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Graph Theory in Paris

Proceedings of a Conference in Memory of Claude Berge

GT04
PARIS

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Jean Fonlupt
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Preface

Following the death of Claude Berge in June 2002, the Equipe Combinatoire, the group founded by Berge in 1975 under the aegis of the C.N.R.S. and in liaison with the Université Pierre et Marie Curie, decided to organise a conference on graph theory in his memory. This meeting, GT04, took place in July 2004. It was the first international conference on graph theory to be held in the Paris region since the memorable meeting in Orsay in 1976, which coincided with Claude's fiftieth birthday. The conference was held in the heart of the Latin Quarter, on one of the campuses of the Université Pierre et Marie Curie, the Couvent des Cordeliers (the former site of a Franciscan convent). Our aim was not only to celebrate the life and achievements of Claude Berge, but also to organise a conference in the line of continuity of the international meetings on graph theory and related topics which had been held successfully in Marseille-Luminy at five-year intervals since 1981.

GT04 brought together many prominent specialists on topics upon which Claude Berge's work has had a major impact, such as perfect graphs and matching theory. The meeting attracted over two hundred graph-theorists, roughly half of whom contributed to the scientific program. Plenary talks were presented by Maria Chudnovsky, Vašek Chvátal, Gérard Cornuéjols, András Frank, Pavol Hell, László Lovász, Jaroslav Nešetřil, Paul Seymour, Carsten Thomassen and Bjarne Toft.

Generous support for the conference was provided by the Université Pierre et Marie Curie, CNRS (Centre National de la Recherche Scientifique), INRIA (Institut National de Recherche en Informatique et en Automatique), the European network DONET (Discrete Optimization NETwork), the Délégation Générale pour l'Armement, France Télécom, ILOG and Schlumberger.

This volume includes contributions from many of the participants. All papers were refereed, and we are pleased to thank those colleagues who assisted us in this task. A short section of open problems presented during the meeting and edited by Rama Murty concludes the book.

The Editors

Adrian Bondy
Jean Fonlupt
Jean-Luc Fouquet
Jean-Claude Fournier
Jorge Ramírez Alfonsín

Claude Berge

Graph Theory
Trends in Mathematics, 1–9

Claude Berge – Sculptor of Graph Theory

Bjarne Toft

Abstract. Claude Berge fashioned graph theory into an integrated and significant part of modern mathematics. As was clear to all who met him, he was a multifaceted person, whose achievements, however varied they might seem at first glance, were interconnected in many ways.

1. Introduction

My purpose here is to present an account of some of Claude Berge's activities and achievements, mainly with regard to his role as a graph theorist. The information upon which I draw is mostly available in published sources. But the account is also personal, in that I shall include some of my own experiences and impressions. As a doctoral student in July 1969, I attended the *Colloquium on Combinatorial Theory and its Applications* in Balatonfüred at Lake Balaton. For me, this was like a dream, with its unique Hungarian charm and hospitality, the presence of many young people and of famous mathematicians like Berge, Erdős, Rényi, Rota, Turán and van der Waerden, to mention just a few. This was my first encounter with Berge, and I admired his French intellectual style, if at a distance. I also learned from him – at this meeting Berge emphasized the importance of hypergraphs, then still something of a novelty. Again in Hungary, at the meeting in Keszthely in June 1973 to celebrate Erdős' 60th birthday, I got to know Berge better. I gave my first conference lecture there, in the same afternoon session as Berge – I was nervous talking about hypergraph colouring with Berge sitting in the front row. But Berge was gracious and reassuring – he was without pretention, invariably putting those in his company at ease. As regards his mathematics, too, Berge had a distinctive manner, attempting always to combine the general with the concrete, and to see things in a general mathematical framework. He introduced hypergraphs not merely to generalize, but also to unify and simplify.

2. The late fifties and early sixties

The period around 1960 seems to have been particularly important and fruitful for Berge. Through the book *Théorie des graphes et ses applications* [2] he had established a mathematical name for himself. In 1959 he attended the first graph theory conference ever in Dobogokő, Hungary, and met the Hungarian graph theorists. He published a survey paper on graph colouring [4]. It introduced the ideas that soon led to perfect graphs. In March 1960 he talked about this at a meeting in Halle in East Germany [6]. In November of the same year he was one of the ten founding members of the OuLiPo (*Ouvroir de Littérature Potentiel*). And in 1961, with his friend and colleague Marco Schützenberger, he initiated the *Séminaire sur les problèmes combinatoires de l'Université de Paris* (which later became the*Equipe combinatoire du CNRS*). At the same time Berge achieved success as a sculptor [7].

3. Games, graphs, topology

Games were a passion of Claude Berge throughout his life, whether playing them – as in favorites such as chess, backgammon and hex – or exploring more theoretical aspects. This passion governed his interests in mathematics. He began writing on game theory as early as 1951, spent a year at the Institute of Advanced Study at Princeton in 1957, and the same year produced his first major book *Théorie générale des jeux à n personnes* [1]. Here, one not only comes across names such as von Neumann and Nash, as one would expect, but also names like König, Ore and Richardson. Indeed, the book contains much graph theory, namely the graph theory useful for game theory. It also contains much topology, namely the topology of relevance to game theory. Thus, it was natural that Berge quickly followed up on this work with two larger volumes, *Théorie des graphes et ses applications* [2] and *Espaces topologiques, fonctions multivoques* [3]. *Théorie des graphes et ses applications* [2] is a master piece, with its unique blend of general theory, theorems – easy and difficult, proofs, examples, applications, diagrams. It is a personal manifesto of graph theory, rather than a complete description, as attempted in the book by König [31]. It would be an interesting project to compare the first two earlier books on graph theory, by Sainte-Laguë [34] and König [31] respectively, with the book by Berge [2]. It is clear that Berge's book is more leisurely and playful than König's, in particular. It is governed by the taste of Berge and might well be subtitled 'seduction into graph theory' (to use the words of Rota from the preface to the English translation of [13]). Among the main topics in [2] are factorization, matchings and alternating paths. Here Berge relies on the fundamental paper of Gallai [25]. Tibor Gallai is one of the greatest graph theorists – he is to some degree overlooked – but not by Berge. Gallai was among the first to emphasize min-max theorems and LP-duality in combinatorics. In [26] one finds for the first time in writing the result (in generalized form) that the complement of a bipartite graph is perfect, attributed by Gallai to König and dated to 1932. But also [2] contains a theorem characterizing the size of a maximum independent set of vertices in a bipartite graph, which is easily seen to be equivalent to the fact that the complement of a bipartite graph is perfect. To notice this non trivial, yet simple, result seems to me to be a major step in the direction of the perfect graph conjectures. The 1959 book *Espaces topologiques, fonctions multivoques* [3] deals with general topology, focussing on what is useful in game theory, optimization theory and combinatorics. It includes a theory of multivalued functions, as stated in the title. And as Berge explains, when combinatorial properties of these functions are studied, it may be called a theory of oriented graphs. One of the theorems of [3] is known as *Berge's Maximum Theorem*. It deals with multivalued continuous mappings. It is very useful in economics and well known among economists. Also here, Berge manages to focus on the essential and useful. At *The History of Economic Thought* Website (http://cepa.newschool.edu/het/) the topic *Continuity and all that* is divided into four sections. Two of these deal with Berge's theory. They are called *Upper and lower semicontinuity of correspondences* and *Berge's Theorem*. In the 1960's two more books by Berge appeared, namely *Programmes, jeux et réseaux de transport* [8] and *Principes de combinatoire* [13]. In the preface to the English version of [13], which came out in 1971, Gian-Carlo Rota said:

> *Two Frenchmen have played a major role in the renaissance of Combinatorics: Berge and Schützenberger. Berge has been the more prolific writer, and his books have carried the word farther and more effectively than anyone anywhere. I recall the pleasure of reading the disparate examples in his first book, which made it impossible to forget the material. Soon after that reading, I would be one of the many who unknotted themselves from the tentacles of the Continuum and joined the then Rebel Army of the Discrete. What are newed pleasure is it to again read Berge in the present book!*

Both books [2] and [3] are now classics, and can still be purchased (in English) as new Dover Paperbacks. In 1970 Berge helped to give graph theory a new aspect by extending it to hypergraphs in the book *Graphes et hypergraphes* [15]. The purpose was to generalize, unify and simplify. The term hypergraph was coined by Berge, following a remark by Jean-Marie Pla, who had used the word hyperedge in a seminar. In 1978 Berge enriched the field once more with his lecture notes *Fractional Graph Theory* [17]. The purpose was again the same – and conjectures changed into elegant theorems in their fractional versions. The 1970 book was later split into two and appeared in the most recent versions as *Graphs* [20] and *Hypergraphs* [19]. In addition to his books Berge edited many collections of papers, some of which have been very influential, such as *Hypergraph Seminar* [16] and *Topics on Perfect Graphs* [18].

