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To the memory of my brilliant teachers and theses advisers, Baudouin Fraeijs de Veubeke of Liège and John Valentine Breakwell of Stanford.

Preface

This book has two main objectives in its comprehensive exposition of the important problem of time-fixed terminal rendezvous around the Earth using chemical propulsion. The first objective is to present the mathematics of relative motion in near-circular orbit, subjected to the perturbations emanating from the oblateness of the Earth, third-body gravity such as due to the moon and the Sun, and atmospheric drag, all in analytic form, suitable for fast trajectory prediction, without the need for numerical integrations, and for further implementation in computer codes that solve efficiently the required impulsive maneuvers. These analytic solutions are put to use to create computer programs that calculate the required impulsive maneuver that initiates the chase trajectory to intercept a passive or non-maneuvering vehicle in a fixed time, where a second impulse is applied at interception to actually rendezvous with that vehicle.

Unlike previous attempts to solve this problem in vacuum and analytic form, the contents of this book provide solutions of this important spaceflight problem, by also considering the various perturbations affecting the trajectories of both active and passive vehicles, in analytic form, but also with very high accuracy, because the classic treatments available in the literature dating back to the early 1960s are inherently and grossly inaccurate. The gross inaccuracies are mainly due to the paradigm of using a single rotating reference frame usually attached to the passive vehicle itself, with respect to which the chase trajectory is constructed through the initiation of the impulsive velocity change imparted to the active maneuvering vehicle.

The paradigm of the single reference frame is cast aside here, and the problem of the time-fixed terminal rendezvous is solved through the use of three different reference frames at different phases of the calculations, which render possible the computations of the initiating impulse with very high accuracy, while also taking into account the precessions of the orbits due to the second zonal harmonic J_2 of the oblate Earth. Examples of rendezvous spanning several hours are thus generated analytically through an iterative scheme, by using a first guess from an analytic solution that makes use of a single reference frame located midway between the active and passive vehicles at the initial time.

After showing the derivations of the analytic second-order approximation to the Euler-Hill equations of relative motion in Chapter 1, and then generating the analytic solutions of relative motion as a function of time, by solving the system of the linearized differential equations with, in turn, the forcing terms for the perturbations due to third-body gravity, zonal harmonics J_2 and J_3, and atmospheric drag, in Chapters 2, 3, 4, and 5, the case of the J_2-perturbed theory is depicted in Chapter 6 by also including the second-order approximation to the solution of the linearized system of equations of relative motion for added accuracy, and still making use of the single mid-point rotating frame to provide the first guess mentioned earlier. This first guess is then used in Chapter 7, where the three different frames are employed, and an iterative scheme applied, to arrive at the highly accurate rendezvous solution in the near-circular orbit case, and for moderate interception times of the order of several hours. Numerical examples are shown throughout the chapters to verify the accuracy of these analytic solutions by comparing them to numerically integrated ones. The J_2 theory is derived in Chapter 2, and later used in Chapters 6 and 7, without showing the derivations again, but nevertheless included in those latter chapters for ease of reading and completeness, and for making each chapter as self-contained as possible.

The second objective of this book is to apply the theory to the exact long-duration time-fixed terminal rendezvous problem around the oblate Earth, for the general elliptic orbit case. The mathematics of relative motion in general elliptic orbit referred to a rotating coordinate frame attached to the passive non-maneuvering vehicle, and that drags and precesses due to atmospheric drag and Earth oblateness, are depicted in Chapter 8 for the coplanar case with drag, and for the noncoplanar case with drag and J_2 effects, in Chapters 9 and 10. The system of the exact nonlinear differential equations are derived in Chapter 9, and the algorithm that provides the solution through numerical integration and through an iterative scheme is shown in Chapter 10 with several examples. Chapters 9 and 10 are, of course, related such that orbit geometry figures depicted in Chapters 8 and 9 are not repeated in Chapter 10. The theory is exact in the sense that the trajectories are generated by numerically integrating the full system of 12 first-order differential equations, without any simplifying assumptions. The higher-order zonal harmonics such as J_3, J_4, etc., are not considered, even though they too can be taken into account to describe a more precise modeling of the oblateness of the Earth. Two-impulse rendezvous trajectories are, thus, produced for various examples of elliptic orbits, and for different interception times spanning several hours, and even days. The last chapter shows how the theories laid out in this book can effectively be applied to station-keep a Walker constellation in Earth orbit, by implementing a series of impulsive maneuvers to counter the drift experienced by the various vehicles, and to maintain the constellation geometry, thus preventing the symmetric configuration of the constellation from getting disrupted, which would, otherwise, affect the coverage generated by their onboard sensors. Many of the examples shown in the book use the analytic solutions generated in Chapter 7 as a starting guess to iterate on, with the full nonlinear, and therefore exact dynamics of the later chapters, to arrive at exact converged solutions that can be flown by actual spacecraft in near-circular orbits. Even though the perturbations due J_3, third-body gravity, and atmospheric drag are available analytically for the near-circular orbit case in the first chapters, the examples at hand do not include them as such, but are easily added to the computer programs to also account for these perturbations. These perturbations can be included in their numerically integrated form, if needed, for the general elliptic orbit case. One of the main advantages of the relative motion theory lies in its being perfectly nonsingular, both for the coplanar, and the J_2-perturbed noncoplanar cases, unlike the equations of motion based on the classical orbital elements which, of course, exhibit singularities for the circular and equatorial orbits. Equinoctial elements are, however, nonsingular for both of these important cases, and they can also be used for guidance applications, even though they are a bit cumbersome and not as efficient and straightforward as the rotating Cartesian relative rotating coordinates theory of this book which allows to compute easily the rendezvous trajectory.

