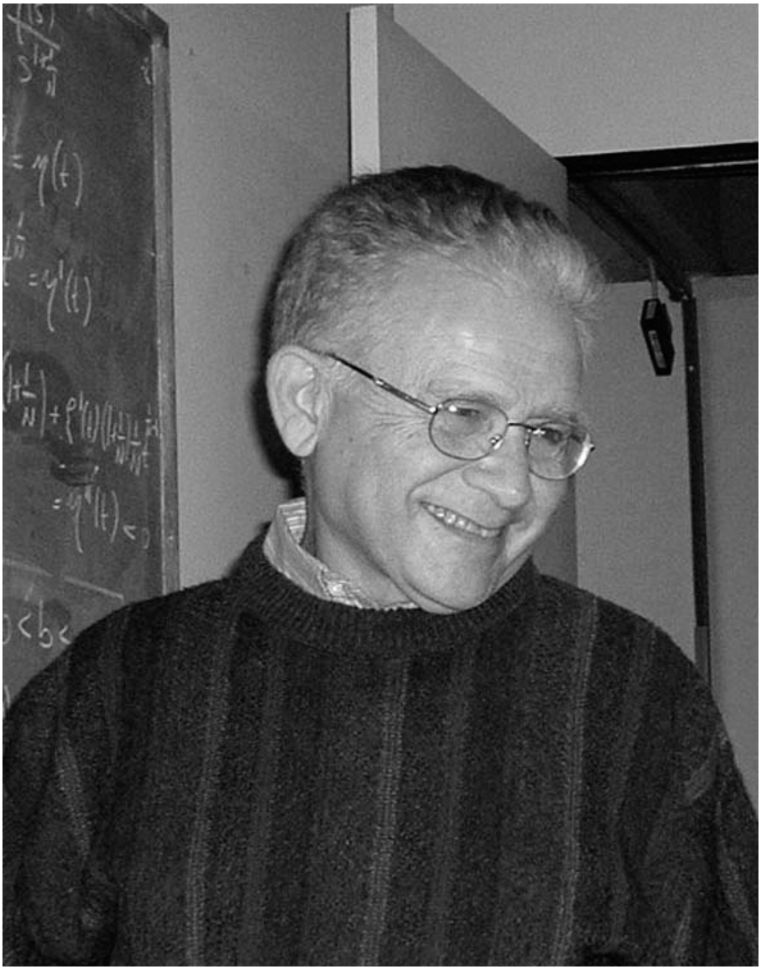

Djairo de Figueiredo

# Dedication

This volume is dedicated to **Djairo G. de Figueiredo** on the occasion of his 70th birthday.

In January 2003 David Costa, Orlando Lopes and Carlos Tomei, colleagues and friends of Djairo, invited us to join the organizing committee for a *Workshop on Nonlinear Differential Equations*, sending us the following message:

*Djairo's career is a remarkable example for the Brazilian community. We are proud of his mathematical achievements and his ability to develop so many successors, through systematic dedication to research, advising activities and academic orchestration. Djairo is always organizing seminars and conferences and is constantly willing to help individuals and the community. It is about time that he should enjoy a meeting without having to work for it.*

How true! Of course we all accepted with great enthusiasm. The workshop took place in Campinas, June 7–11, 2004. It was a wonderful conference, with the participation of over 100 mathematicians from all over the world.

The wide range of research interests of Djairo is reflected by the articles in this volume. Through their contributions, the authors express their appreciation, gratitude and friendship to Djairo.

We are happy that another eminent Brazilian mathematician, Jacob Palis from IMPA, has accepted our invitation to give an appreciation of Djairo's warm personality and his excelling work.

The editors:

Thierry Cazenave
David Costa
Raúl Manásevich
Orlando Lopes
Paul Rabinowitz
Bernhard Ruf
Carlos Tomei

# On Djairo de Figueiredo. A Mathematician

J. Palis

Djairo is one of the most prominent Brazilian mathematicians.

From the beginning he was a very bright student at the engineering school of the University of Brazil, later renamed Federal University of Rio de Janeiro. He turned out to be a natural choice to be awarded one of the not so many fellowships, then offered by our National Research Council - CNPq, for Brazilians to obtain a doctoral degree abroad. While advancing in his university courses, he participated at this very engineering school in a parallel mathematical seminar, conducted by Mauricio Peixoto. Mauricio, who was the *catedrático* of Rational Mechanics and about to become a world figure, suggested to Djairo to get a PhD in probability and statistics.

Actually, Elon Lima, also one of our world figures, tells me that he had the occasion to detect Djairo's talent some years before at a boarding house in Fortaleza, where they met by pure chance. Djairo was 15 years old and Elon, then a high school teacher and an university freshman, just a few years older. Full of enthusiasm for mathematics, one day Elon initiated a private course to explain the construction of the real numbers to the young fellow and one of his colleagues. That Djairo was able to fully understand such subtle abstract piece of mathematics, tells us of both his talent as well as that of Elon for learning and teaching. They both went to Rio, one to initiate and the other to complete their university degrees. Amazingly, for a while again they lived under the same roof, in *Casa do Estudante do Brasil* (curiously, two of my brothers were also staying there at the time), and continued to talk about mathematics. First, Elon departed to the University of Chicago and Djairo, a couple of years later, to the Courant Institute at the University of New York, where they obtained their PhDs.

At Courant, it happened that Djairo did not get a degree neither in probability nor in statistics, as it's so common among us not to strictly follow a well meant advice, in this case by Peixoto to him. Djairo was instead enchanted by the charm of partial differential equations, under the guidance of Louis Nirenberg. Louis and him became friends forever. He was to become an authority on elliptic partial differential equations, linear and nonlinear, individual ones or systems of them. His thesis appeared in Communications on Pure and Applied Mathematics, a very distinguished journal.

He then returned briefly to Rio de Janeiro staying at the Instituto de Matemática Pura e Aplicada – IMPA. Soon, he went to Braslia to start a "dream" University, together with his colleague Geraldo Ávila, as advised by Elon Lima to the founder of it, Darcy Ribeiro. In 1965 he returned to the United States. This time, he went to the University of Wisconsin and right after to the University of Ilinois for perhaps a longer stay than he might have thought at first: unfortunately, undue external and undemocratic pressure led to a serious crisis at his home institution. In this period he developed collaborations with Felix Browder on the theory of monotone operators and with L.A. Karlovitz on the geometry of Banach spaces and applications, a bit different from his main topic of research as mentioned above.

After spending another year at IMPA, Djairo went back to the University of Braslia in the early 70's, having as a main goal to rebuilt as possible the initial exceptionally good scientific atmosphere. He did so together with Geraldo Ávila and, subsequently, other capable colleagues. Their efforts bore good fruits. He retired from Brasília in the late 80's and faced a new challenge: to upgrade mathematics in the University of Campinas by his constant and stimulating activity, high scientific competence and dedicated work. He has been, again from the beginning, a major figure at this new environment. And he continues to be so today, when we are celebrating his 70's Anniversary.

To commemorate this especial occasion for Brazilian mathematics, a high level Conference was programmed. More than one hundred of his friend mathematicians took part on it, including forty-three foreigners from thirteen countries. Also, a number of his former PhD students and several grand-students.

In his career, Djairo produced about eighty research articles published in very good journals. His range of co-authors is rather broad, among them Gossez, Gupta, Pierre-Louis Lions, Nussbaum, Mitidieri, Ruf, Jianfu, Costa, Felmer, Miyagaki, and, as mentioned above, Felix Browder. He is a wonderful, very inspiring lecturer at all levels, from introductory to frontier mathematics. Such a remarkable feature spreads over the several books he has written. Among them are to be mentioned Análise de Fourier e EDP – Projeto Euclides, much appreciated by a wide range of students, including engineering ones, and Teoria do Potencial – Notas de Matemática, both from IMPA.