4. Perfect graphs

In 1960 Berge wrote a survey paper [4] on graph colouring, a topic not treated in depth in [2]. The paper was reviewed in *Mathematical Reviews* (MR 21, 1608) by Gabriel Andrew Dirac. Berge moved here into an area that Dirac knew like the back of his hand – and where Dirac had thought a lot about how to best prove and present results. Also Dirac, with his Hungarian background, did graph theory in the style of König, and he always maintained that König's book was the best source for learning graph theory. So Dirac was not particularly fond of Berge's more leisurely style, and his review of [4] was quite critical. Seen from today's perspective it seems too harsh. Here the first hints of perfect graphs appeared. As Berge wrote (my translation):

> *We shall here determine certain categories of graphs for which the chromatic number equals the clique number.*

In 1976 I attended the conference in Orsay to celebrate Berge's 50th birthday. I asked him about his reaction to Dirac's negative review. He said that he was just surprised and that he was on good terms with Dirac both before and after 1960. This was his nature – he took no offence. And Dirac invited Berge to visit Aarhus as one of the first graph theory guests after his appointment in Denmark in 1970. Dirac did his utmost to please Berge and make the visit a success, as I witnessed with my own eyes. I can still see the backs of Berge and Dirac, under an umbrella, disappearing in the fog and rain down Ny Munkegade in Aarhus. In 1959 in Dobogokő, Berge met Gallai, who told him about his work about graphs in which every odd cycle has two non-crossing chords, to be published in [27]. Berge saw immediately the importance of Gallai's work and included some of it in [4]. Berge called a graph in which every cycle has a chord a 'Gallai graph' (the terminology commonly used now is 'chordal graph' or 'rigid circuit graph') and proved the new result (in today's terminology) that such a graph is perfect, and he also included a proof of the theorem of Hajnal and Surányi that their complements are perfect – this they had presented in Dobogokő [29]. Berge called line-graphs of bipartite multigraphs 'pseudo-Gallai graphs'. Such graphs, and also their complements, are shown to be perfect. One property of these graphs is that all odd cycles of length at least 5 have chords. Berge remarked that, to establish perfectness, it is not enough to require only that all odd cycles of length at least five have chords, as the complement of the 7-cycle shows. He attributed this observation to A. Ghouila-Houri. Berge lectured about all these ideas at the colloquium on graph theory in Halle in East Germany, in March 1960, and wrote an extended abstract [6]. Here he defined a 'Gallai graph' as in [4] and a 'semi-Gallai graph' as one in which each odd cycle of length at least five has a chord. Berge mentions the result of Hajnal and Surànyi [29] that the complement of a Gallai graph is perfect and his own result that a Gallai graph itself is perfect. Moreover he notices that bipartite graphs, line-graphs of bipartite graphs and a class of Shannon graphs are perfect semi-Gallai graphs. At the end he says that it would seem natural to conjecture that all semi-Gallai graphs are perfect, but he then again exhibits the Ghouila-Houri counterexample (the complement of the 7-cycle). Berge does not mention complements of line-graphs of bipartite graphs nor complements of bipartite graphs in the Halle abstract. The abstract [6] is in German. Berge had given it the title (English translation) *Colouring of Gallai and semi-Gallai graphs.* The referee apparently asked Gallai if this was appropriate, and Gallai in his modest style answered that there was a misunderstanding and that he had never looked thoroughly at these classes. So Dirac, involved with the editing, on his way to take up a professorship at Ilmenau in East Germany, suggested the change of title to the one the paper ended up with. In [21] Berge mentions that the strong perfect graph conjecture was stated in Halle in March 1960 at the end of his lecture in the form, that if a graph and its complement are both semi-Gallai graphs (such graphs are now called Berge graphs) then the clique number and chromatic number are equal. It is clear from the abstract that this was certainly a natural question with which to end the lecture, but it also seems clear that the focus was not yet on the conjecture. In 1961 Berge spent the summer at a symposium on combinatorial theory at the RAND Cooperation in Santa Monica in California. There he presented his results, including some new ones on unimodular graphs, and he had many fruitful discussions, among others with Alan J. Hoffman. It seems likely that the whole English terminology, as we know it, was created here. On his return to Paris Berge wrote an English version of the theory of perfect graphs, and he sent it to Hoffman for comments. This manuscript, with some improvements suggested by Hoffman, Gilmore and McAndrew, appeared as *Some classes of perfect graphs* ([9], [12] and [14]). The 1963 paper [9] consists of lecture notes from the Indian Statistical Institute in Calcutta, which Berge visited in March and April 1963. Berge himself had at some point forgotten the existence of these published notes – they are not mentioned in the preprint [21], where he had some difficulties explaining why these important ideas from 1960 had to wait so long to get published. So he was pleased when I sent him a copy of [9]. I discovered [9] in the fine library of the University of Regina, Canada, in 1993. The published lecture notes are however not rare and are present in several libraries around the world. So my bet on the first publication using the term perfect graph and mentioning explicitly the perfect graph conjectures is Berge's paper [9] from 1963. This paper contains the whole basic theory of perfect graphs in the still commonly used terminology. Now, 40 years later the strong perfect graph conjecture has finally been proved [24]. In addition to [6], an abstract [10] from a meeting in Japan in September 1963 is often mentioned as an original source for perfect graphs. However the abstract is very short (nine lines) and mentions neither perfect graphs nor the perfect graph conjectures. At the meeting in Japan Berge did however distribute a manuscript. Judging from titles and from [21] this was the manuscript later to be presented and discussed at Ravello, Italy, in June 1964, and published in 1966 [11]. This paper (in French) contains the strong perfect graph conjecture and acknowledges Paul C. Gilmore's influence.

5. Problems

Berge was influential as an author of books, conference lecturer, thesis director and seminar organizer – the weekly seminar held at the *Maison des sciences de l'homme* on boulevard Raspail was legendary – but he was less so as a problem poser. His problems are relatively few and sometimes seem accidental. The book [2] has an appendix with fourteen unsolved problems. It might be interesting to examine these more closely with modern eyes. There are exceptions to this. In particular the perfect graph conjectures have had a huge impact. There are at least two other widely known circles of problems due to Berge: *The Berge Path Partition Conjectures* and *Berge's Hypergraph Edge Colouring Problems*. See [23] or [30] for a more detailed description. Finally, Berge edited for a period (1960–64) a magazine column [5] of brain teasers. It might also be interesting to take a closer look at these (unfortunately I did not have access to this material).

6. OuLiPo

The OuLiPo is a group of French writers and intellectuals, who experiment with literature. When one writes literature, poetry or music, one imposes on oneself certain restrictions. The main idea of this workshop for potential literature is to make these restrictions of a more precise mathematical nature (as Schönberg did in music and Lewis Carroll did in some of his writings). Martin Gardner wrote two columns on OuLiPo in *Scientific American* in the late 1970's, later to be completely rewritten and included in the book [28]. He said that

> *the most sophisticated and amusing examples of literary word play have been produced by the whimsical, slightly mad French group called the Oulipo.*

There are some books in English about and by the group, among them [32] and [33]. In both these books Berge is prominently featured. Berge was active in OuLiPo and wrote several articles, responsible as he was for "combinatory analysis". His most well-known OuLiPo work is the short story *Who killed the Duke of Densmore?* [22], which is a classical crime story, where the solution however requires knowledge of the Theorem of Hajós, characterizing interval graphs (Berge had heard György Hajós lecture about this theorem in Halle in March 1960). With this theorem it is possible to see that the set of events cannot have taken place as described by the participants, because the overlap graph of the events as described is not an interval graph. So at least one of the participants must be lying. But only the removal of one particular vertex of the corresponding overlap graph changes this into an interval graph, so this reveals the culprit. Berge pays tribute to the author Lewis Carroll and also to Carroll's alter ego, the mathematician C.L. Dodgson, both of whom are represented in the Duke's library. In an interval graph the sequence of events cannot be fully determined since a suitable sequential ordering of the intervals may be reversed. One of the persons in the short story contemplates the possibility of writing a novel with a set of events corresponding to an interval graph, where the two possible orderings in time would give two different solutions to the plot. Maybe Berge himself tried to create such an interesting (possible?) sequence of events? Berge spoke to Adrian Bondy of his wish to write a detective story in which the reader is the murderer, or the author, or the publisher... In [33] there are other interesting contributions, for example the ultimate lipogram, a book where not only one letter, but all letters have been avoided (where however the list of contents, footnotes, index, errata list, foreword and afterword, not being part of the text of the book itself, do use letters!), and a paper by the famous author Raymond Queneau (who had attended Berge's graph theory seminars around 1960), based on Hilbert's axioms for plane geometry, where point has been replaced by word and line by sentence, thus providing a foundation for literature (Hilbert told us that the words point and line are undefined and may be called anything). The OuLiPo group surely had/have a lot of fun, and their meetings, which take place inpublic, are packed. This was in particular so for their meeting in Berge's memory, at which he was officially 'excused' for his absence.