This book can be used to design terminal two-impulse time-fixed rendezvous trajectories, both for the near-circular and the more general elliptic orbit cases, to carry out trade studies useful in designing both actual vehicles and missions, to fly actual missions by ground-generated or autonomous onboard-generated solutions for rendezvous and docking, formation-keeping, inspection, spacecraft servicing, and relocation applications, among others, in low-Earth orbit, as well as higher orbits such as the geostationary orbit, and to instruct students as well as researchers in spaceflight guidance studies, both in universities that provide aerospace and mechanical engineering curriculae, and practitioners in research laboratories and aerospace companies that design and fly such missions. The theories in this book can be extended to account for even more precise Earth oblateness models, as well as standard atmosphere models using tabular data, instead of the exponential models used here, and to the multi-impulse rendezvous applications through the methods of optimal control, for truly minimum-fuel solutions, if desired, being understood that the full nonlinear differential equations for the dragging and precessing reference frames make it very convenient and straightforward, to calculate the impulsive maneuvers in an efficient manner.

It is appropriate to mention the teachings of my late teacher and thesis adviser, Professor John Valentine Breakwell in the Department of Aeronautics and Astronautics at Stanford University, who made the relative motion mathematics referred to rotating frames, in his classes, a central part of his various graduate courses, providing the inspiration to carry out the research in this book, with obvious essential applications in spaceflight guidance for civilian and military programs, especially in autonomous mode. The complex and difficult typesetting of this book was carried out by Mary Villanueva, previously of The Aerospace Corporation and currently of Raytheon Space and Airborne Systems in El Segundo, California, while the transcription of the figures from print and electronic versions to Adobe Illustrator files was carried out mainly by Yvonne Craig of The Rand Corporation in Arlington,

Virginia, and by Jason Perez, previously of The Aerospace Corporation. Gratitude is expressed to all three specialists. The research provided in this book was initially started at the NASA Jet Propulsion Laboratory in Pasadena, California, then, followed at Ford Aerospace in Palo Alto, California, and later at The Aerospace Corporation under contract with the United States Air Force Space and Missile Systems Center.

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Contents

Chapter 1
The Second-Order Analytic Approximation to the Solution of the Euler-Hill Equations of Relative Motion

1.1 Introduction

The first-order solution of the problem of relative motion of a spacecraft in near-circular orbit is known to degrade in accuracy, when compared to the numerically integrated exact solution, at greater distances from the origin of the rotating reference frame. These solutions have been developed to study the problem of the terminal rendezvous guidance where an active spacecraft at several hundred km from its rendezvous target centered at the rotating frame, must maneuver to intercept the target in a given time. In References [1, 2], the relative motion technique was used for a different purpose, namely to describe the future motion of a spacecraft relative to the rotating frame, as it is perturbed by the Earth zonal harmonics J_2 and J_3, and by the luni-solar gravity effects. Although these perturbations have a small effect on the spacecraft motion which would not wander to great distances from the origin of the frame, it is felt that because of the presence of initial non-zero velocities, the subsequent motion may drift to considerable distances from the origin of the frame, thereby, degrading the accuracy of the analytic first-order solution of the equations of motion. The initial velocities exist because the orbit determination-generated osculating orbit is necessarily elliptical in nature with small eccentricity, such that at time zero, or epoch, a reference circular orbit having the same radius as the radial distance of the actual spacecraft is assumed, to describe the future motion of the vehicle itself, which, unlike the frame, experiences the various perturbations just mentioned. Second-order corrections to the linear solution of Reference [3] have been obtained in References [4, 5] to extend the region of accuracy of the analytic solutions at greater distances from the origin. This chapter rederives the second-order solutions, resolving the errors of Reference [4], and the typographical errors of Reference [5], by adopting the nomenclature of these two references in defining the coordinates. The radial coordinate is depicted as y, and the "tangential" coordinate by x, although x is pointing in the opposite direction of motion. The out-of-plane or