On the way to all such achievements, he was elected Member of the Brazilian Academy of Sciences and The Academy of Sciences for the Developing World - TWAS. He is a Doctor Honoris Causa of the Federal University of Paraiba and Professor Emeritus of the University of Campinas. He has also been distinguished with the Brazilian Government Commend of Scientific Merit – Grand Croix.

Above all, Djairo is a sweet and very gregarious person. We tend to remember him always smiling

Rio de Janeiro, 3 de Agosto de 2005.

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# Remarks on a Class of Neumann Problems Involving Critical Exponents

Emerson A. M. Abreu[1], Paulo Cesar Carrião[2] and Olimpio Hiroshi Miyagaki[3]

*Dedicated to Professor D.G.Figueiredo on the occasion of his* $70^{th}$ *birthday*

**Abstract.** This paper deals with a class of elliptic problems with double critical exponents involving convex and concave-convex nonlinearities. Existence results are obtained by exploring some properties of the best Sobolev trace constant together with an approach developed by Brezis and Nirenberg.

**Mathematics Subject Classification (2000).** 35J20, 35J25, 35J33, 35J38.

**Keywords.** Sobolev trace exponents, elliptic equations, critical exponents and boundary value problems.

## 1. Introduction

This paper deals with a class of elliptic problems with double critical exponents involving convex and concave-convex nonlinearities of the type

$$-\Delta u = u^{2^*-1} + f(x,u) \quad \text{in} \quad \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \nu} = u^{2_*-1} + g(x,u) \quad \text{on} \quad \partial\Omega, \tag{2}$$

$$u > 0 \quad \text{in} \quad \Omega, \tag{3}$$

where $\Omega \subset I\!R^N$, $(N \geq 3)$, is a bounded smooth domain, $\frac{\partial u}{\partial \nu}$ is the outer unit normal derivative, $f$ and $g$ have subcritical growth at infinity, $2^* = \frac{2N}{N-2}$ and $2_* = \frac{2(N-1)}{N-2}$ are the limiting Sobolev exponents for the embedding $H_0^1(\Omega) \subset L^{2^*}(\Omega)$ and $H^1(I\!R_+^N) \hookrightarrow L^{2_*}(\partial I\!R_+^N)$, respectively, where $I\!R_+^N = \{(x,t) : x \in I\!R^{N-1}, \;\; t > 0\}$.

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In a famous paper [6], Brezis and Nirenberg proved some existence results for (1) and (3) with Dirichlet boundary condition and $f$ satisfying the following conditions:

$$f(x,0)=0 \quad \text{and} \quad \lim_{s\to+\infty}\frac{f(x,s)}{s^{2^*-1}}=0, \tag{$f0$}$$

$$\begin{array}{c}\text{there exists some function } h(s) \text{ such that}\\ f(x,s)\geq h(s)\geq 0, \text{ for a.e. } x\in\omega \quad \forall s\geq 0,\end{array} \tag{$f1$}$$

where $\omega$ is some nonempty open set in $\Omega$ and the primitive $H(s)=\int_0^s h(t)dt$ satisfies

$$\lim_{\epsilon\to 0}\int_0^{\epsilon^{-1}}\epsilon^2 H[(\frac{\epsilon^{-1}}{1+s^2})^{\frac{N-2}{2}}]s^{N-1}ds=\infty, \tag{$f2$}$$

$$\begin{array}{c} f:\Omega\times[0,\infty)\longrightarrow I\!R \text{ is measurable in } x\in\Omega, \text{ continuous in}\\ s\in[0,\infty), \text{and} \sup\limits_{x\in\Omega,\ s\in[0,M]}|f(x,s)|<\infty, \text{ for all } M>0,\end{array} \tag{$f3$}$$

$$\begin{array}{c} f(x,s)=a(x)s+f_1(x,s) \text{ with } a\in L^\infty(\Omega),\\ \lim\limits_{s\to 0}\dfrac{f_1(x,s)}{s}=0 \text{ and } \lim\limits_{s\to\infty}\dfrac{f_1(x,s)}{s^{2^*-1}}=0, \text{ uniformly in } x.\end{array} \tag{$f4$}$$

Actually, in spite of the embedding $H_0^1(\Omega)\subset L^{2^*}(\Omega)$ not being compact any longer, they were able to get some compactness condition, proving that the critical level of the Euler-Lagrange functional associated to (1) with Dirichlet boundary condition lies below the critical number $\frac{1}{N}S^{N/2}$, where

$$S=\inf\{\int_\Omega|\nabla u|^2dx:\ \int_\Omega|u|^{2^*}dx=1,\quad 0\neq u\in H_o^1(\Omega)\}.$$

Still in the Dirichlet condition case, in [2] Ambrosetti, Brezis and Cerami treated a situation involving concave and convex nonlinearities. Recently, Garcia-Azorero, Peral and Rossi in [12] studied a concave-convex problem involving subcritical nonlinearities on the boundary.

The problem (1)–(3) with $f=g=0$ was first studied in [11, Theorem 3.3], which was generalized in [8] (see also [9] ). In [18] symmetric properties of solutions were obtained, but, basically in these papers it was proved that every positive solution $w_\epsilon$ of the partial differential equation with nonlinear boundary condition

$$\left\{\begin{array}{lll} -\Delta u=N(N-2)u^\alpha & \text{in} & I\!R^N_+,\\ \frac{\partial u}{\partial t}=cu^\beta & \text{on} & \partial I\!R^N_+,\end{array}\right. \tag{$E$}$$

with $\alpha=2^*-1$ and $\beta=2_*-1$, verifies

$$w_\epsilon(x,t)=(\frac{\epsilon}{\epsilon^2+|(x,t)-(x_0,t_0)|^2})^{\frac{N-2}{2}}$$

for some $\epsilon>0$ where $(N-2)t_0\epsilon^{-1}=c$. Equivalently the minimizing problem

$$S_0=\inf\{|\nabla u|^2_{2,I\!R^N_+}:|u|^2_{2_*,I\!R^N_+}+|u|^2_{2^*,I\!R^N_+}=1\}$$

is attained by the above function $w_\epsilon(x,t)$, where $|u|_{a,\Omega}$ denotes the usual $L^a(\Omega)$-norm.

We would like to mention the papers [4, 7, 14, 19] for more information about the Sobolev trace inequality, as well as related results involving the Yamabe problem. Still in $I\!\!R^N_+$, in [9] a nonexistence result for (E) was proved for the case that one of the inequalities $\alpha \leq 2^* - 1$, $\beta \leq 2_* - 1$, is strict (see also [13]).

When the domain $\Omega$ is unbounded, by applying the concentration compactness principle, Lions in [15] studied some minimization problems related to (1)–(3) with linear perturbations. Recently in [5], (see also [10]) a quasilinear problem was studied involving a subcritical nonlinearity in $\Omega$ and a perturbation of a critical situation on $\partial\Omega$, while in [16], the critical case is treated and some existence results for (1)–(3) with $f = 0$, $g(s) = \delta s$, $\delta > 0$ and $N \geq 4$ were proved ( see also in [17] when $\Omega$ is a ball).

On the other hand, making $f(x,s) = \lambda s$, $g(x,s) = \mu s$, with $\lambda$, $\mu \in I\!\!R$, in (1)-(3), and arguing as in the proof of Pohozaev's identity, more exactly, multiplying the first equation (1) by $x.\nabla u$, we obtain

$$\begin{aligned} 0 &= \operatorname{div}(\nabla u(x.\nabla u) - x\frac{|\nabla u|^2}{2} + x(\frac{\lambda}{2}u^2 + \frac{1}{2^*}u^{2^*})) \\ &\quad + \frac{N-2}{2}|\nabla u|^2 - N(\frac{\lambda}{2}u^2 + \frac{1}{2^*}u^{2^*}). \end{aligned}$$

Integrating this equality over $\Omega$, we have

$$\begin{aligned} 0 &= \int_{\partial\Omega}(< x.\nu > (\frac{|\nabla u|^2}{2} + (\frac{\lambda}{2}u^2 + \frac{1}{2^*}u^{2^*})) \\ &\quad + \frac{N-2}{2}(\int_{\partial\Omega} \mu u^2 + u^{2_*}) - \lambda \int_\Omega u^2. \end{aligned}$$

From this identity we can conclude that, for instance, if $\Omega$ is star-shaped with respect to the origin in $I\!\!R^N$, $\lambda = 0$ and $\mu \geq 0$, then any solution of (1)–(3) vanishes identically.