7. Sculpture

In our modern everyday life we are surrounded and bombarded by (too) beautiful, flawless pictures, sculptures and designs. In this stream Claude Berge's sculptures catch our attention, with their authenticity and honesty. They are not pretending to be more than they are. Berge catches again something general and essential, as he did in his mathematics. The sculptures may at first seem just funny, and they certainly have a humorous side. But they have strong personalities in their unique style – you come to like them as you keep looking at them – whether one could live with them if they came alive is another matter! The book *Sculptures multipètres* [7] gives a good impression of Berge's early sculptures, made partly from stones he found in the Seine. It was prefaced by Philippe Soupault, a well-known surrealist writer.

8. Conclusion

Claude Berge's greatest scientific achievement is that he gave graph theory a place in mathematics at large by revealing and emphasizing its connections to set theory, topology, game theory, operations research, mathematical programming, economics and other applications. The influence came mainly through his books, but also via his lectures, discussions at conferences and seminars, and of course very strongly through his many students. He generalized, unified, simplified, and combined in a unique way the general and the concrete. He will for a long time remain an inspirational force. So let us continue to play games and enjoy graph theory Claude Berge style!

9. Acknowledgment

I am indebted to a large number of people who helped me prepare my lecture at the GT04 meeting in Paris in July 2004 on which this paper is based. In particular I wish to thank Jorge Ramírez Alfonsín, Adrian Bondy, Kathie Cameron, Jack Edmonds, Michel LasVergnas, Douglas Rogers and Gert Sabidussi.

References

[1] C. Berge, *Théorie générale des jeux à n personnes, Memorial des sciences mathématiques* **138**, Gauthier-Villars (Paris) 1957.

[2] C. Berge, *Théorie des graphes et ses applications*, Dunod (Paris) 1958. Translation: *The Theory of Graphs and its Applications*, Meuthen (London) and Wiley (New York)1962. Republished as: *The Theory of Graphs*, Dover (New York) 2001.

[3] C. Berge, *Espaces topologiques, fonctions multivoques*, Dunod (Paris) 1959 and 1962. Translation: *Topological Spaces: including a Treatment of Multi-valued Functions, Vector Spaces and Convexity*, Oliver and Boyd (Edinburgh and London) 1963. Republished as: *Topological Spaces*, Dover (New York) 1997.

[4] C. Berge, Les problèmes de coloration en théorie des graphes, *Publ. Inst. Statist. Univ. Paris* **9** (1960), 123–160.

[5] C. Berge, Problèmes plaisants et delectables, Column in *Le Journal de l'A.F.I.R.O.*, 1960–64.

[6] C. Berge, Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind, *Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg*, Math.-Nat. **X/1** (1960), 114–115.

[7] C. Berge, *Sculptures multipètres*, présenté par Philippe Soupault, 1000 exemplaires numérotés, sur les presses de Lanord, imprimeur à Paris, 1962.

[8] C. Berge and A. Ghouila-Houri, *Programmes, jeux et réseaux de transport*, Dunod 1962. Translation: *Programming, games and transportation networks*, Methuen and Wiley 1965.

[9] C. Berge, Some classes of perfect graphs, in: *Six Papers on Graph Theory*, Indian Statistical Institute, Calcutta, 1963, 1–21.

[10] C. Berge, Sur une conjecture relative au problème des codes optimaux de Shannon, in: *Union Radio Scientifique Internationale, XIVe Assemblée Générale, Tokyo Sept. 9–20, 1963*,Volume **XIII-6**, *Ondes et Circuits Radio électriques*, U.R.S.I. Bruxelles 1963, 317–318.

[11] C. Berge, Une application de la Théorie des Graphes à un Problème de Codage, in: *Automata Theory* (ed. E.R. Caianiello), Academic Press 1966, 25–34.

[12] C. Berge, Some classes of perfect graphs, in: *Graph Theory and Theoretical Physics* (ed. F. Harary), Academic Press 1967, 155–165.

[13] C. Berge, *Principes de combinatoire*, Dunod 1968. Translation: *Principles of Combinatorics*, Academic Press 1971.

[14] C. Berge, Some classes of perfect graphs, in:*Combinatorial Mathematics and its Applications, Proceedings Conf. Univ. North Carolina, Chapel Hill 1967*, Univ. North Carolina Press 1969, 539–552.

[15] C. Berge, *Graphes et hypergraphes*, Dunod 1970. Translation: *Graphs and Hypergraphs*, North-Holland 1973.

[16] *Hypergraph Seminar*, Proceedings First Working Sem. Ohio State Univ., Columbus Ohio 1972 (ed. C. Berge and D. Ray-Chaudhuri), *Lecture Notes in Mathematics* **411**, Springer-Verlag 1974.

[17] C. Berge, *Fractional Graph Theory*, Indian Statistical Institute Lecture Notes No. **1**, The MacMillan Company of India 1978.

[18] *Topics on Perfect Graphs* (edited by C. Berge and V. Chvátal), *North-Holland Mathematical Studies* **88**, *Annals of Discrete Math.* **21**, North-Holland 1984.

[19] C. Berge, *Hypergraphs*, North-Holland 1989.

[20] C. Berge, *Graphs*, North-Holland 1991.

[21] C. Berge, *The history of perfect graphs*, Equipe Combinatoire **94/02**, Mars 1994, preprint, 8 pages.

[22] C. Berge, *Qui a tué le duc de Densmore?* Bibliotèhques Oulipienne No **67**, limited edition of 150 copies, 1994, English translation: *Who killed the Duke of Densmore?* in [33].

[23] C. Berge, Motivation and history of some of my conjectures, *Discrete Math.* **165/166** (1997), 61–70.

[24] M. Chudnovsky, N. Robertson, P. Seymour and R. Thomas, The strong perfect graph theorem, *Annals of Mathematics*, to appear.

[25] T. Gallai, On factorization of graphs, *Acta Math.Acad. Sci. Hung.* **I** (1950), 133–153.

[26] T. Gallai, Maximum-Minimum Sätze über Graphen, *Acta Math. Acad. Sci. Hung.* **IX** (1958), 395–434.

[27] T. Gallai, Graphen mit triangulierbaren ungeraden Vielecken, *Publ. Math. Inst. Hung. Acad. Sci.* **7**, (1962), 3–36.

[28] M. Gardner, *Penrose Tiles to Trapdoor Ciphers and the Return of Dr. Matrix*, Freeman 1989, updated, revised and republished by the Mathematical Association of America 1997.

[29] A. Hajnal and J. Surányi, Über die Auflösung von Graphen in vollständige Teilgraphen, *Ann.Univ. Budapest* **1** (1958), 53–57.

[30] T.R. Jensen and B. Toft, *Graph Coloring Problems*, Wiley Interscience 1995.

[31] D. König, *Theorie der endlichen und unendlichen Graphen*, Teubner, Leipzig 1936.

[32] *OULIPO a primer of potential literature* (W.F. Motte ed.), University of Nebraska Press 1986, and Dalkey ArchivePress, Illinois State University 1998.

[33] *OULIPO Laboratory.* Papers by Raymond Queneaux, Italo Calvino, Paul Fournel, Claude Berge, Jaques Jouet and Harry Mathews, Atlas Anti-classics 1995.

[34] M.A. Sainte-Laguë, *Les réseaux (ou graphes), Mémorial des sciences mathématiques*, Fascicule **XVIII**, Gauthier-Villars (Paris) 1926.

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Graph Theory
Trends in Mathematics, 11–19

k-path-connectivity and mk-generation: an Upper Bound on m

M. Abreu and S.C. Locke

Abstract. We consider simple connected graphs for which there is a path of length at least k between every pair of distinct vertices. We wish to show that in these graphs the cycle space over $\mathbb{Z}_2$ is generated by the cycles of length at least mk, where $m = 1$ for $3 \leq k \leq 6$, $m = 6/7$ for $k = 7$, $m \geq 1/2$ for $k \geq 8$ and $m \leq 3/4 + o(1)$ for large k.

Keywords. k-path-connectivity, cycle space, k-generation.