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z coordinate, is along the orbital angular momentum vector. In References [1, 2], x is along the radial direction, y along the "tangential" direction in the direction of motion, and z along the orbital angular momentum vector. When the second-order expressions developed here are added to the first-order solutions of the perturbed motion of References [1, 2], the differences in the coordinates must be properly accounted for.

1.2 Derivation of the Equations of Motion in Rotating Rectangular Coordinates for an Elliptic Orbit

Let $\widehat{\mathbf{x}}, \widehat{\mathbf{y}}, \widehat{\mathbf{z}}$ represent an inertial coordinate system centered at the Earth, and let a reference elliptic orbit be defined by Ω, i, ω, with eccentricity e, and semimajor axis a as in Figure 1.1. Then $\widehat{\boldsymbol{\theta}}, \widehat{\mathbf{r}}, \widehat{\mathbf{h}}$ define a rotating frame with $\widehat{\mathbf{r}}$, a unit vector along the radial direction $\mathbf{r}_0$, $\widehat{\boldsymbol{\theta}}$ a unit vector in the orbital plane and in the direction opposite to the motion, and $\widehat{\mathbf{h}}$ a unit vector along the angular momentum vector to complete the right-handed system. This reference frame is attached to the reference elliptic orbit at O$'$ and rotates at the angular rate $\dot{\theta}$ which is a function of angular position θ. The perigee is at P, and $\theta^* = \theta - \omega$ is the true anomaly. The spacecraft position can be

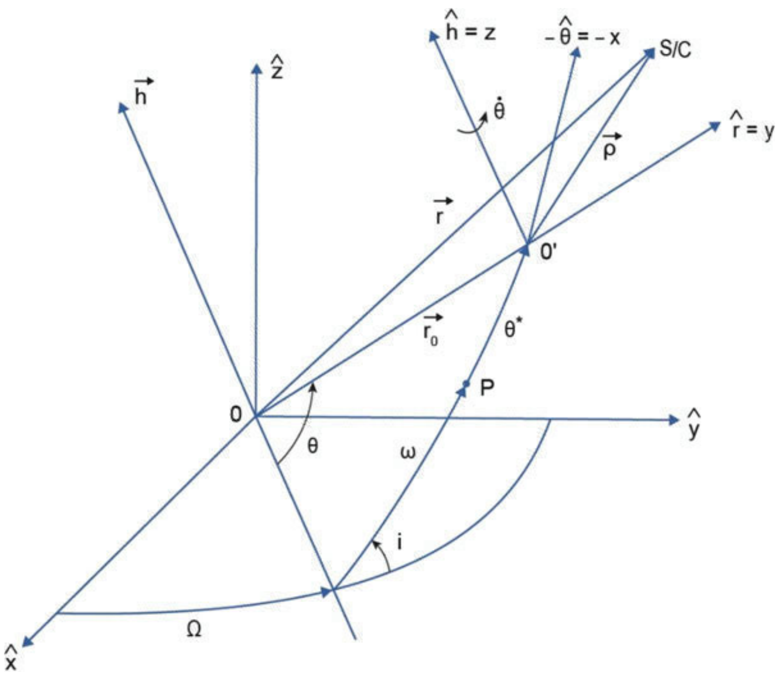

Figure 1.1 Geometry of rotating frame.