We would like to point out that hereafter $\int_\Omega f$ and $\int_{\partial\Omega} g$ mean $\int_\Omega f(x)dx$ and $\int_{\partial\Omega} g(y)d\sigma$, respectively.

Motivated by the above papers and remarks, in order to state our first result, we make some assumptions on $f$ and $g$, namely,

$$\begin{gathered} g(y,s) = b(y)s + g_1(y,s), \;\; y \in \partial\Omega, \;\; s \in I\!\!R \;\text{ and }\; b \in L^\infty(\partial\Omega) \\ g(y,s) \geq 0, \;\forall y \in \partial w \cap \partial\Omega \neq \emptyset, \end{gathered} \tag{$g1$}$$

$$\lim_{s\to 0} \frac{g_1(y,s)}{s} = 0 \;\text{ and }\; \lim_{s\to\infty} \frac{g_1(y,s)}{s^{2_*-1}} = 0, \;\text{ uniformly in }\; y, \tag{$g2$}$$

$$0 < \Theta_1 \leq \inf\{||u||^2 - 2\int_{\partial\Omega} bu^2 : ||u|| = 1\} \;\text{ for some } \Theta_1 \in I\!\!R, \tag{$g3$}$$

$$\begin{gathered}\text{there exists } \varrho \geq 1 \text{ with } a+\varrho > 0 \text{ on a subset of } \Omega \\ \text{of positive Lebesgue measure in } I\!\!R^N \text{ such that} \\ 0 < \Theta_2 \leq \inf\{||u||^2 - 2\textstyle\int_\Omega (\varrho + a)u^2 : ||u|| = 1\}, \quad \text{for some } \Theta_2 \in I\!\!R,\end{gathered} \tag{$f5$}$$

where $||u||^2 = |\nabla u|^2_{2,\Omega} + |u|^2_{2,\Omega}$ denotes the usual norm in $H^1(\Omega)$.

Our first result is the following.

**Theorem 1.1 (Convex case).** *Assume that* $(g_1)$–$(g_3)$ *and* $(f_0)$–$(f_5)$ *hold. Then problem* (1)–(3) *possesses at least one positive solution.*

**Remark 1.1.** The above result still holds when $f = 0$ and $g$ verify the condition

$$\begin{gathered}\text{there exists some function } p(s) \text{ such that} \\ g(y,s) \geq p(s) \geq 0, \text{ for a.e. } \; y \in \partial\Omega \cup \partial w, \quad \forall s \geq 0, \\ \text{and the primitive } P(s) = \textstyle\int_0^s p(t)dt \;\text{ satisfies} \\ \lim_{\epsilon\to 0} \textstyle\int_0^{\epsilon^{-1}} \epsilon P[(\frac{\epsilon^{-1}}{1+s^2})^{\frac{N-2}{2}}]s^{N-2}ds = \infty.\end{gathered}$$

Because, since $(N-2)t_o = c\epsilon$, we have

$$\begin{aligned} &\frac{1}{\epsilon^{N-2}} \int_{B_R(x_o,t_o)\cap\{t=0\}} P[(\frac{A\epsilon}{\epsilon^2 + |x-x_o|^2 + |t-t_o|^2})^{N-2/2}] \\ &= \frac{1}{\epsilon^{N-2}} \int_{B_R(x_o,t_o)\cap\{t=0\}} P[(\frac{A\epsilon}{d\epsilon^2 + |x-x_o|^2})^{N-2/2}] \\ &= \epsilon B \int_0^{R/d\epsilon} P[(\frac{A\epsilon^{-1}}{1+r^2})^{N-2/2}]r^{N-2}dr, \end{aligned}$$

where $d = (c/(N-2))^2 + 1$ and $A, B > 0$.

**Remark 1.2.** In [16, 17] it was proved that the functional levels $c = c(\delta)$ where the Palais–Smale sequence can converge are close to the critical number $c(0) = \frac{1}{N}S^{N/2}$, when $\delta$ goes to $+\infty$. In our work, we are going to use the number $S_0$, which verifies the inequality $S_0 < S$. So, for $\delta$ large enough, we obtain

$$\bar{S} \equiv (\frac{1}{2} - \frac{1}{2_*}) \max\{S_0^{\frac{2^*}{2^*-2}}, S_0^{\frac{2_*}{2_*-2}}\} \leq c(\delta) < \frac{1}{N}S^{N/2}.$$

Assuming some condition on $F$ and $G$, as in [6, page 462], the functional level of our solution $u$ [see Remark 3.1 below] is less than the number $\bar{S}$.

Since with the techniques used here the case $f = 0$, $g(s) = \delta s$ can be treated, combining this fact with our result we have a multiplicity result, when $N \geq 4$.

Finally, we would like to point out that the hypothesis ($g3$) and the structure of the problem studied in this paper includes the main hypothesis in [16, 17], so we also obtain at least one solution if $N = 3$.

Next, we treat the concave-convex case. For this we define

$$f(x,s) = a(x)s + \lambda f_1(x,s), \;\; g(y,s) = b(y)s + \mu g_1(y,s), \;\; \lambda, \mu > 0,$$

with $a \in L^\infty(\Omega)$, $b \in L^\infty(\partial\Omega)$, and we will assume that

$$\lim_{s\to 0} \frac{f_1(x,s)}{s^q} = 0 \text{ and } \lim_{s\to\infty} \frac{f_1(x,s)}{s^{2^*-1}} = 0, \text{ uniformly in } x, \tag{$f6$}$$

$$\lim_{s\to 0} \frac{g_1(y,s)}{s^\tau} = 0 \text{ and } \lim_{s\to\infty} \frac{g_1(y,s)}{s^{2_*-1}} = 0, \text{ uniformly in } y, \tag{$g4$}$$

where $1 < q, \tau < 2$.

We state our result in this case:

**Theorem 1.2 (Concave-convex case).** *Assume that* $(g_1)$, $(g_3)$, $(g_4)$, $(f_0)$, $(f_1)$, $(f_2)$, $(f_3)$, $(f_5)$ *and* $(f_6)$ *hold. Then problem* (1)–(3) *has at least one positive solution with* $\lambda, \mu > 0$, *sufficiently small.*

**Remark 1.3.** In our forthcoming paper [1] we obtained some multiplicity results in the concave-convex case.

The paper is divided up as follows. In Section 2 some preliminary results will be stated. In Section 3 we shall deal with the convex case, and the concave-convex case will be treated in the last section.

## 2. Preliminary results

In this section, we are going to state some preliminary remarks. Since we are concerned with the existence of a positive solution, we can assume

$$f_1(x,s) = 0, \quad x \in \Omega, \quad s \le 0 \quad \text{and} \quad g_1(y,s) = 0, \quad y \in \partial\Omega, \quad s \le 0.$$

Define the Euler–Lagrange functional $\Phi : H^1(\Omega) \to I\!\!R$, associated to problem $(P)$,

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega (\frac{1}{2}au^2 + F_1(x,u) + \frac{|u|^{2^*}}{2^*}) - \int_{\partial\Omega} (\frac{1}{2}bu^2 + G_1(y,u) + \frac{|u|^{2_*}}{2_*})$$

where $F_1(x,u) = \int_0^u f_1(x,t)dt$ and $G_1(x,u) = \int_0^u g_1(x,t)dt$. It is standard to see that $\Phi \in C^1$ and

$$\begin{aligned} \Phi'(u)v &= \int_\Omega \nabla u \nabla v - \int_\Omega (auv + f_1(x,u)v + |u|^{2^*-2}uv) \\ &\quad - \int_{\partial\Omega} (buv + g_1(y,u)v + |u|^{2_*-2}uv), \quad u, v \in H^1(\Omega). \end{aligned}$$

The proofs of our results are made by employing the variational techniques, and the best constant $S_0$ introduced by Escobar will play an important role in our arguments.