1. Introduction

For basic graph-theoretic terms, we refer the reader to Bondy and Murty [5]. All graphs considered are simple (without loops or multiple edges). For a graph G, we use $V(G)$ for the vertex set of G, $E(G)$ for the edge set, and $\varepsilon(G) = |E(G)|$. For a set $X \subseteq V(G)$, $G[X]$ denotes the subgraph of G induced by X. For a path P, the length of P is $\varepsilon(P)$. If P has ends x and y, we call P an *(x, y)-path.* For $u, v \in V(P)$ with u preceding v on P, $P[u, v]$ denotes the subpath of P from u to v. For a positive integer k, an *$(x, y : k)$-path* is an (x, y)-path of length at least k. A simple connected graph is *k-path-connected* if between every pair of distinct vertices, there is an $(x, y : k)$-path. It is easy to see that every maximal 2-connected subgraph of a k-path-connected graph is itself k-path-connected, and we may therefore restrict our study to graphs which are 2-connected and k-path-connected. Given a subgraph H of G, $d_H(x, y)$ will denote the distance in H between x and y (i.e., the length of the shortest (x, y)-path in H). Recall that $\kappa(G)$ is the (vertex) connectivity of G and that $N(x)$ is the neighborhood of a vertex $x \in V(G)$.

A cycle is a connected, 2-regular graph. For a cycle C, the length of C is $\varepsilon(C)$. We use the term *k^+-cycle* to refer to a cycle of length at least k. Given $x, y \in V(C)$, if $d_C(x, y) = \max\{d_C(x', y') : x', y' \in V(C)\}$ then x and y are said to be *antipodal*

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vertices of C. If $\varepsilon(C)$ is even there exists a unique antipodal for each vertex of C and an edge joining two antipodal vertices is called a *diameter* of C. If $\varepsilon(C)$ is odd there exist exactly two antipodals for each vertex of C and an edge joining two antipodal vertices is called a *near-diameter* of C. We use the standard notation $(v_1, \ldots, v_t)$ for the cycle C with edges v_iv_{i+1} for $i = 1, \ldots, t$ and the edge v_tv_1. This notation induces a natural orientation for the cycle from which we may denote by $C[v_i, v_j]$ the path on C with edges v_sv_{s+1} for $s = i, i+1, \ldots, j-1 \mod \varepsilon(C)$; and by $C^-[v_i, v_j]$ the path on C with edges v_sv_{s-1} for $s = i, i-1, \ldots, j+1 \mod \varepsilon(C)$. Let $\{x, y\}$ and $\{x', y'\}$ be two pairs of all distinct vertices of C. Then $\{x', y'\}$ is said to be a *separating* pair for the pair $\{x, y\}$ if the vertices appear on C in the order $xx'yy'$ or $xy'yx'$. A *chord* is an edge joining two non consecutive vertices of a cycle. Two chords are said to be crossing if the end vertices of one is a separating pair for the end vertices of the other. The *circumference* of a graph is the length of its longest cycle.

The *cycle space*, $Z(G)$, of a graph G is the vector space of edge sets of Eulerian subgraphs of G. A graph G is k-*generated* if the cycle space of G over $\mathbb{Z}_2$ is generated by the cycles of length at least k. A 2-connected graph G is a k-*generator* if it is both k-generated and $(k-1)$-path-connected. In [8] it was established that any 2-connected graph which contains a k-generator must itself be a k-generator.

The relation between long paths, cycle space, k-path-connectivity and k-generation of a graph has been studied by several authors. In particular, Bondy [4] conjectured that *if G is a 3-connected graph with minimum degree at least d and at least $2d$ vertices, then every cycle of G can be written as the symmetric difference of an odd number of cycles, each of whose lengths are at least $2d-1$* and Hartman [6] proved that *if G is a 2-connected graph with minimum degree d, where G is not K_{d+1} if d is odd, then the cycles of length at least $d+1$ generate the cycle space of G*. Locke [8, 9] partially proved Bondy's conjecture and gave ideas to extend the results presented. Furthermore, Locke [7, 8] gave another proof of Hartman's theorem and together with Barovich [3] generalized that result by considering fields other than $\mathbb{Z}_2$. Locke and Teng in [10] give some results on odd sums of long cycles in 2-connected graphs.

The families of graphs studied by Locke in [7, 8] turned out to be k-path-connected and $(k+1)$-generated. So in [9] he conjectured:

Conjecture 1.1. *For some constant m, $0 < m \leq 1$, every k-path-connected graph is mk-generated.*

From [2] we recall that a k-path-connected graph G (other than K_1) must have a cycle of length at least $k+1$ in each block. Thus, G is t-generated for $t \leq \left\lfloor \frac{k+3}{2} \right\rfloor$. This immediately improves the lower bound, so every k-path-connected graph is $\left\lfloor \frac{k+3}{2} \right\rfloor$-generated, for $k \geq 1$. While noting that any $(2k-3)^+$-cycle is a k-generator, implies that we only need to study k-path-connected graphs which contain cycles of length less than or equal to $2k-4$.

Locke in [9] proved that $m = 1$ for $3 \le k \le 5$ and Abreu, Labbate, Locke in [1] proved the following result:

Theorem 1.2. *Let G be a 2-connected, 6-path-connected graph with $|V(G)| \ge 9$ and minimum degree at least 3. Then G is 6-generated.*

In the next section we complete the proof of the following:

Theorem 1.3. *Let G be a 2-connected, 6-path-connected graph. Then G is 6-generated.*

The dodecahedron is an 18-path-connected graph but is only 17-generated [9], so it will not be possible to prove in general that a k-path-connected graph is k-generated.

However, in Section 3 we present a family of graphs that is $(4a+3)$-path-connected and $(3a+3)$-generated but not $(3a+4)$-generated for $a \ge 1$. This family allows us to prove

Theorem 1.4. *Let G be a 2-connected, k-path-connected graph, and $k \ge 7$. Then G is mk-generated where*

(i) *$m = 6/7$ for $k = 7$, and*
(ii) *$m \le 3/4 + o(1)$ for large k.*

2. 6-path-connected graphs

We first recall some results from [1].

Theorem 2.1. *Let G be a 2-connected, $(k-1)$-path-connected graph with a cycle C of length $2k-4$, then G is k-generated if one of the following holds:*

(1) *G is at least 3-connected,*
(2) *There are no diameters of C in $E(G)$,*
(3) *There is exactly one diameter of C in $E(G)$,*
(4) *There are at least three diameters of C in $E(G)$.*

Theorem 2.2. *Let G be a 2-connected, $(k-1)$-path-connected graph with a cycle C of length $2k-5$, then G is k-generated if one of the following holds:*

(1) *There are no near-diameters of C in $E(G)$,*
(2) *There is exactly one near-diameter of C in $E(G)$,*
(3) *There are at least three pairwise crossing near-diameters of C in $E(G)$.*

Now we present a couple of new results on 2-connected, $(k-1)$-path-connected graphs with a cycle of length at least $(2k-4)$.

Lemma 2.3. *Let G be a 2-connected, $(k-1)$-path-connected graph with a cycle $C = (v_1, v_2, \ldots, v_{2k-4})$ of length $(2k-4)$. If there is a diameter v_1v_{k-1} and a (v_i, v_j)-path P, internally disjoint from C where v_1v_{k-1} separates $\{v_i, v_j\}$, then either $|i-j| = k-2$ and P is a diameter or G is a k-generator.*

Proof. If $|i-j| = k-2$ and P is a diameter, there is nothing to prove. We may assume that $j > i$. Let $C_1 = C[v_1, v_i] \cup P \cup C[v_j, v_1]$ and $C_2 = C[v_i, v_j] \cup P$. $\varepsilon(C_1)+\varepsilon(C_2) = \varepsilon(C)+2\varepsilon(P) = 2k-4+2\varepsilon(P)$. If $\varepsilon(P) > 1$, then $\varepsilon(C_1)+\varepsilon(C_2) \geq 2k$, so $\max\{\varepsilon(C_1), \varepsilon(C_2)\} \geq k$. If $\varepsilon(P) = 1$, then $\varepsilon(C_1) + \varepsilon(C_2) = 2(k-1)$ and either $\varepsilon(C_1) = \varepsilon(C_2) = k-1$, in which case $|i-j| = k-2$ and P is a diameter, or $\max\{\varepsilon(C_1), \varepsilon(C_2)\} \geq k$. In both cases in which P is not a diameter, we can conclude that there is $C_3 \in \{C_1, C_2\}$ such that $\varepsilon(C_3) \geq k$. Therefore $C \cup P$ is k-generated.

Let $C_4 = C[v_1, v_i] \cup P \cup C^-[v_j, v_{k-1}] \cup v_{k-1}v_1$ and $C_5 = C^-[v_1, v_j] \cup P \cup C[v_i, v_{k-1}] \cup v_{k-1}v_1$. $\varepsilon(C_4) + \varepsilon(C_5) = \varepsilon(C) + 2\varepsilon(P) + 2 = 2k-4+2+2\varepsilon(P)$. Since $\varepsilon(P) \geq 1$, $\max\{\varepsilon(C_4), \varepsilon(C_5)\} \geq k$ and therefore, there is $C_6 \in \{C_4, C_5\}$ such that $\varepsilon(C_6) \geq k$. Hence $C \cup P \cup \{v_1v_{k-1}\}$ is k-generated.