referred to the rotating frame by the vector $\boldsymbol{\rho}$, and to the inertial frame by the vector $\mathbf{r} = \mathbf{r}_0 + \boldsymbol{\rho}$. Because $\mathbf{r}_0$ and $\mathbf{r}$ obey the following differential equations as described in the inertial system (Reference [5]),

$$\overset{I}{\ddot{\mathbf{r}}}_0 = -\frac{\overset{I}{\mu}\,\ddot{\mathbf{r}}_0}{r_0^3} \tag{1.1}$$

$$\overset{I}{\ddot{\mathbf{r}}} = -\frac{\overset{I}{\mu}\,\mathbf{r}}{r^3} \tag{1.2}$$

Then with μ standing for the gravity constant of the Earth,

$$\overset{I}{\ddot{\boldsymbol{\rho}}} = -\mu\left[\frac{\overset{I}{\mathbf{r}}}{r^3} - \frac{\overset{I}{\mathbf{r}_0}}{r_0^3}\right] \tag{1.3}$$

This acceleration can also be written directly in terms of the rotating frame as

$$\overset{I}{\ddot{\boldsymbol{\rho}}} = \overset{R}{\ddot{\boldsymbol{\rho}}} + 2\,\overset{R}{\boldsymbol{\omega}} \times \overset{R}{\dot{\boldsymbol{\rho}}} + \overset{R}{\boldsymbol{\omega}} \times \left(\overset{R}{\boldsymbol{\omega}} \times \overset{R}{\boldsymbol{\rho}}\right) + \overset{R}{\boldsymbol{\omega}} \times \overset{R}{\boldsymbol{\rho}} \tag{1.4}$$

where the angular velocity vector in the rotating frame has the form

$$\overset{R}{\boldsymbol{\omega}} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

Writing $\boldsymbol{\rho}$ and $\mathbf{r}_0$ in terms of their coordinates in the rotating frame,

$$\boldsymbol{\rho} = x\widehat{\boldsymbol{\theta}} + y\widehat{\mathbf{r}} + z\widehat{\mathbf{h}} \tag{1.5}$$

$$\mathbf{r}_0 = r_0\widehat{\mathbf{r}} \tag{1.6}$$

then

$$\overset{R}{\boldsymbol{\omega}} \times \overset{R}{\boldsymbol{\rho}} = x\dot{\theta}\widehat{\mathbf{r}} - y\dot{\theta}\,\widehat{\boldsymbol{\theta}} \tag{1.7}$$

$$\overset{R}{\boldsymbol{\omega}} \times \left(\overset{R}{\boldsymbol{\omega}} \times \overset{R}{\boldsymbol{\rho}}\right) = -y\dot{\theta}^2\widehat{\mathbf{r}} - x\dot{\theta}^2\widehat{\boldsymbol{\theta}} \tag{1.8}$$

$$\overset{R}{\boldsymbol{\omega}} \times \overset{R}{\boldsymbol{\rho}} = x\ddot{\theta}\,\widehat{\mathbf{r}} - y\ddot{\theta}\,\widehat{\boldsymbol{\theta}} \tag{1.9}$$

$$\overset{R}{\boldsymbol{\omega}} \times \overset{R}{\boldsymbol{\rho}} = \dot{x}\dot{\theta}\,\widehat{\mathbf{r}} - \dot{y}\dot{\theta}\,\widehat{\boldsymbol{\theta}} \tag{1.10}$$

such that Equation (1.4) becomes

$$\overset{I}{\ddot{\boldsymbol{\rho}}} = \left(\ddot{y} + 2\dot{x}\dot{\theta} - y\dot{\theta}^2 + x\ddot{\theta}\right)\widehat{\mathbf{r}} + \left(\ddot{x} - 2\dot{y}\dot{\theta} - x\dot{\theta}^2 - y\ddot{\theta}\right)\widehat{\boldsymbol{\theta}} + \ddot{z}\widehat{\mathbf{h}} \tag{1.11}$$

If Equation (1.3) is resolved into the rotating system directly and compared to Equation (1.11), the second-order differential equations in x, y, z can be readily obtained by observing also that

$$r^3 = \left[x^2 + (y + r_0)^2 + z^2\right]^{3/2} \tag{1.12}$$

These exact equations are given by the following three equations, in perfect agreement with Reference [5].