## 3. Convex case

In this section, we shall adapt some arguments made in the proof of Theorem 2.1 in [6]. From $(f4)$ we can fix $\varrho \geq 1$ large enough so that

$$-f(x,u) \leq \varrho u + u^{2^*-1} \ \ a.e. \ \ x \in \Omega, \forall u \geq 0.$$

Define the functional on $H^1(\Omega)$ by

$$\begin{aligned}\Phi(u) &= \int_\Omega (\frac{1}{2}|\nabla u|^2 + \frac{1}{2}\varrho u^2 - \frac{1}{2}\varrho u_+^2 - \frac{1}{2^*}u_+^{2^*} - F(x,u_+)) \\ &\quad - \int_{\partial\Omega} (G(y,u_+) + \frac{1}{2_*}u_+^{2_*}), \ u_+ = \max\{u,0\}.\end{aligned}$$

It is standard to prove that $\Phi \in C^1$.

Now $\Phi$ verifies the mountain pass geometry, namely

**Lemma 3.1.** $\Phi$ *verifies*

i) *There exist positive constants* $\rho$ *and* $\beta$ *such that*

$$\Phi(u) \geq \beta, \ \ ||u|| = \rho.$$

ii) *There exist a positive constant* $R > \rho$, *and* $u_0 \in H^1(\Omega)$ *such that*

$$\Phi(u_0) < 0, \ ||u_0|| > R.$$

*Proof.* From $(f4)$, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$F(x,u) \leq \frac{1}{2}au^2 + \frac{C_\epsilon}{2^*}u^{2^*} + \frac{1}{2}\epsilon u^2 \ \text{ for a.e. } \ x \in \Omega, \forall u \geq 0.$$

Similarly from $(g1)$ and $(g2)$, there exists some constant $D_\epsilon > 0$, such that

$$G(y,u) \leq \frac{1}{2}bu^2 + \frac{D_\epsilon}{2_*}u^{2_*} + \frac{1}{2}\epsilon u^2 \ \text{ for a.e. } \ y \in \partial\Omega, \forall u \geq 0.$$

Therefore

$$\begin{aligned}\Phi(u) &\geq \frac{1}{4}||u||^2 + \frac{1}{4}||u||^2 + \int_\Omega (-\frac{1}{2}(\varrho + a)u^2 - \frac{1}{2^*}u_+^{2^*} - \frac{C_\epsilon}{2^*}u_+^{2^*} - \frac{1}{2}\epsilon u_+^2) \\ &\quad + \int_{\partial\Omega} (-\frac{1}{2}bu_+^2 - \frac{1}{2_*}u_+^{2_*} - \frac{D_\epsilon}{2_*}u_+^{2_*} - \frac{1}{2}\epsilon u_+^2).\end{aligned}$$

From $(g3)$ and $(f5)$ follows that

$$\frac{1}{4}||u||^2 - \frac{1}{2}\int_\Omega (\varrho + a)u^2 \geq \frac{\Theta_2}{4}||u||^2$$

and

$$\frac{1}{4}||u||^2 - \frac{1}{2}\int_{\partial\Omega} bu^2 \geq \frac{\Theta_1}{4}||u||^2.$$

Thus

$$\Phi(u) \geq C_1||u||^2 - C_2||u||^{2^*} - C_3||u||^{2_*}, \ \ C_1, \ C_2, \ C_3 > 0.$$

This proves (i).

The proof of (ii) follows observing that for fixed $0 \neq u \in H^1(\Omega)$,

$$\Phi(tu) \to -\infty \text{ as } t \to \infty.$$

This proves Lemma 3.1. □

From Lemma 3.1, applying the mountain pass theorem due to Ambrosetti and Rabinowitz [3], there is a $(PS)_c$ sequence $\{u_n\} \subset H^1(\Omega)$ such that

$$\Phi(u_n) \to c,\ \Phi'(u_n) \to 0, \text{ in } H^{-1}(\Omega) \text{ as } n \to \infty,$$

where

$$c = \inf_{h\in\Gamma} \sup_{t\in[0,1]} \Phi(h(t)) > 0,$$

with

$$\Gamma = \{h \in C([0,1], H^1(\Omega)) :\ h(0) = 0 \text{ and } \ h(1) = u_0\}.$$

The following estimate is the crucial step of our proof.

**Lemma 3.2.**

$$c < (\frac{1}{2} - \frac{1}{2_*}) \max\{S_0^{\frac{2^*}{2^*-2}}, S_0^{\frac{2_*}{2_*-2}}\} \equiv \bar{S}.$$

First we are going to complete the proof of Theorem 1.1, postponing the proof of this result.

*Proof of Theorem* 1.1. First of all we shall prove that there exists a positive constant $C > 0$ such that

$$||u_n|| \leq C,\ \forall n \in I\!N.$$

Indeed, since

$$\Phi(u_n) = c + o(1), \tag{4}$$

$$\Phi'(u_n)u_n = \langle \xi_n, u_n\rangle \text{ with } \xi_n \to 0 \text{ in } H^{-1}(\Omega). \tag{5}$$

Taking (4)$-\frac{1}{2}$(5) we infer that

$$\begin{aligned} \tfrac{1}{N}\textstyle\int_\Omega (u_{n+})^{2^*} + \tfrac{1}{2(N-1)}\int_{\partial\Omega}(u_{n+})^{2_*} \quad \leq \quad & \textstyle\int_\Omega(-F(x,u_{n+}) + \frac{1}{2}f(x,u_{n+})u_{n+}) \\ & + \textstyle\int_{\partial\Omega}(-G(y,u_{n+}) + \frac{1}{2}g(y,u_{n+})u_{n+}) \\ & + c + \tfrac{1}{2}||\xi_n||||u_n||. \end{aligned} \tag{6}$$

From $(f4)$ and $(g2)$, for all $\epsilon > 0$, there exist $A_\epsilon, B_\epsilon > 0$ such that

$$\frac{1}{2}f(x,u_{n+})u_{n+} - F(x,u_{n+}) \leq C\epsilon u_+^{2^*} + A_\epsilon u^2,\ \forall x \in \Omega,$$

$$\frac{1}{2}g(y,u_{n+})u_{n+} - G(y,u_{n+}) \leq C\epsilon u_+^{2_*} + B_\epsilon u^2,\ \forall y \in \partial\Omega.$$

For $\epsilon$ sufficiently small, from (6) we obtain

$$\int_\Omega (u_{n+})^{2^*} + \int_{\partial\Omega}(u_{n+})^{2_*} \leq c + C_1||u_n||,\ \forall n \in I\!N,\ \ C_1 > 0. \tag{7}$$

Combining (7) with (4) we reach that $||u_n||$ is bounded.

Now, passing to the subsequence if necessary, we can assume $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$.

Passing to the limit in $\Phi'(u_n)v = o(1)$, $v \in H^1(\Omega)$, as $n \to \infty$, we have

$$\Phi'(u)v = 0.$$

By the maximum principle it follows that $u \geq 0$ on $\Omega$ and $u$ is a positive solution, provided that

**Claim** $u \neq 0$.

Suppose that $u = 0$. Then since $\Omega$ is bounded,

$$\int_\Omega f(x, u_{n+})u_{n+} \to 0 \text{ and } \int_{\partial\Omega} g(y, u_{n+})u_{n+} \to 0, \text{ as } n \to \infty. \tag{8}$$

Combining (8) with (5), we obtain

$$\int_\Omega |\nabla u_n|^2 - \int_\Omega (u_{n+})^{2^*} - \int_{\partial\Omega} (u_{n+})^{2_*} = o(1),$$

and we can assume

$$\int_\Omega |\nabla u_n|^2 \to l, \quad \int_\Omega (u_{n+})^{2^*} \to l_1 \text{ and } \int_{\partial\Omega} (u_{n+})^{2_*} \to l_2, \text{as } n \to \infty,$$

with $l = l_1 + l_2$.