Now we need to prove that if P is not a diameter, $H = C \cup P \cup \{v_1v_{k-1}\}$ is $(k-1)$-path-connected. For $x, y \in V(H) - V(C)$, with $x \neq y$, there are $x', y' \in V(C)$ such that there is an (x, x')-path P_1 in H and an a (y, y')-path P_2 in H with $\varepsilon(C[x', y']) \geq k-2$ or $\varepsilon(C^-[x', y']) \geq k-2$, and with P_1 disjoint from P_2. Therefore, either $\varepsilon(P_1 \cup C[x', y'] \cup P_2) \geq k$ or $\varepsilon(P_1 \cup C^-[x', y'] \cup P_2) \geq k$. Thus, there is an $(x, y : k)$-path in H.

For $x \in V(H) - V(C)$ and $y \in V(C)$, there is an $x' \in V(C)$ and an (x, x')-path P_3 with either $\varepsilon(P_3 \cup C[x', y]) \geq k-2$ or $\varepsilon(P_3 \cup C^-[x', y]) \geq k-2$, which gives us an $(x, y : k-1)$-path in H.

For $x, y \in V(C)$ with x and y not antipodal on C, there already is an $(x, y : k-1)$-path in $C \subseteq H$. Hence, we need only consider the case in which x and y are antipodal. If $\{x, y\} \neq \{v_1, v_{k-1}\}$, we may assume, without loss of generality, that $v_1xv_{k-1}y$ appear in this order on C and either $C^-[x, v_1] \cup v_1v_{k-1} \cup C[v_{k-1}, y] \geq k-1$ or $C[x, v_{k-1}] \cup v_1v_{k-1} \cup C^-[v_1, y] \geq k-1$. Hence there is an $(x, y : k-1)$-path in $C \cup P \cup \{v_1v_{k-1}\}$.

If $\{x, y\} = \{v_1, v_{k-1}\}$ we may assume, without loss of generality, that $x = v_1$ and $y = v_{k-1}$ and then either $C[v_1, v_i] \cup P \cup C^-[v_j, v_{k-1}] \geq k-1$ or $C^-[v_1, v_j] \cup P \cup C[v_i, v_{k-1}] \geq k-1$. Giving an $(x, y : k-1)$-path in H.

Therefore when P is not a diameter, $C \cup P \cup \{v_1v_{k-1}\}$ is $(k-1)$-path-connected and k-generated, hence a k-generator. □

Lemma 2.4. *Let G be a 2-connected, k-path-connected graph with a cycle $C = (v_1, v_2, \ldots, v_{2k-4})$ of length $2k-4$. If G contains two consecutive diameters of C, then G is a k-generator.*

Proof. Without loss of generality, suppose these two consecutive diameters are v_1v_{k-1} and v_2v_k. Suppose there is a (v_i, v_j)-path P crossing $\{v_2, v_{k-1}\}$. Thus P crosses at least one of v_1v_{k-1} or v_2v_k. By the previous lemma, $\varepsilon(P) = 1$, and the unique edge e of P is a diameter of C. But then, $e \notin \{v_1v_{k-1}, v_2v_k\}$, and $\{e, v_1v_{k-1}, v_2v_k\}$ is a set of three diameters of G, which by Theorem 2.1 implies that G is a k-generator.

Hence, we may assume that $G - \{v_2, v_{k-1}\}$ is disconnected. Since G is k-path-connected, there is a $(v_2, v_{k-1} : k)$-path Q in G. If $V(Q) \cap \{v_3, \ldots, v_{k-2}\} = \emptyset$, then $Q \cup C[v_2, v_{k-1}]$ is a $(2k-3)^+$-cycle, hence a k-generator. However if $V(Q) \cap \{v_3, \ldots, v_{k-2}\} \neq \emptyset$, then $V(Q) \cap \{v_k, \ldots, v_{2k-4}, v_1\} = \emptyset$, and $Q \cup C[v_{k-1}, v_2]$ is a $(2k-3)^+$-cycle, hence a k-generator. Thus, G is a k-generator. □

We now return to the discussion of a 2-connected, 6-path-connected graph G which is not a 6-generator. If G contains a 9^+-cycle, then G is a 6-generator. We may therefore assume G has no 9^+-cycle.

Suppose G has an 8-cycle $C = (v_1, v_2, \ldots, v_8)$. By Theorem 2.1, we may assume that G has exactly two diameters, and by Lemma 2.4 we may assume, without loss of generality, that these diameters are v_1v_5 and v_3v_7. From Lemma 2.3, no chord, which is not itself a diameter, can cross a diameter. Also, by Lemma 2.3, there can be no (v_i, v_j)-path P internally-disjoint from C if $\{v_i, v_j\}$ separates $\{v_1, v_5\}$ or if $\{v_i, v_j\}$ separates $\{v_3, v_7\}$.

Suppose that there is an edge xy in $G - V(C)$. Then, we can find a (v_i, v_j)-path P containing xy, and internally disjoint from C, with $v_i \neq v_j$. Since P cannot cross either diameter, without loss of generality, $\{v_i, v_j\} \subseteq \{v_1, v_2, v_3\}$. But then, $P \cup C$ contains a 9^+-cycle, contradicting our hypotheses about G. Hence, $\varepsilon(G - V(C)) = 0$. Suppose there is a vertex $x \in V(G) - V(C)$. Then, we can find a (v_i, v_j)-path P containing x, and internally disjoint from C, with $v_i \neq v_j$. Since P cannot cross either diameter, $\{v_i, v_j\} \subseteq \{v_{2m+1}, v_{2m+2}, v_{2m+3}\}$, for some $m \in \{0, 1, 2, 3\}$, subscripts modulo 8. Since G can have no cycle of length exceeding 8, $\{v_i, v_j\} = \{v_{2m+1}, v_{2m+3}\}$, and $\varepsilon(P) = 2$. Note that $N(x) \subseteq V(C)$, and thus $N(x) \subseteq \{v_1, v_3, v_5, v_7\}$. If $N(x) \neq \{v_{2m+1}, v_{2m+3}\}$, then (reversing the direction along the cycle, if necessary) we may assume that $xv_{2m+5} \in E(G)$, and $v_{2m+1}xv_{2m+5}$ is a path of length exceeding one and crossing the diameter $v_{2m+3}v_{2m+7}$, in contradiction to our assumption. Thus, $|N(x)| = 2$.

We now have a characterization of G. The graph G is an eight cycle $C = (v_1, v_2, \ldots, v_8)$, together with two crossing diameters, v_1v_5 and v_3v_7 and possibly some vertices of degree two joined to $\{v_{2m+1}, v_{2m+3}\}$, for various choices of m, and also possibly some chords $\{v_{2m+1}v_{2m+3}\}$, again for various choices of m. But, this graph has no (v_1, v_5)-path of length at least 6. This violates our assumptions about G.

This concludes all of the cases in which the circumference of G is 8, leaving only the cases where the circumference of G is 7, since a $(k-1)$-path-connected graph must contain a k^+-cycle. Suppose that G has a vertex v of degree 2, with $N(v) = \{x, y\}$. There is an $(x, y : 6)$-path P in G, and $P \cup xvy$ is an 8^+-cycle. Therefore, we may assume that $\delta(G) \geq 3$.

Let $C = (v_1, v_2, \ldots, v_7)$ be a 7-cycle in G and let $R_m = \{v \in V(G) : d(v, V(C)) = m\}$ be the set of vertices of G at distance exactly m from the cycle C. In previous work [1], we showed that $|R_2| = 0$ (or G contains an 8^+-cycle) and that $\varepsilon(G[R_1]) = 0$, (or G contains an 8^+-cycle). But then, for any $v \in R_1$, $H = C \cup \{v\} \cup \{vw : w \in N(v)\}$ is a 6-generator. Hence, $|R_1| = 0$, and $V(G) = V(C)$.

What remains is to check the Hamiltonian graphs on 7 vertices to see that any of which are 6-path-connected are also 6-generated. In order to do this we first prove some general results.