$$\ddot{x} - 2\dot{y}\dot{\theta} - x\dot{\theta}^2 - y\ddot{\theta} + \mu x\left[x^2 + (y + r_0)^2 + z^2\right]^{-3/2} = 0 \tag{1.13}$$

$$\ddot{y} + 2\dot{x}\dot{\theta} - y\dot{\theta}^2 + x\ddot{\theta} + \mu(y + r_0)\left[x^2 + (y + r_0)^2 + z^2\right]^{-3/2} - \mu r_0^{-2} = 0 \tag{1.14}$$

$$\ddot{z} + \mu z\left[x^2 + (y + r_0)^2 + z^2\right]^{-3/2} = 0 \tag{1.15}$$

The eccentricity and the semimajor axis of the reference elliptic orbit enter into these equations through the angular velocity $\dot{\theta}$ and acceleration $\ddot{\theta}$. These latter two parameters are obtained as follows:

The orbit equations given in polar form are

$$\ddot{r}_0 - r_0\dot{\theta}^2 = -\mu/r_0^2 \tag{1.16}$$

$$r_0\ddot{\theta} + 2\dot{r}_0\dot{\theta} = 0 \tag{1.17}$$

The last equation is of course equivalent to $\frac{d}{dt}\left(r_0^2\dot{\theta}\right) = 0$, yielding the constant angular momentum $h = r_0^2\dot{\theta}$, which in turn gives

$$\dot{\theta} = \frac{h}{r_0^2} = \frac{\mu^{1/2}(1 + ec_{\theta^*})^2}{a^{3/2}(1 - e^2)^{3/2}} \tag{1.18}$$

where $h^2 = \mu a(1 - e^2)$, and $r_0 = \left(h^2/\mu\right)(1 + ec_{\theta^*})^{-1}$ has been used. Because θ^* is a function of θ, the angular position, $\dot{\theta}$ is also a function of θ such that along an elliptic reference orbit, $\ddot{\theta}$ the non-zero angular acceleration is obtained from Equation (1.17)

$$\ddot{\theta} = -\frac{2r_0\dot{\theta}}{r_0} \tag{1.19}$$

Because $\dot{r}_0 = -\frac{\mu}{h}es_{\theta^*}$ by direct differentiation of r_0, then

$$\ddot{\theta} = \frac{-2\mu e s_{\theta^*}(1 + e c_{\theta^*})^3}{a^3(1 - e^2)^3} \tag{1.20}$$

For a circular reference orbit, $\dot{\theta} = \mu^{1/2} a^{-3/2} = n$, the mean motion, and $\ddot{\theta} = 0$ as expected. Given the initial conditions on the position and velocity components and in view of Equations (1.18) and (1.20), the exact differential equations of motion relative to an elliptic reference orbit can now be integrated numerically. These equations are shown in Equations (1.13) through (1.15).

For spacecraft in near-circular orbits, it is appropriate to consider a circular reference orbit, in which case, the equations of motion simplify to take the following form:

$$\ddot{x} - 2n\dot{y} - n^2 x + n^2 x \left[\left(1 + \frac{y}{r_0}\right)^2 + \frac{x^2}{r_0^2} + \frac{z^2}{r_0^2} \right]^{-3/2} = 0 \tag{1.21}$$

$$\ddot{y} + 2n\dot{x} - n^2(y + r_0) + n^2(y + r_0) \left[\left(1 + \frac{y}{r_0}\right)^2 + \frac{x^2}{r_0^2} + \frac{z^2}{r_0^2} \right]^{-3/2} = 0 \tag{1.22}$$

$$\ddot{z} + n^2 z \left[\left(1 + \frac{y}{r_0}\right)^2 + \frac{x^2}{r_0^2} + \frac{z^2}{r_0^2} \right]^{-3/2} = 0 \tag{1.23}$$

Here r_0 is the radius of the reference cicular orbit. If x, y, z are considered to be small compared to r_0, then, the term in brackets appearing in the preceding expressions can be expanded as

$$\left[\left(1 + \frac{y}{r_0}\right)^2 + \frac{x^2}{r_0^2} + \frac{z^2}{r_0^2} \right]^{-3/2} = (1 + \eta)^{-3/2} \simeq 1 - \frac{3}{2}\eta + \frac{15}{8}\eta^2 - \frac{35}{16}\eta^3 + \cdots$$

If terms up to the third-order in x/r_0 etc. ..., are retained, then,

$$\begin{aligned}(1 + \eta)^{-3/2} \simeq & 1 - \frac{3y}{r_0} - \frac{3}{2}\left(\frac{x}{r_0}\right)^2 - \frac{3}{2}\left(\frac{z}{r_0}\right)^2 + 6\left(\frac{y}{r_0}\right)^2 \\ & + \frac{15}{2}\left(\frac{y}{r_0}\right)\left(\frac{x}{r_0}\right)^2 + \frac{15}{2}\left(\frac{y}{r_0}\right)\left(\frac{z}{r_0}\right)^2 - 10\left(\frac{y}{r_0}\right)^3 + O_4\end{aligned} \tag{1.24}$$