Again, since $\Omega$ is bounded, we have

$$\int_\Omega F(x, u_{n+}) \to 0, \quad \int_{\partial\Omega} G(y, u_{n+}) \to 0, \text{as } n \to \infty,$$

and thus from (4), we infer that

$$\frac{l}{2} - \frac{l_1}{2^*} - \frac{l_2}{2_*} = c. \tag{9}$$

So, we can assume that $l > 0$ (if not the proof is completed).

By definition of $S_0$, we obtain

$$l \geq S_0(l_1^{\frac{2}{2^*}} + l_2^{\frac{2}{2_*}}).$$

Since $l = l_1 + l_2$, we have

$$\begin{aligned} 1 &\geq S_0((\frac{l_1}{l})^{\frac{2}{2^*}} \frac{1}{l^{\frac{2^*-2}{2^*}}} + (\frac{l_2}{l})^{\frac{2}{2_*}} \frac{1}{l^{\frac{2_*-2}{2_*}}}) \\ &\geq \min\{\frac{1}{l^{\frac{2^*-2}{2^*}}}, \frac{1}{l^{\frac{2_*-2}{2_*}}}\} S_0((\frac{l_1}{l})^{\frac{2}{2^*}}) + (\frac{l_2}{l})^{\frac{2}{2_*}}) \\ &\geq \min\{\frac{1}{l^{\frac{2^*-2}{2^*}}}, \frac{1}{l^{\frac{2_*-2}{2_*}}}\} S_0(\frac{l_1 + l_2}{l})^{\frac{2}{2^*}} \\ &\geq \min\{\frac{1}{l^{\frac{2^*-2}{2^*}}}, \frac{1}{l^{\frac{2_*-2}{2_*}}}\} S_0. \end{aligned}$$

From this inequality, we reach

$$\max\{l^{\frac{2^*-2}{2^*}}, l^{\frac{2_*-2}{2_*}}\} \geq S_0.$$

That is,

$$l \geq \max\{S_0^{\frac{2^*}{2^*-2}}, S_0^{\frac{2_*}{2_*-2}}\}. \tag{10}$$

From (9), we obtain

$$c \geq (\frac{1}{2} - \frac{1}{2_*})l \geq \bar{S}$$

which is a contradiction. This proves that $u \neq 0$. □

**Remark 3.1.** The solution $u$ of the problem (1)–(3) obtained above satisfies either

$$\Phi(u) = c, \tag{11}$$

or

$$\Phi(u) \leq c - \bar{S}. \tag{12}$$

Indeed, we use the same technique as in [6]. Therefore, take a sequence $u_n$ as in the proof above such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$ and $u_n \to u$ a.e. in $\overline{\Omega}$. So, defining $v_n = u_n - u$, it is not difficult to see that

$$\Phi(u) + \int_\Omega \Big(\frac{1}{2}|\nabla v_n|^2 - \frac{1}{2^*}(v_{n+})^{2^*}\Big) - \int_{\partial\Omega} \frac{1}{2_*}(v_{n+})^{2_*} = c + o(1), \tag{13}$$

and

$$\int_\Omega \Big(|\nabla v_n|^2 - (v_{n+})^{2^*}\Big) - \int_{\partial\Omega} (v_{n+})^{2_*} = o(1).$$

Then, by passing to a subsequence if necessary we obtain

$$\int_\Omega |\nabla v_n|^2 \to l, \quad \int_\Omega (v_{n+})^{2^*} \to l_1, \quad \text{and} \quad \int_{\partial\Omega} (v_{n+})^{2_*} \to l_2.$$

Hence $l = l_1 + l_2$.
From (10) and (13) we conclude (11) or (12).

Now, we will prove Lemma 3.2.

*Proof of Lemma* 3.2. It is sufficient to prove that there exists $v_0 \in H^1(\Omega)$, $v_0 \geq 0$ on $\bar{\Omega}$, $v_0 \neq 0$ on $\Omega$, such that

$$\sup_{t \geq 0} \Phi(tv_0) < \bar{S}.$$

First of all we will state some estimates. Consider the cut-off function $\varphi \in C^\infty(\bar{\Omega})$ such that $0 \leq \varphi \leq 1$, $(x,t) \in \Omega \subset I\!R^{N-1} \times I\!R$ and $\varphi(x,t) = 1$ on a neighborhood $U$ of $(x_0, t_0)$ such that $U \subset w \subset \Omega$.

Define

$$u_\epsilon(x,t) = w_\epsilon(x,t)\varphi(x,t)$$

and

$$v_\epsilon(x,t) = \frac{u_\epsilon(x,t)}{(|u_\epsilon|^2_{2^*,\Omega} + |u_\epsilon|^2_{2_*,\partial\Omega})^{\frac{1}{2}}}.$$

The following estimates are proved by combining [16, Lemma 5.2] with the argument used in the proof of [6, Lemma 1.1]:

$$|\nabla v_\epsilon|^2_{2,\Omega} = S_0 + O(\epsilon^{N-2}), \tag{14}$$

$$|u_\epsilon|^{2^*}_{2^*,\Omega} = |u_1|^{2^*}_{2^*,\mathbb{R}^N_+} + O(\epsilon^N), \tag{15}$$

$$|u_\epsilon|^{2_*}_{2_*,\partial\Omega} = |u_1|^{2_*}_{2_*,\mathbb{R}^{N+}} + O(\epsilon^{N-1}), \tag{16}$$

$$|v_\epsilon|^2_{2,\Omega} = \begin{cases} o(\epsilon) & \text{for} \quad N \geq 4 \\ O(\epsilon) & \text{for} \quad N = 3. \end{cases} \tag{17}$$

As we mentioned before, it is sufficient to show that

$$\sup_{s\geq 0} \Phi(s\tilde{v_\epsilon}) < \bar{S},$$

where $\tilde{v_\epsilon}(x,t) = \alpha v_\epsilon(x,t)$ with $\alpha > 0$ to be chosen later on.

Notice that

$$|\nabla \tilde{v_\epsilon}|^2_{2,\Omega} = \alpha^2 |\nabla v_\epsilon|^2_{2,\Omega} \equiv X^2_\epsilon, \tag{18}$$

$$|\tilde{v_\epsilon}|^{2^*}_{2^*,\Omega} = \alpha^{2^*} |v_\epsilon|^{2^*}_{2^*,\Omega} \equiv A^{2^*}_\epsilon, \tag{19}$$

$$|\tilde{v_\epsilon}|^{2_*}_{2_*,\partial\Omega} = \alpha^{2_*} |v_\epsilon|^{2_*}_{2_*,\partial\Omega} \equiv B^{2_*}_\epsilon. \tag{20}$$

Thus substituting these equalities in to the expression of $\Phi(s\tilde{v_\epsilon})$ we have

$$\Phi(s\tilde{v_\epsilon}) = \frac{s^2}{2}X^2_\epsilon - \frac{s^{2^*}}{2^*}A^{2^*}_\epsilon - \frac{s^{2_*}}{2_*}B^{2_*}_\epsilon - \int_\Omega F(x,t,s\tilde{v_\epsilon}) - \int_{\partial\Omega} G(y,t,s\tilde{v_\epsilon}).$$

Since $\Phi(s\tilde{v_\epsilon}) \to -\infty$ as $s \to \infty$, there exists $s_\epsilon > 0$ such that

$$\sup_{s\geq 0} \Phi(s\tilde{v_\epsilon}) = \Phi(s_\epsilon \tilde{v_\epsilon}) \tag{21}$$

(If $s_\epsilon = 0$ the proof is finished.). From (21) we obtain

$$s_\epsilon X^2_\epsilon - s^{2^*-1}_\epsilon A^{2^*}_\epsilon - s^{2_*-1}_\epsilon B^{2_*}_\epsilon = \int_\Omega f(x,t,s_\epsilon\tilde{v_\epsilon})\tilde{v_\epsilon} + \int_{\partial\Omega} g(y,t,s_\epsilon\tilde{v_\epsilon})\tilde{v_\epsilon}. \tag{22}$$