Lemma 2.5. *Let G be a graph, C a cycle of length $2k-5$ in G and S a set of chords of C in G. Then $C+S$ is $(k-1)$-path-connected if and only if for every antipodal pair of vertices $\{v_i, v_{i+k-3}\}$ there is a chord $v_sv_t \in S$ separating the pair $\{v_i, v_{i+k-3}\}$.*

Proof. Given a pair $\{v_i, v_{i+k-3}\}$, if there is chord v_sv_t that separates it, without loss of generality we may assume that the vertices appear in the order $v_iv_sv_{i+k-3}v_t$ on C. Then the paths $P_1 = C[v_i, v_s] \cup v_sv_t \cup C^-[v_t, v_{i+k-3}]$ and $P_2 = C^-[v_i, v_t] \cup v_tv_s \cup C[v_s, v_{i+k-3}]$ satisfy $\varepsilon(P_1) + \varepsilon(P_2) = \varepsilon(C) + 2 = 2k-3$, implying that $\max\{\varepsilon(P_1), \varepsilon(P_2)\} \geq k-1$. Therefore there is path $P_3 \in \{\varepsilon(P_1), \varepsilon(P_2)\}$ which is a $(v_i, v_{i+k-3} : k-1)$-path.

For the converse, let $\{v_i, v_{i+k-3}\}$ be a pair of vertices for which there is no chord in S separating it. This implies that $C+S-\{v_i, v_{i+k-3}\}$ is disconnected with exactly two components A_1 and A_2. For $i=1,2$, let $H_i = A_i \cap (C+S)$. Then the longest path in H_i for $i=1,2$ has length $k-2$. Therefore $C+S$ is not $(k-1)$-path-connected. □

Let H be a 2-connected graph and $s,t \in V(H)$. The graph H is said to be $\{s,t\}$*-near-*$(k-1)$*-path-connected* if $k \geq 3$, there is an $(s,t : k-2)$-path in H, and for every pair of distinct vertices $x,y \in V(H)$ with $\{x,y\} \neq \{s,t\}$, there is an $(x,y : k-1)$-path in H. If H is also k-generated, then it is said to be an $\{s,t\}$*-near-*k*-generator*. In [1] the following lemma was proved.

Lemma 2.6. *Let G be a 2-connected, $(k-1)$-path-connected graph that contains an $\{s,t\}$-near-k-generator H. Then, G contains a k-generator (and then G is a k-generator).*

Remark 2.7. In a cycle $C = (v_1, \ldots, v_{2k-5})$ of length $2k-5$ there are $(x,y : k-1)$-paths among all pairs of non-antipodal vertices $x,y \in C$. A near-diameter $e = v_iv_{i+k-3}$ separates every pair of antipodal vertices $\{v_j, v_{j+k-3}\}$ except for the pairs $p_1 = \{v_i, v_{i+k-2}\}$ and $p_2 = \{v_{i-1}, v_{i+k-3}\}$. Then any chord e' which is different from the near-diameters $e_1 = v_iv_{i+k-2}$ and $e_2 = v_{i-1}v_{i+k-3}$ and which crosses e, also crosses at least one of e_1, e_2. Therefore $C+e+e'$ is either a k-generator or a p_i-near-k-generator for some $i=1,2$.

Remark 2.8. In particular if $C = (v_1, \ldots, v_7)$ is a 7-cycle and if a 2-chord crosses a 3-chord (near-diameter) in C, we have a near-6-generator. To prove this, suppose without loss of generality that the 2-chord is the edge $e = v_2v_7$ and the 3-chord is the edge $e' = v_1v_4$. The cycles $C_1 = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)$, $C_2 = (v_2, v_3, v_4, v_5, v_6, v_7)$ and $C_3 = (v_1, v_2, v_7, v_6, v_5, v_4)$ generate the cycle space of $C+e+e'$, so it is 6-generated and by Remark 2.7 it is near-5-path-connected.

Let G be 6-path-connected Hamiltonian graph on 7 vertices and let $C = (v_1, \ldots, v_7)$ be a 7-cycle in G. The possible near-diameters of C are of the form

$e_j = v_j v_{j+3}$, subscripts modulo 7. We record the vector $F = (f_1, f_2, \ldots, f_7)$, where $f_j = 1$ if $e_j \in E(G)$ and $f_j = 0$ if $e_j \notin E(G)$. We list a representative of the equivalence class $[F]$ of F under the dihedral group D_7, acting on C. We write $C + F$ for the graph whose vertex set is $\{v_1, v_2, \ldots, v_7\}$ and whose edges are the edges of C, together with the edges $\{e_j : f_j = 1\}$. Those patterns which force three pairwise crossing near-diameters are marked with the symbol $\surd$, as are those with zero or one near-diameters (hence covered by Theorem 2.2). For each of the remaining patterns we study the addition of chords which are not near-diameters to $C + F$ and in each case we get either a 6-generator or a graph which is not 6-path-connected. Each pattern is marked with the corresponding observation below in which it is studied.

F	$\lvert[F]\rvert$		F	$\lvert[F]\rvert$	
$(0,0,0,0,0,0,0)$	1	$\surd$	$(1,1,1,1,0,0,0)$	7	$\surd$
$(1,0,0,0,0,0,0)$	7	$\surd$	$(1,1,1,0,1,0,0)$	14	$\surd$
$(1,1,0,0,0,0,0)$	7	Obs. 2.9	$(1,1,0,1,1,0,0)$	7	Obs. 2.12
$(1,0,1,0,0,0,0)$	7	Obs. 2.10	$(1,1,0,1,0,1,0)$	7	Obs. 2.12
$(1,0,0,1,0,0,0)$	7	Obs. 2.11	$(1,1,1,1,1,0,0)$	7	$\surd$
$(1,1,1,0,0,0,0)$	7	$\surd$	$(1,1,1,1,0,1,0)$	7	$\surd$
$(1,1,0,1,0,0,0)$	14	Obs. 2.12	$(1,1,1,0,1,1,0)$	7	$\surd$
$(1,1,0,0,1,0,0)$	7	Obs. 2.9	$(1,1,1,1,1,1,0)$	7	$\surd$
$(1,0,1,0,1,0,0)$	7	Obs. 2.10	$(1,1,1,1,1,1,1)$	1	$\surd$

Observation 2.9. Let $F_1 = (1,1,0,0,0,0,0)$. By Remark 2.8 and Lemma 2.5 the graphs $C + F_1 + v_2v_7$, $C + F_1 + v_4v_6$, $C + F_1 + v_1v_3$ and $C + F_1 + v_3v_5$ are 6-generators or near-6-generators. On the other hand for $F_2 = (1,1,0,0,1,0,0)$, by Lemma 2.5 $C + F_2 + \{v_2v_4, v_5v_7, v_1v_6\}$ is not 5-path-connected, since there is no $(v_1, v_5 : 5)$-path, therefore neither is $C + F_1 + \{v_2v_4, v_5v_7, v_1v_6\}$.

Observation 2.10. Let $F_3 = (1,0,1,0,0,0,0)$. By Remark 2.8 and Lemma 2.5 the graphs $C + F_3 + v_2v_7$, $C + F_3 + v_5v_7$, $C + F_3 + v_2v_4$ and $C + F_3 + v_3v_5$ are 6-generators. On the other hand $C + F_3 + \{v_1v_3, v_4v_6, v_1v_6\}$ is not 6-path-connected. For $F_4 = (1,0,1,0,1,0,0)$ we have that $C + F_4 + \{v_1v_3, v_1v_6\}$ is not 6-path-connected, since there is no $(v_3, v_6 : 6)$-path, while by Remark 2.8 $C + F_4 + v_4v_6$ is a near-6-generator.

Observation 2.11. Let $F_5 = (1,0,0,1,0,0,0)$. By Remark 2.8 and Lemma 2.5 the graphs $C + F_5 + v_1v_6$, $C + F_5 + v_2v_7$ and $C + F_5 + v_3v_5$ are 6-generators or near-6-generators. On the other hand, by Lemma 2.5 $C + F_5 + \{v_1v_3, v_2, v_4, v_4v_6, v_5v_7\}$ is not 5-path-connected, since there is no $(v_1, v_4 : 5)$-path.

Observation 2.12. Let $F_6 = (1,1,0,1,0,0,0)$ and note that $C + F_6$ is contained in $C + F_0 + v_4v_7$ where $F_0 = (1,0,0,0,0,0,0)$. Therefore $C + F_6$ is a 6-generator, as well as $C + F_7$ and $C + F_8$ with $F_7 = (1,1,0,1,1,0,0)$ and $F_8 = (1,1,0,1,0,1,0)$, which contain $C + F_6$.

