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Density Evolution Under Delayed Dynamics

An Open Problem

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Preface

This monograph has arisen out of a number of attempts spanning almost five decades to understand how one might examine the evolution of densities in systems whose dynamics are described by differential delay equations. Though we have no definitive solution to the problem, we offer this contribution in an attempt to define the problem as we see it and to sketch out several obvious attempts that have been suggested to solve the problem and which seem to have failed. We hope that by being available to the general mathematical community, they will inspire others to consider—and hopefully solve—the problem. Serious attempts have been made by all of the authors over the years and we have made reference to these where appropriate. Much of the material in this contribution has been previously published, but not all of it. A serious attempt has been made by Taylor (2004) and should be consulted for a variety of issues, some of which are repeated here. Along the same lines, many of the topics we raise here are also highlighted in Mitkowski and Mitkowski (2012), and in Mitkowski (2021).

The material is organized as follows:

Chapter 1 offers motivating examples to show why we want to study density evolution in systems with delayed dynamics.

In Chap. 2, we review briefly what is known about density evolution in systems with finite dimensional dynamics, starting with a description of the connection between dynamics and densities in Sect. 2.1. Section 2.3 reviews the situation for the commonly known situation in which the dynamics are described by ordinary differential equations. Section 2.4 briefly considers dynamics described by stochastic differential equations, while Sect. 2.5 does the same for finite-dimensional maps. Section 2.6 concludes with a description of the dynamic density evolution behaviors of ergodicity, mixing, exactness, and asymptotic periodicity.

Chapter 3 motivates the study of the dynamics of ensembles of differential delay equations through some simple numerical examples. Section 3.1 relates the formal "density evolution" problem for differential delay equations to what is actually measured in an experimental setting. Section 3.2 gives numerical evidence for the existence of interesting ergodic properties of density evolution dynamics in the presence of delays. Chapter 4 considers the real mathematical problems involved

ranging from the proper nature of the underlying space to the problem of defining a density and highlights all of the problems attendant in doing so. Chapter 5 outlines an approach that has been tried based on the Hopf functional. Section 5.1 introduces the notion of Hopf functionals, and Sect. 5.2 applies this to the specific case of delay differential equations.

Chapter 6 considers the problem reformulated as the method of steps. Finally, Chap. 7 considers the approximations to the delay problem, first examining a high-dimensional map approximation to the delay equation. Chapter 8 is devoted to developing approximate Liouville-like equations and an examination of invariant densities for differential delay equations. We conclude in Chap. 9.

We are indebted to numerous colleagues with whom we have discussed this problem over the years. We would like, in particular, to thank Andrzej Lasota (1932–2006), André Longtin, and Helmut Schwegler. The impetus for writing this is in large part due to a month-long workshop "Short Thematic Program on Delay Differential Equations" held at the Fields Institute (Toronto, May, 2015) and organized by Prof. Jianhong Wu (York University). Many colleagues there offered comments and suggestions, and for that we thank them, while other colleagues have generously shared their work with us ahead of publication.

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Contents

Part I
Introduction and Background to Density Evolution Problems

Chapter 1 is an introductory section in which we set the stage for defining the problems of density evolution under the action of differential delay equations and motivating why we want to look at it.

Chapter 2 (page 9) talks briefly about density evolution in systems with finite-dimensional dynamics, starting with a description of the connection between dynamics and densities in Sect. 2.1. Section 2.3 reviews the situation for the commonly known situation in which the dynamics are described by ordinary differential equations. Section 2.4 briefly considers dynamics described by stochastic differential equations, while 2.5 does the same for finite-dimensional maps. This chapter concludes in Sect. 2.6 with a description of the dynamic density evolution behaviors of ergodicity, mixing, exactness, and asymptotic periodicity.

Chapter 1
Introduction and Motivation

In examining the dynamical behavior of a system there are fundamentally two options available to the experimentalist.

1. In the first option s/he will examine the dynamical trajectories of individuals, be they fundamental particles in a cloud chamber or cells in a petri dish or animals in an ecological experiment. In this case the experimentalist may be interested in replicating the experiment many times, and building up a statistical description of the observed behavior under the assumption (among others) that the trajectory behavior will be replicated between trials given the same initial conditions.
2. In the second option this approach will be forsaken for one in which the evolving statistics of large populations are examined. This is, of course, most familiar in statistical mechanics, but is also important in many other areas. The advantage of this approach is that if one can understand the dynamics of density evolution, then many interesting statistical quantities can be computed, and the results compared with experimental results.

Which approach is taken is sometimes a matter of choice, but often dictated by the nature of the individual units being studied as well as the types of experiments that are possible.