By using the hypotheses on $f$ and $g$ it follows that

$$X^2_\epsilon \geq s^{2^*-2}_\epsilon A^{2^*}_\epsilon + s^{2_*-2}_\epsilon B^{2_*}_\epsilon.$$

So, from (18)–(20) we have

$$\begin{aligned} \min\{s^{2^*-2}_\epsilon, s^{2_*-2}_\epsilon\} &\leq \frac{X^2_\epsilon}{A^{2^*}_\epsilon + B^{2_*}_\epsilon} \\ &= \frac{\alpha^2 |\nabla v_\epsilon|^2_{2,\Omega}}{\alpha^{2^*}|v_\epsilon|^{2^*}_{2^*,\Omega} + \alpha^{2_*}|v_\epsilon|^{2_*}_{2_*,\partial\Omega}} \\ &\leq \frac{\alpha^2}{\min\{\alpha^{2^*}, \alpha^{2_*}\}}\left(\frac{|\nabla v_\epsilon|^2_{2,\Omega}}{|v_\epsilon|^{2^*}_{2^*,\Omega} + |v_\epsilon|^{2_*}_{2_*,\partial\Omega}}\right). \end{aligned} \tag{23}$$

Therefore from (14)–(16) and (23) we have

$$\lim_{\epsilon\to 0} \frac{X^2_\epsilon}{A^{2^*}_\epsilon + B^{2_*}_\epsilon} \leq \frac{1}{\min\{\alpha^{2^*-2}, \alpha^{2_*-2}\}}\left(\frac{S_0}{|u_1|^{2^*}_{2^*,\mathbb{R}^N_+} + |u_1|^{2_*}_{2_*,\mathbb{R}^{N-1}}}\right).$$

Now choosing $\alpha > 0$ such that

$$\frac{1}{\min\{\alpha^{2^*-2}, \alpha^{2_*-2}\}}\Big(\frac{1}{|u_1|^{2^*}_{2^*,\mathbb{R}^N_+} + |u_1|^{2_*}_{2_*,\mathbb{R}^{N-1}}}\Big) \leq 1,$$

from (23) results

$$\min\{s_\epsilon^{2^*-2}, s_\epsilon^{2_*-2}\} \leq S_0,$$

that is

$$s_\epsilon \leq \max\{S_0^{(2^*-2)^{-1}}, S_0^{(2_*-2)^{-1}}\}. \tag{24}$$

Also

$$X_\epsilon^2 \leq S_0 + O(\epsilon^{N-2}). \tag{25}$$

Since the critical level $c > 0$, we can assume that $s_\epsilon \geq c_0 > 0$, $\forall \epsilon > 0$.

From (22), we obtain

$$s_\epsilon^{2^*} A_\epsilon^{2^*} + s_\epsilon^{2_*} B_\epsilon^{2_*} \geq s_\epsilon^2 X_\epsilon^2 + O(\epsilon), \tag{26}$$

where in the above inequality we used the following facts

$$\int_\Omega \frac{f(x,t,s_\epsilon\tilde{v_\epsilon})\tilde{v_\epsilon}}{s_\epsilon} \to 0 \ \text{ and } \ \int_{\partial\Omega} \frac{g(x,t,s_\epsilon\tilde{v_\epsilon})\tilde{v_\epsilon}}{s_\epsilon} \to 0, \ \text{ as } \ n \to \infty.$$

Now inserting (26) into the expression of $\Phi(s_\epsilon\tilde{v_\epsilon})$, from (17) we infer that

$$\begin{aligned} \Phi(s_\epsilon\tilde{v_\epsilon}) &\leq \frac{s_\epsilon^2}{2}X_\epsilon^2 - \min\{\frac{1}{2^*}, \frac{1}{2_*}\}s_\epsilon^2 X_\epsilon^2 \\ &\quad - \int_\Omega F(x,t,s_\epsilon\tilde{v_\epsilon}) - \int_{\partial U} G(y,t,s_\epsilon\tilde{v_\epsilon}) + O(\epsilon) \\ &\leq (\frac{1}{2} - \frac{1}{2^*})s_\epsilon^2 X_\epsilon^2 - \int_\Omega F(x,t,s_\epsilon\tilde{v_\epsilon}) + O(\epsilon) \\ &= (\frac{1}{2} - \frac{1}{2_*})s_\epsilon^2 X_\epsilon^2 + h(x,t,s_\epsilon\tilde{v_\epsilon}) + O(\epsilon), \end{aligned}$$

where $h(x,t,s_\epsilon\tilde{v_\epsilon}) = -\int_\Omega F(x,t,s_\epsilon\tilde{v_\epsilon})$ and $N \geq 3$.

Arguing as in [6] together with $(f2)$ we can assume

$$\frac{h(x,t,s_\epsilon\tilde{v_\epsilon})}{\epsilon^{N-2}} \to -\infty, \ \text{ as } \ \epsilon \to \infty. \tag{27}$$

But from (24)

$$s_\epsilon^2 \leq \max\{S_0^{2(2^*-2)^{-1}}, S_0^{2(2_*-2)^{-1}}\},$$

and since (25) holds, we have

$$s_\epsilon^2 X_\epsilon^2 \leq \max\{S_0^{2^*(2^*-2)^{-1}}, S_0^{2_*(2_*-2)^{-1}}\} + O(\epsilon^{N-2}), \ \text{for } N \geq 3. \tag{28}$$

Therefore, from (27) and (28) we conclude that

$$\Phi(s_\epsilon\tilde{v_\epsilon}) < \bar{S}.$$

This proves Lemma 3.2. □

## 4. Concave-convex case

First of all, notice that by using the embeddings $H^1(\Omega) \hookrightarrow L^t(\Omega)$ with $t = q, 2^*$, and $H^1(\Omega) \hookrightarrow L^r(\partial\Omega)$ with $r = \tau, 2_*$, we have

$$\begin{aligned} \Phi(u) &\geq \frac{1}{4}(||u||^2 - 2\int_\Omega (\varrho + a)u^2) + \frac{1}{4}(||u||^2 - 2\int_{\partial\Omega} bu^2) \\ &\quad - \int_\Omega (\frac{\lambda}{q}|u|^q + \frac{1}{2^*}|u|^{2^*}) - \int_{\partial\Omega} (\frac{\mu}{\tau}|u|^\tau + \frac{1}{2_*}|u|^{2_*}) \\ &\geq C_1||u||^2 - \lambda C_2||u||^q - \mu C_3||u||^\tau - C_4||u||^{2_*} - C_5||u||^{2^*}, \end{aligned}$$

for some positive constants $C_i$ (i=1, 2, …, 5).

Define

$$h(t) \equiv h_{\lambda\mu}(t) = C_1t^2 - \lambda C_2t^q - \mu C_3t^\tau - C_4t^{2_*} - C_5t^{2^*}.$$

Thus

$$\Phi(u) \geq h(||u||).$$

Take the cut-off function $\xi : I\!R_+ \to [0, 1]$ nonincreasing, smooth, such that

$$\xi(t) = \begin{cases} 1 & \text{if } t \leq R_0 \\ 0 & \text{if } t \geq R_1, \end{cases}$$

where $0 < R_0 = R_0(\lambda, \mu)$ and $0 < R_1 = R_1(\lambda, \mu)$ are chosen such that $h(s) \leq 0$ for $s \in [0, R_0]$ and $s \in [R_1, \infty]$; and $h(s) \geq 0$ for $s \in [R_0, R_1]$.