This completes the proof of Theorem 1.3 □

3. The $M^+_{r,s,t}$ family

Let $X = \{u_1, u_2, u_3, u_4\}$ be the vertices of a copy of K_4. We replace each edge u_iu_j by a path P_{ij}, where $\varepsilon(P_{13}) = \varepsilon(P_{24}) = r$, $\varepsilon(P_{12}) = \varepsilon(P_{34}) = s$, $\varepsilon(P_{14}) = \varepsilon(P_{23}) = t$, with $1 \le r < s < t$. The resulting graph, $M_{r,s,t}$, has only seven cycles. Four of these cycles are of length $r+s+t$ and the other three have lengths: $2r+2s$, $2r+2t$ and $2s+2t$. Any basis for the cycle space uses at least one cycle of length $r+s+t$, and thus $M_{r,s,t}$ is $(r+s+t)$-generated but not $(r+s+t+1)$-generated. The cycle $P_{12} \cup P_{23} \cup P_{34} \cup P_{14}$ contains a (u_i, u_{i+1})-path of length at least $2s+t$, for $i = 1, 2, 3, 4$ (with $u_5 = u_1$). The cycle $P_{13} \cup P_{23} \cup P_{24} \cup P_{14}$ contains a (u_1, u_3)-path of length at least $2s+t$, and a (u_2, u_4)-path of length at least $2s+t$. The graph $M_{r,s,t}$ is at most $(\min\{2s+t, 2t+r\})$-path-connected. However, in general, the path-connectivity of $M_{r,s,t}$ is strictly less than this. For example, an $M_{1,a+1,2a+1}$ can't be more than $(4a+2)$-path-connected since for $x \in V(P_{23})$ at distance a from u_2 on P_{23}, the longest (x, u_4)-path in $M_{1,a+1,2a+1}$ has length $4a+2$.

We would like to modify $M_{r,s,t}$ to yield a graph with higher path-connectivity but which is not $(r+s+t+1)$-generated. Let $M^+_{r,s,t} = M_{r,s,t} \cup Q_{14} \cup Q_{23}$, where Q_{14} is a (u_1, u_4)-path and Q_{23} is a (u_2, u_3)-path, each of length t, internally disjoint from each other and from $M_{r,s,t}$. The graph $M^+_{r,s,t}$ has only 19 cycles. Eight of these cycles have length $r+s+t$, two have length $2t$, one has length $2r+2s$, four have length $2t+2r$ and four have length $2t+2s$. Any basis for the cycle space uses at least one cycle of length $r+s+t$, and thus $M^+_{r,s,t}$ is $(r+s+t)$-generated but not $(r+s+t+1)$-generated.

Lemma 3.1. *For $a \ge 1$, the graph $M^+_{1,a+1,2a+1}$ is $(4a+3)$-path-connected.*

Proof. For distinct vertices $x, y \in X$. As before we realize that we can always find an $(x, y : 4a+3)$-path in $M^+_{1,a+1,2a+1}$.

For $x \in V(P_{ij})$ we have $(x, u_i : 4a+3)$-paths and $(x, u_j : 4a+3)$-paths making use of the $(u_i, u_j : 4a+3)$-path found in the previous case, together with a segment of P_{ij}. Similarly for $x \in V(Q_{ij})$.

For $x \in V(M^+_{1,a+1,2a+1}) - X$ but $x \notin V(P_{ij}) \cup V(Q_{ij})$ there are $(x, u_i : 4a+3)$-paths and $(x, u_j : 4a+3)$-paths using two paths of length $2a+1$. For example, for $x \in V(P_{23})$, $P_{23}[x, u_3] \cup Q^-_{23} \cup u_2u_4 \cup P^-_{14}$ is an $(x, u_1 : 4a+3)$-path.

For distinct vertices $x, y \in V(M^+_{1,a+1,2a+1}) - X$. If $x, y \in V(P_{ij})$ we may assume without loss of generality that the vertices appear in the order u_i, x, y, u_j in $M^+_{1,a+1,2a+1}$. Consider the path $P_1 = P_{ij}[u_i, x] \cup P_{ik} \cup P_{kl} \cup P_{lj} \cup P^-_{ij}[u_j, y]$, where P_{kl} is a path of length $a+1$, and P_{ik} and P_{lj} are paths of length $2a+1$. Then P_1 is an $(x, y : 4a+3)$-path. Similarly for $x, y \in V(Q_{ij})$

If $x \in V(P_{ij})$, $y \notin V(P_{ij})$ and $\varepsilon(P_{ij}) = a+1$. Then if $y \in V(P_{kl})$ with $\varepsilon(P_{kl}) = a+1$, let $P_2 = P^-_{ij}[u_i, x] \cup P_{ik} \cup u_ku_l \cup P_{jl} \cup P^-_{kl}[y, u_l]$ where P_{ik} and P_{jl} are paths of length $2a+1$. Then P_2 is an $(x, y : 4a+3)$-path. If $y \in V(P_{ik})$ with $\varepsilon(P_{ik}) = 2a+1$, let $P_3 = P_{ij}[x, u_j] \cup P_{jl} \cup u_lu_i \cup Q_{ik} \cup P^-_{ik}[u_k, y]$ where P_{jl} is a path of length $2a+1$. Then P_3 is an $(x, y : 4a+3)$-path. Similarly if $y \in V(Q_{ik})$ or $y \in V(Q_{jl})$ or $y \in V(P_{jl})$ with $\varepsilon(P_{jl}) = 2a+1$.

If $x \in V(P_{ij})$, $y \notin V(P_{ij})$ and $\varepsilon(P_{ij}) = 2a+1$. Then if $y \in V(Q_{ij})$, let $P_4 = P_{ij}^{-}[u_i, x] \cup P_{ik} \cup P_{kl} \cup P_{jl}^{-} \cup Q_{ij}^{-}[u_j, y]$ where P_{ik} and P_{jl} are paths of length $a+1$ and P_{kl} is a path of length $2a+1$. Then P_4 is an $(x, y : 4a+3)$-path. Similarly if $y \in V(P_{kl})$ or $y \in V(Q_{kl})$. Analogously if $x \in V(Q_{ij})$ and $y \notin V(Q_{ij})$. □

Proof of Theorem 1.4. (i) For $a = 1$, $M^{+}_{1,a+1,2a+1}$ is a 7-path-connected graph which is only 6-generated, hence $m = 6/7$ for $k = 7$.

(ii) We have found a family of graphs which are $(4a+3)$-path-connected and $(3a+3)$-generated but not $(3a+4)$-generated. Since these graphs are also $(4a+2)$, $(4a+1)$ and $(4a)$-path-connected, we have, for $0 \le b \le 3$, graphs which are $(4a+b)$-path-connected and $(3a+3)$-generated but not $(3a+4)$-generated. Now, $\frac{3a+3}{4a+b}$ has a limit of approximately $\frac{3}{4}$, and approaching $\frac{3}{4}$ as a increases, hence $m \le 3/4 + o(1)$ for large k. □

We may now conclude that for $k \ge 8$ every k-path-connected graph is mk-generated for some constant m, with $\frac{1}{2} \le m \le \frac{3}{4}$.

References

[1] M. Abreu, D. Labbate and S.C. Locke, 6-path-connectivity and 6-generation, Discrete Math. **301** (2005), no. 1, 20–27.

[2] S.C. Locke, Personal communication with M. Abreu (2000).

[3] M. Barovich and S.C. Locke, The cycle space of a 3-connected Hamiltonian graph. Discrete Math. **220**, (2000), 13–33.

[4] J.A. Bondy, Personal communication with S.C. Locke (1979).

[5] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, Elsevier, North-Holland, 1976.

[6] I.B.-A. Hartman, Long cycles generate the cycle space of a graph, Europ. J. Combin. **4** (1983), 237–246.

[7] S.C. Locke, A basis for the cycle space of a 2-connected graph, Europ. J. Combin. **6** (1985), 253–256.

[8] S.C. Locke, A basis for the cycle space of a 3-connected graph, Cycles in Graphs, Annals of Discrete Math. **27** (1985), 381–397.

[9] S.C. Locke, Long Paths and the Cycle Space of a Graph, Ars Combin. **33**, (1992), 77–85.

[10] S.C. Locke and C. Teng, Odd sums of long cycles in 2-connected graphs, Congressus Numerantium **159**, (2002), 19–30.

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Automated Results and Conjectures on Average Distance in Graphs

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Abstract. Using the *AutoGraphiX* 2 system, a systematic study is made on generation and proof of relations of the form

$$\underline{b}_n \leq \bar{l} \oplus i \leq \bar{b}_n$$

where $\bar{l}$ denotes the average distance between distinct vertices of a connected graph G, i one of the invariants: diameter, radius, girth, maximum, average and minimum degree, $\underline{b}_n$ and $\bar{b}_n$ are best possible lower and upper bounds, functions of the order n of G and $\oplus \in \{-, +\times, /\}$. In 24 out of 48 cases simple bounds are obtained and proved by the system. In 21 more cases, the system provides bounds, 16 of which are proved by hand.