To illustrate these two points of view, consider the following, much-studied, example of a simple deterministic system whose evolution exhibits a species of random behavior. For a given real number x_0 (the "initial state" of the system) between 0 and 1, let x_1, x_2, x_3, etc. be defined by repeated application of the formula

$$x_{n+1} = 4x_n(1 - x_n), \quad n = 0, 1, 2, \ldots \tag{1.1}$$

One can view this formula as prescribing the evolution of the state of the system, x_n, at discrete times $n = 0, 1, 2, \ldots$. The evolution of this system is deterministic, in that once the initial state is specified equation (1.1) uniquely determines the

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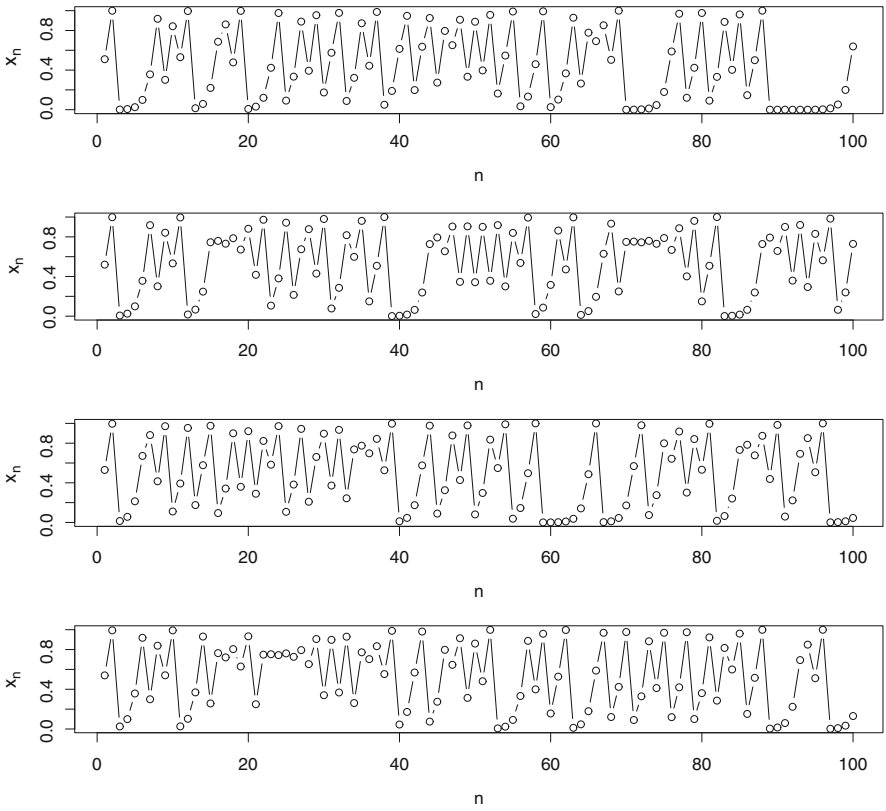

Fig. 1.1 Numerical trajectories for the map $x \mapsto 4x(1-x)$. The initial conditions differ only slightly for each trajectory

sequence of values $\{x_0, x_1, x_2, \ldots\}$ (i.e., the *trajectory* of the system) for all time. Thus, for example if $x_0 = 0.51$ we obtain

$$x_1 = .9996, x_2 \approx .0026, x_3 \approx .0064, x_4 \approx .025, x_5 \approx .099, \text{ etc.}$$

The qualitative behavior of this system is most easily appreciated graphically, as in Fig. 1.1 which plots x_n vs. n for typical trajectories obtained for different choices of x_0. Each of these trajectories is erratic, and random in the sense that no regularity is apparent. Furthermore, it can be seen by comparing the graphs in Fig. 1.1 that two nearly identical initial states yield radically different time evolutions. This phenomenon, termed "sensitive dependence on initial conditions" (Lorenz 1963), imposes strong limits on the predictability of this system over long periods of time: a small error in the determination of the initial condition rapidly becomes amplified to the extent that reliable prediction of the future states of the system eventually becomes impossible. Thus, despite being entirely deterministic, trajectories of

this simple system have some hallmarks of essentially *random* phenomena: their behavior is irregular and unpredictable.

The mechanisms underlying the random character of this system are reasonably well understood (see, e.g., Collet and Eckmann 1980), the key notion being sensitivity to initial conditions and its consequences. However, within this framework it is difficult to approach questions of the type "what is the asymptotic behavior of a *typical* trajectory of this system?" Indeed, the very nature of sensitivity to initial conditions would seem to preclude any notion of "typical" behavior, since even very similar initial conditions eventually lead to their own very particular, uncorrelated evolutions.

However, different conclusions are reached if one takes a probabilistic point of view. Suppose that instead of being precisely determined, the initial state x_0 has associated with it some uncertainty. In particular, suppose we know the initial probability density, f, giving the probabilities of all possible values that x_0 can take. Then it makes sense to ask, "what will be the probability density of x_1, the new state after one iteration of the map (1.1)?" A precise answer to this question can be found using analytical methods described in Chap. 2. For an approximate answer, it suffices to simulate a large ensemble of different initial states x_0 distributed according to f, evolve these states forward under the map (1.1), and approximate the transformed density of the ensemble by constructing a histogram of the ensemble of values x_1. One can then proceed, in the same fashion, to determine the probability densities of subsequent states x_2, x_3, etc. Thus, even if the initial state x_0 is not known precisely, it is at least possible to give a probabilistic description of the system's evolution in terms of the evolution of a probability density.