Now, setting $\varphi(u) = \xi(||u||)$, $u \in H^1(\Omega)$, define the truncated functional

$$\Phi_\varphi(u) \geq \frac{1}{2}||u||^2 - \int_\Omega (F(x, \varphi(u)u) + \frac{u^{2^*}}{2^*}\varphi(u)) - \int_{\partial\Omega} (G(y, \varphi(u)u) + \frac{u^{2_*}}{2_*}\varphi(u)).$$

Then $\Phi_\varphi \in C^1(B(0, R_0), I\!R)$ $(B(0, R_0) \subset H^1(\Omega))$ and

$$\Phi_\varphi(u) \geq h_{\lambda\mu\varphi}(||u||),$$

where

$$h_{\lambda\mu\varphi}(t) = C_1t^2 - \lambda C_2t^q - \mu C_3t^\tau - C_4t^{2_*}\xi(t) - C_5t^{2^*}\xi(t)$$

and

$$h_{\lambda\mu}(t) = h_{\lambda\mu\varphi}(t) \text{ if } t \leq R_0.$$

Notice that if $\Phi_\varphi(u) \leq 0$, then $||u|| \leq R_0$ for some $R_0 > 0$, thus $\Phi = \Phi_\varphi$.

The next Lemma gives us the compactness conditions for our proof.

**Lemma 4.1.** *For* $\lambda, \mu > 0$ *sufficiently small,* $\Phi_\varphi$ *satisfies condition* $(PS)_c$*, namely, every sequence* $(u_k) \subset H^1(\Omega)$ *satisfying* $\Phi_\varphi(u_k) \to c$ *and* $\Phi'_\varphi(u_k) \to 0$ *in* $H^{-1}(\Omega)$ *is relatively compact, provided*

$$c \in (-\bar{S}, 0) \quad \lambda, \mu > 0 \textit{ small enough.}$$

*Proof.* According to the remarks above, we are going to prove the lemma for $\lambda, \mu > 0$ small enough such that

$$\Phi_{\varphi}(u) \geq h_{\lambda\mu\varphi}(||u||) \geq -\bar{S}. \tag{29}$$

Let $(u_k) \subset H^1(\Omega)$ such that

$$\Phi_{\varphi}(u_k) = \Phi(u_k) \to c, \ \text{as } k \to \infty,$$

$$\Phi'_{\varphi}(u_k) = \Phi'(u_k) \to 0 \ \text{ in } \ H^{-1}(\Omega), \ \text{as } k \to \infty,$$

with $||u_k|| \leq R_0$. Then, we can assume that

$$\begin{array}{rcl} u_k & \rightharpoonup & u, \ \text{(weakly) in } \ L^{2^*}(\Omega) \text{ and } \ L^{2_*}(\partial\Omega), \\ u_k & \to & u, \ \text{(strongly) in } \ L^{q}(\Omega) \text{ and } \ L^{\tau}(\partial\Omega), \\ u_k & \to & u, \ \text{(a.e.) in } \ \overline{\Omega}. \end{array}$$

Thus

$$\frac{1}{2}\int_{\Omega}(|\nabla u_k|^2 + \varrho|u_k|^2) - \int_{\Omega}(\frac{a}{2}|u_k|^2 + F_1(x,u_k) + \frac{1}{2^*}|u_k|^{2^*}) - \int_{\Omega}\varrho u_{k+}^2$$
$$- \int_{\partial\Omega}(\frac{b}{2}|u_k|^2 + G_1(y,u_k) + \frac{1}{2_*}|u_k|^{2_*}) = c + o(1),$$

and

$$\left\{ \begin{array}{rcl} -\triangle u_k + \varrho u_k - (au_k + f_1(x,u_k) + |u_k|^{2^*-2}u_k) - \varrho u_{k+} & = & \eta_k, \\ \frac{\partial u_k}{\partial \nu} - (bu_k + g_1(y,u_k) + |u_k|^{2_*-2}u_k) & = & \nu_k, \end{array} \right.$$

where $\eta_k, \nu_k \to 0$, in $H^{-1}(\Omega)$. That is, $\Phi'_{\varphi}(u)u = 0$.

Letting $v_k = u_k - u$, by the Brezis and Lieb lemma, we have

$$\Phi_{\varphi}(u) + \int_{\Omega}|\nabla v_k|^2 - \frac{1}{2^*}|v_k|^{2^*}_{2^*,\Omega} - \frac{1}{2_*}|v_k|^{2_*}_{2_*,\partial\Omega} = c + o(1), \tag{30}$$

and

$$\int_{\Omega}|\nabla v_k|^2 - |v_k|^{2^*}_{2^*,\Omega} - |v_k|^{2_*}_{2_*,\partial\Omega} = o(1). \tag{31}$$

Making (30) $-\frac{1}{2^*}$(31) and (30) $-\frac{1}{2_*}$(31) we reach

$$\frac{1}{N}\int_{\Omega}|\nabla v_k|^2 = (\frac{1}{2_*} - \frac{1}{2^*})|v_k|^{2_*}_{2_*,\partial\Omega} + c + o(1) - I_{\varphi}(u), \tag{32}$$

$$\frac{1}{2(N-1)}\int_{\Omega}|\nabla v_k|^2 = (\frac{1}{2_*} - \frac{1}{2^*})|v_k|^{2^*}_{2^*,\Omega} + c + o(1) - I_{\varphi}(u). \tag{33}$$

From (32) and (33), we can assume (passing if necessary to a subsequence) that

$$\int_{\Omega}|\nabla v_k|^2 \to l \geq 0, \ \text{as } k \to \infty,$$

$$|v_k|^{2^*}_{2^*,\Omega} \to l_1 \geq 0, \quad |v_k|^{2_*}_{2_*,\partial\Omega} \to l_2 \geq 0, \ \text{as } k \to \infty.$$

Moreover, from (31) we have $l = l_1 + l_2$.

From (30) and (10) we have

$$\Phi_\varphi(u) = c - \frac{l}{2} + \frac{l_1}{2^*} + \frac{l_2}{2_*} < c - \frac{l}{2} + \frac{1}{2_*} l \leq c - \bar{S}.$$

On the other hand combining with (29), we conclude that $c > 0$, which is a contradiction. This completes the proof of Lemma 4.1. ☐

*Proof of Theorem* 1.2. For $\lambda, \mu > 0$ sufficiently small, by applying the Ekeland variational principle for the functional $\Phi_\varphi$, we will find a global minimum $u \in H^1(\Omega)$ for $\Phi_\varphi$, that is,

$$\Phi_\varphi(u) = \inf_{H^1(\Omega)} \Phi_\varphi = \Phi_\varphi(|u|).$$

So, we can assume $u > 0$ in $\Omega$. ☐

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# Existence of Solutions for a Class of Problems in $I\!R^N$ Involving the $p(x)$-Laplacian

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**Abstract.** In this work, we study the existence of solutions for a class of problems involving $p(x)$-Laplacian operator in $I\!R^N$. Using variational techniques we show some results of existence for a class of problems involving critical and subcritical growth.

**Keywords.** Variational methods, Sobolev embedding, quasilinear operator.

## 1. Introduction

In this paper, we consider the existence of solutions for the following class of quasilinear problems:

$$\begin{cases} -\Delta_{p(x)}u + u^{p(x)-1} = \lambda u^{q(x)} \text{ in } I\!R^N, \\ u \geq 0,\ u \neq 0 \text{ and } u \in W^{1,p(x)}(I\!R^N), \end{cases} \tag{$P_\lambda$}$$

where $p, q : I\!R^N \to I\!R$ are measurable functions satisfying some growth conditions, $\lambda$ is a positive parameter and $\Delta_{p(x)}$ is the $p(x)$-Laplacian operator given by

$$\Delta_{p(x)}u = div(|\nabla u(x)|^{p(x)-2}\nabla u(x)).$$

Motivated by the papers of Fan at al. [4, 5, 6, 7] and references therein, we consider this class of operators due to the following facts:

- *This operator appears in some physical problems, for example, in the theory of elasticity in mechanics.*
- *This operator has interesting properties, such as, it is not homogeneous if the function p is not constant. This fact implies some difficulties, as for example, we can not use the Lagrange Multiplier Theorem in a lot of problems involving this operator.*

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In this work, our main objective is to study the behavior of the functions $q(x)$ and $p(x)$ at infinity to get positive solutions for the problem $(P_\lambda)$ in $\mathbb{R}^N$. We would like to mention that the results contained in this work are preliminary and other situations have been considered by the authors.