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1. Introduction

Classical books on graph theory, such as Berge's *Graphs and Hypergraphs* [4], present many lower and upper bounds on graph invariants (*i.e.,* numerical functions of graphs which do not depend on the numbering of vertices or edges) in terms of the graph's number n of vertices and/or m of edges. So it appears to be a naturel challenge to see if such bounds can be discovered automatically [8] by some computer system, (and if not, if such a system can provide substantial help, *e.g.*, by discovering extremal graphs for a given expression).

Recently, using the AutoGraphiX 2 (AGX 2) software [1, 5, 6], a systematic study has been performed [2] on automated generation of bounds of the following form:

$$\underline{b}_n \leq i_1 \oplus i_2 \leq \bar{b}_n$$

where $\underline{b}_n$ and $\bar{b}_n$ are expressions depending only on the order n of the graphs under study, i_1 and i_2 are graph invariants and $\oplus$ belongs to $\{+, -, \times, /\}$. Moreover, it is requested that the bound $\underline{b}_n$ and $\bar{b}_n$ be best possible in the strong sense that for all n (except very small values, due to border effects) there exists a graph for which the bound is tight. The proposed form generalizes formulae of the well-known Nordhaus-Gaddum [12] form, in that i_1 and i_2 are independent invariants instead of the same one in G and its complement $\overline{G}$ and that the operations $-$ and $/$ are considered in addition to $+$ and $\times$.

In the present paper we report in detail on results of the comparison of average distance in graphs with six other invariants: diameter, radius, girth, maximum, average and minimum degree.

These results fall into the following categories:

(a) *Automated complete results*: structural conjectures on the family of extremal graphs, algebraic expression of the bound, automated proof of this bound's validity and tightness (it turns out that such results are frequently obtained in a simple way; they are therefore referred to as *observations*);
(b) *Automated complete conjectures*: structural conjectures and algebraic relations obtained as above, but without automated proof. Some conjectures are proved by hand (and referred to as *propositions*), others remain open;
(c) *Semi-automated conjectures*: structural conjectures obtained automatically, but algebraic relations derived from them by hand; of those some are proved and some remain open;
(d) *Automated structural conjectures*, for which algebraic expressions have not been found (or do not exist);
(e) *No results*, as the (presumably) extremal graphs do not present any regularity.

In order to enable an informed evaluation of the results obtained, they are all presented. Simple ones are briefly listed. Their main interest is that they can enrich the database of relations used in the automated proofs. Other results are given with full proofs or with indications about how to prove them if a previous proof technique carries over.

The paper is organized as follow: each of the next six sections presents a comparison of average distance with diameter, radius, girth, maximum, average and minimum degree respectively. Brief conclusions are given in the last section. Observations made by AGX 2 are collected in the Appendix.

2. The diameter

The diameter D of a graph $G = (V, E)$ is defined by $D = \max\{d(u, v), u, v \in V\}$, where $d(u, v)$ is the distance between u and v in G. A *diametric* path in G is a path between two vertices u and v such that $d(u, v) = D$.

Automated results obtained by AGX 2, in 6 cases out of 8, when comparing the average distance $\overline{l}$ and the diameter D are given in Table 2 of the Appendix.

The following proposition was obtained automatically by AGX 2 and then proved by hand.

Proposition 2.1. *For any connected graph on at least* 3 *vertices,*

$$D - \bar{l} \le \frac{2n-4}{3}.$$

The bound is attained if and only if the graph is a path.

Proof. Let G be a connected graph of diameter D and average distance $\bar{l}$, and H a subgraph of G induced by a diametric path. Let

$$\sigma = \sum_{u,v \in V} d(u,v) \quad \text{and} \quad \sigma_H = \sum_{u,v \in V(H)} d(u,v),$$

where $V(H)$ is the set of vertices of H. It is easy to see that

$$\sigma \ge \sigma(H) = D \cdot (D+1) \cdot (D+2)/6$$

and

$$\bar{l} \ge D \cdot (D+1) \cdot (D+2)/(3n(n-1)).$$

Thus

$$\begin{aligned} D - \bar{l} &\le D - \tfrac{D\cdot(D+1)\cdot(D+2)}{3n(n-1)} \\ &\le \tfrac{3n(n-1)\cdot D - D\cdot(D+1)\cdot(D+2)}{3n(n-1)} \\ &\le \tfrac{-D^3 - 3D^2 + (3n(n-1)-2)\cdot D)}{3n(n-1)}. \end{aligned}$$

Easy algebraic manipulations show that this last expression is an increasing function of D. It thus reaches its maximum if and only if $D = n-1$, *i.e.,* if G is a path. □

Before stating the next conjecture, let us define the family of graphs called bugs [10]. A *bug* Bug_{p,k_1,k_2} is a graph obtained from a complete graph K_p by deleting an edge uv and attaching paths P_{k_1} and P_{k_2} at u and v, respectively. A bug is *balanced* if $|k_1 - k_2| \le 1$. (In a bug, $n = p + k_1 + k_2$ and $m = \frac{p(p-1)}{2} + k_1 + k_2 - 1$).

Conjecture 2.2. *Among all connected graphs on at least* 3 *vertices,* $D/\bar{l}$ *is maximum for a balanced bug.*

3. The radius

The eccentricity of a vertex v in a graph $G = (V,E)$ is defined by $\mathrm{ecc}(v) = \max\{d(u,v), u \in V\}$, where $d(u,v)$ is the distance between u and v in G. The radius of G is the minimum of its eccentricities, *i.e.,* $r = \min\{\mathrm{ecc}(v), v \in V\}$.

Automated results obtained by AGX 2, in 4 cases out of 8, when comparing the average distance $\bar{l}$ and the radius r are given in Table 2 of the Appendix.

The following proposition was obtained as a conjecture using *AGX* 2 in automated mode, and then proved by hand.

Proposition 3.1. *For any connected graph on at least* 3 *vertices,*

$$\bar{l}/r \leq 2 - \frac{2}{n}\cdot$$

The bound is attained if and only if the graph is a star.

Proof. If G is a connected graph of radius r, it contains a spanning tree T of the same radius $r(T) = r$. It is obvious that $\bar{l}(T) \geq \bar{l}$, where $\bar{l}(T)$ and $\bar{l}$ denote the average distance in T and G respectively, with equality if and only if $G \equiv T$. So $\bar{l}/r$ is maximum for a tree and we can assume that G is a tree.

Let m_i denote the number of vertex pairs in G at distance i, for $i = 1, \ldots, D$ where D is the diameter of G. We have:

$$\begin{array}{rcl} \bar{l} & = & 2 \cdot (m_1 + 2m_2 + 3m_3 + \cdots + Dm_D)/(n(n-1)) \\ \bar{l} & \leq & (2n - 2 + D((n(n-1)) - 2n + 2))/(n(n-1)) \\ \bar{l} & \leq & D - 2(D-1)/n. \end{array}$$

Then we obtain:

$$\bar{l}/r \ \leq \ \frac{D}{r} - \frac{(D-1)}{r} \cdot \frac{2}{n}\cdot$$

Since G is assumed to be a tree, we have [4] $D = 2r$ or $D = 2r - 1$, thus

$$\bar{l}/r \leq 2 - \frac{4}{n} + \frac{2}{rn}$$

which is largest for $r = 1$ and the bound follows.

Now, let G be a tree such that: $\bar{l}/r \ = \ 2 - \frac{2}{n}\cdot$ Because of

$$\bar{l}/r \ = \ 2 - 2/n \ \leq \ D/r - (D-1)/r \cdot 2/n \ \leq \ 2 - 2/n$$

necessarily

$$D/r - (D-1)/r \cdot 2/n \ = \ 2 - 2/n$$

which implies $D/r = 2$ and $D - 1 = r$, *i.e.,* $D = 2$ and $r = 1$. The star is the unique tree satisfying these conditions. □

Before stating the conjectures about $\bar{l} - r$ and $\bar{l}/r$, let us define the family of graphs called bags [10]. A *bag* $\mathrm{Bag}_{p,k}$ is a graph obtained from a complete graph K_p by replacing an edge uv with a path P_k (as P_k has $k-2$ internal vertices, for bags $n = p + k - 2$ and $m = \frac{p(p-1)}{2} + k - 2$). A bag is *odd* if k is odd, otherwise it is *even*.

Conjecture 3.2. *For given $n \geq 3$, among all connected graphs on n vertices,*

$$\bar{l} - r \geq \begin{cases} \frac{-n(n-2)}{4(n-1)} & \text{if } n \text{ is even,} \\ \frac{8-(n-1)^3}{4n(n-1)} & \text{if } n \text{ is odd.} \end{cases}$$

The bound is attained for a cycle if n is even and for a bag $\mathrm{Bag}_{4,n-2}$ *if n is odd.*

Conjecture 3.3. *For given $n \geq 3$, among all connected graphs on n vertices, $\bar{l}/r$ is minimum for bags.*