The graphs in Fig. 1.2 show a particular choice for the probability density f of the initial state x_0, together with the subsequent densities of states x_1, x_2, x_3, x_4, obtained by numerical simulation of an ensemble of 10^6 initial values distributed according to f, iterated forward under the map (1.1). The striking feature of this figure is that the sequence of densities rapidly approaches an *equilibrium* or *invariant density* that does not change under further iteration. Moreover, the invariant density appears to be unique. This is supported by Fig. 1.3, which shows how a different choice of initial density evolves toward the same equilibrium density as before.

A different but related statistical approach to this system is to focus on the statistics of a single trajectory. For a given initial state x_0, by iterating $x_{n+1} = 4x_n(1 - x_n)$ we obtain an arbitrarily long sequence $\{x_n\}$ like the one illustrated in Fig. 1.1. A histogram of this sequence reveals the long-term frequency with which the trajectory visits various parts of the interval $[0, 1]$. Figure 1.4 shows such a histogram, for a trajectory of length 10^6. Remarkably, this histogram reproduces the invariant density shown in Figs. 1.2 and 1.3, which arises in a different context. Moreover, the same histogram is obtained for almost *any* choice of initial state.[1]

[1]There are exceptions, such as $x_0 = 0$, that yield trajectories with different (periodic) asymptotic behavior. These exceptions are very rare: in fact they constitute a set of Lebesgue measure 0.

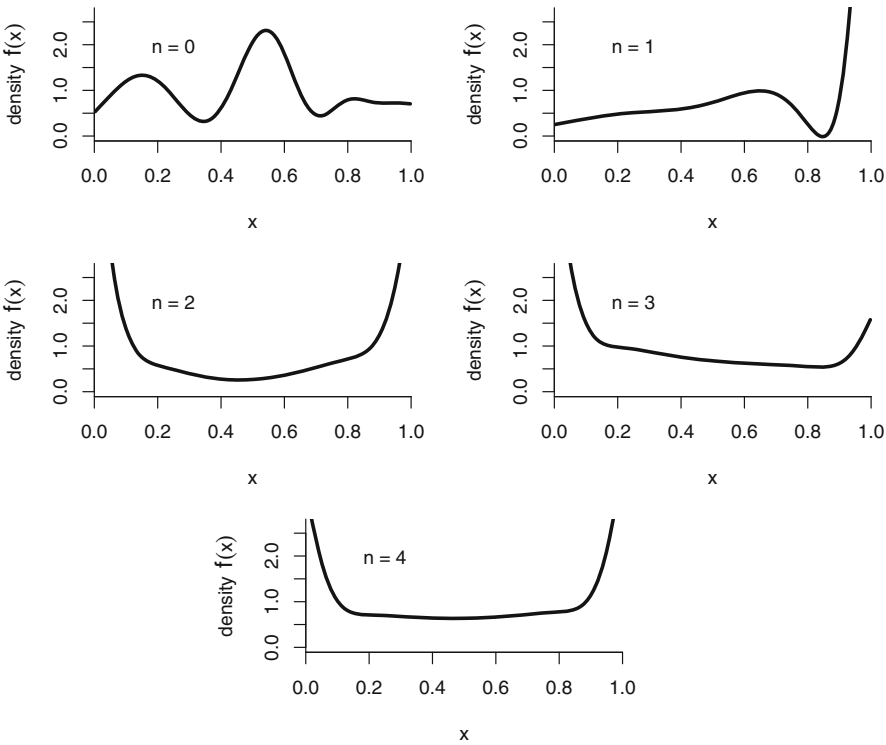

Fig. 1.2 Simulated evolution of an ensemble density f under iterations of the map $x \mapsto 4x(1-x)$

Thus, the invariant density describes the behavior of "typical" trajectories, i.e., those whose statistics are described by this particular probability density.

A probabilistic or ensemble treatment of dynamical systems provides a point of view complementary to one given in terms of the evolution of individual trajectories. The iterated map (1.1) is just one example of a system that behaves erratically on the level of individual trajectories, but has very regular asymptotic properties when considered at the level of probability densities. This observation appears to hold for many other systems. Moreover, it turns out that the converse holds as well: various regularity properties at the level of probability densities imply various degrees of disorder in the evolution of individual trajectories.

For a large class of systems in which the underlying dynamics are described by differential equations, or stochastic differential equations, or maps, there is a large *corpus* of methods that have been developed with which one can approach both of the types of data collection outlined above and the connection of that data to underlying dynamical systems theory.

However, many problems in the physical, and especially the biological, sciences involve the dynamic behavior of individual entities whose dynamics involve signif-