This paper is organized in the following way: In Section 1, we recall some results involving the space $W^{1,p(x)}(\mathbb{R}^N)$ which can be found in [4, 5, 6, 7]. In Section 2, we study problem $(P_\lambda)$ considering a $p(x)$-subcritical case. In that section we get existence of solutions for two different behaviors of $q(x)$ at infinity. In Section 3, we work with the $p(x)$-critical growth case. Depending of the behavior of the function $q(x)$ at infinity in relation to the number $2^*$, we show the existence of a solution for the problem

$$\begin{cases} -\Delta_{p(x)}u = u^{q(x)} \text{ in } \mathbb{R}^N, \\ u \geq 0, u \neq 0 \text{ and } u \in D^{1,p(x)}(\mathbb{R}^N). \end{cases} \tag{$P_*$}$$

## 2. A Short Review on the Spaces $W^{k,p(x)}(\mathbb{R}^N)$

In this section, we remember the definitions and some results involving the spaces $W^{k,p(x)}(\mathbb{R}^N)$, which can be found in the papers [4, 5, 6, 7]. Moreover, in the end of this section we write the relations between the functions $p(x)$ and $q(x)$ explored in all this work.

### 2.1. Definitions and technical results

Throughout this section, $\Omega$ is assumed to be an open domain in $\mathbb{R}^N$, which may be unbounded, with cone property and $p : \overline{\Omega} \to \mathbb{R}$ a measurable function satisfying

$$1 < p_- := essinf_{x\in\overline{\Omega}} p(x) \leq p_+ := esssup_{x\in\overline{\Omega}} p(x) < \frac{N}{k}$$

where $k$ is a given positive integer verifying $kp < N$.

Set

$$L^{p(x)}(\Omega) := \Big\{u : u : \Omega \to \mathbb{R} \text{ is measurable function}, \int_\Omega |u|^{p(x)}dx < \infty\Big\}.$$

We can introduce the norm on $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} := \inf\Big\{\lambda > 0 : \int_\Omega \Big|\frac{u}{\lambda}\Big|^{p(x)} dx \leq 1\Big\},$$

and $L^{p(x)}$ becomes a Banach space.Moreover, it is easy to prove the following result.

**Lemma 2.1** *Let* $\{u_n\}$ *be a sequence in* $L^{p(x)}(\Omega)$. *Then,*

$$|u_n - u|_{p(x)} \to 0 \Leftrightarrow \int_\Omega |u_n - u|^{p(x)}dx \to 0 \ \ as \ \ n \to \infty.$$

For any positive integer $k$, set

$$W^{k,p(x)}(\Omega) := \Big\{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k\Big\}.$$

We can define the norm on $W^{k,p(x)}(\Omega)$ by

$$\|u\| = |u|_{p(x)} + \sum_{|\alpha|\le k} |D^\alpha u|_{p(x)}$$

and $W^{k,p(x)}(\Omega)$ also becomes a Banach space.

Hereafter, let us denote by $D^{1,p(x)}(I\!R^N)$ the closure of $C_0^\infty(I\!R^N)$ in relation to the norm

$$\|u\|_* = |\nabla u|_{p(x)}.$$

In what follows, we state some results involving these spaces.

**Theorem A.** *The spaces* $L^{p(x)}(\Omega)$ , $W^{k,p(x)}(\Omega)$ *and* $D^{1,p(x)}(I\!R^N)$ *are both separable and reflexive.*

**Theorem B.** *If* $p : \overline{\Omega} \to I\!R$ *is a Lipschitz continuous function and* $q : \overline{\Omega} \to I\!R$ *is a measurable function satisfying*

$$p(x) \le q(x) \le p^*(x) = \frac{Np(x)}{N - kp(x)}, \quad a.e \ \ x \in \overline{\Omega},$$

*then there is a continuous embedding* $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

Hereafter, let us denote by $L_+^\infty(\Omega)$ the set

$$L_+^\infty(\Omega) := \Big\{p : p \in L^\infty(\Omega), \inf_{x\in\Omega} p(x) > 1\Big\},$$

and if $f, g \in L_+^\infty(\Omega)$, let us denote by $f(x) << g(x)$ the property

$$\inf_{x\in\Omega}(g(x) - f(x)) > 0.$$

**Theorem C.** *Let* $\Omega$ *be bounded,* $p : \overline{\Omega} \to I\!R$ *be a continuous functions, and* $q$ *any measurable function defined in* $\Omega$ *satisfying*

$$p(x) \le q(x), \ \ a.e \ x \in \overline{\Omega}$$

*and*

$$q(x) << p^*(x).$$

*Then, there is a compact embedding* $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

### 2.2. Hypotheses involving the functions $p(x)$ and $q(x)$

In this paper, we will assume that $p$ and $q$ are continuous functions satisfying

$$p^- \leq p(x) \leq p^+ = ||p||_\infty \text{ a.e. in } I\!R^N, \tag{$H_1$}$$

$$q^- \leq q(x) \leq q^+ = ||q||_\infty \text{ a.e. in } I\!R^N, \tag{$H_2$}$$

with $p^-, q^- > 1,\ \ p^+ < q^- + 1$ and

$$p(x) - 1 << q(x) << p^*(x) - 1. \tag{$H_3$}$$

and

$$p(x) \geq m \text{ a.e. in } I\!R^N,\ \ p(x) \equiv m \text{ for all } |x| \geq R, \tag{$H_4$}$$

Observe that $m = p^-$. The behavior of the function $p(x)$ at infinity implies the following results:

**Lemma 2.2** *Condition* $(H_4)$ *implies that there exists a continuous embedding between* $W^{1,p(x)}(I\!R^N)$ *and* $W^{1,m}(I\!R^N)$.

*Proof.* In fact, for each $u \in W^{1,p(x)}(I\!R^N)$, we have

$$|\nabla u|^m(x) \leq (1 + |\nabla u|^{p(x)})\chi_{B_R}(x) + |\nabla u|^m(1 - \chi_{B_R})(x)$$

and

$$|u|^m(x) \leq (1 + |u|^{p(x)})\chi_{B_R}(x) + |u|^m(1 - \chi_{B_R})(x)$$

for all $x \in I\!R^N$, where $B_R = B_R(0) = \{x \in I\!R^N : |x| \leq R\}$, and $\chi_{B_R}$ is the characteristic function of $B_R$. The above inequalities together with Lebesgue's theorem imply that the identity application between $W^{1,p(x)}(I\!R^N)$ and $W^{1,m}(I\!R^N)$ is continuous. □

## 3. The Mountain Pass Geometry

Since $W^{1,p(x)}(I\!R^N)$ has different properties than those explored for the case when $p(x)$ is a constant function, a careful analysis is necessary in the mountain pass geometry, and in particular, for the Palais–Smale condition.

The functional of Euler–Lagrange related to problem $(P_\lambda)$ is given by

$$I_\lambda(v) = \int_{I\!R^N} \left[\frac{|\nabla v|^{p(x)}}{p(x)} + \frac{|v|^{p(x)}}{p(x)}\right] dx - \lambda \int_{I\!R^N} \frac{v_+^{q(x)+1}}{q(x)+1} dx.$$

Hereafter, let us denote by $I_\infty$, the Euler–Lagrange functional related to the problem

$$\left\{\begin{array}{l} -\Delta_m u + u^{m-1} = \lambda u^s \ \text{ in } \ I\!R^N, \\ u \geq 0,\ u \neq 0 \text{ and } \ u \in W^{1,m}(I\!R^N), \end{array}\right. \tag{$P_\infty$}$$

which is given by

$$I_\infty(v) = \int_{I\!R^N} \left[\frac{|\nabla v|^m}{m} + \frac{|v|^m}{m}\right] dx - \lambda \int_{I\!R^N} \frac{v_+^{s+1}}{s+1} dx$$

where $N < m$, $s \in (m-1, m^*-1)$ and $m^* = \frac{Nm}{N-m}$.