Then, the equations of motion containing third-order terms in the expansions are obtained as:

$$\ddot{x} - 2n\dot{y} - 3n^2\frac{xy}{r_0} - \frac{3n^2}{2}\frac{x^3}{r_0^2} - \frac{3n^2}{2}\frac{xz^2}{r_0^2} + 6n^2\frac{xy^2}{r_0^2} = 0 \tag{1.25}$$

$$\begin{aligned}\ddot{y} + 2n\dot{x} - 3n^2 y - \frac{3n^2}{2}\frac{x^2}{r_0} - \frac{3n^2}{2}\frac{z^2}{r_0} + 3n^2\frac{y^2}{r_0} + 6n^2\frac{yx^2}{r_0^2}\\ + 6n^2\frac{yz^2}{r_0^2} - 4n^2\frac{y^3}{r_0^2} = 0\end{aligned} \tag{1.26}$$

$$\ddot{z} + n^2 z\left(1 - \frac{3y}{r_0} - \frac{3}{2}\frac{x^2}{r_0^2} - \frac{3}{2}\frac{z^2}{r_0^2} + \frac{6y^2}{r_0^2}\right) = 0 \tag{1.27}$$

If only second-order terms are retained, then,

$$\ddot{x} - 2n\dot{y} - 3n^2\frac{xy}{r_0} = 0 \tag{1.28}$$

$$\ddot{y} + 2n\dot{x} - 3n^2 y - \frac{3}{2}n^2\frac{x^2}{r_0} - \frac{3}{2}\frac{n^2 z^2}{r_0} + 3n^2\frac{y^2}{r_0} = 0 \tag{1.29}$$

$$\ddot{z} + n^2 z\left(1 - \frac{3y}{r_0}\right) = 0 \tag{1.30}$$

If second-order terms are also neglected, then the Euler-Hill equations are obtained as:

$$\ddot{x} - 2n\dot{y} = 0 \tag{1.31}$$

$$\ddot{y} + 2n\dot{x} - 3n^2 y = 0 \tag{1.32}$$

$$\ddot{z} + n^2 z = 0 \tag{1.33}$$

in which the x and y motion is decoupled from the out-of-plane z motion, and whose general solution is given by

$$x_1 = k_1\, s_{nt} + k_2\, c_{nt} + k_3\, nt + k_4 \tag{1.34}$$

$$y_1 = k_5\, c_{nt} + k_6\, s_{nt} + k_7 \tag{1.35}$$

$$z_1 = k_8\, c_{nt} + k_9\, s_{nt} \tag{1.36}$$

with the coefficients given in terms of the initial conditions as

$$k_1 = 4\frac{\dot{x}_0}{n} - 6y_0 \tag{1.37}$$

$$k_2 = -2\dot{y}_0/n \tag{1.38}$$

$$k_3 = 6y_0 - 3\dot{x}_0/n \tag{1.39}$$

$$k_4 = 2\dot{y}_0/n + x_0 \tag{1.40}$$

$$k_5 = 2\dot{x}_0/n - 3y_0 \tag{1.41}$$

$$k_6 = \dot{y}_0/n \tag{1.42}$$

$$k_7 = 4y_0 - 2\frac{\dot{x}_0}{n} \tag{1.43}$$

$$k_8 = z_0 \tag{1.44}$$

$$k_9 = \dot{z}_0/n \tag{1.45}$$

These equations were used in Reference [3] to solve the problem of the terminal rendezvous guidance which was used in the Gemini program. These equations give accurate results only in the vicinity of the origin of the Euler-Hill frame, and break down rather quickly at greater distances. This is critical for the rendezvous problem, because the active spacecraft must apply the correct velocity change, several hundred km away from the passive target vehicle, which is considered to be fixed at the origin O′ of the frame. London, in Reference [4], provided the second-order correction to these equations to extend their accuracy to greater distances from the origin, by solving Equations (1.28) through (1.30), which retain the second-order terms in the expansions, using the method of successive approximations. In Reference [5], Anthony and Sasaki extended these results to the case of the elliptic orbit with small eccentricity, and recovered the second-order solutions of London, pointing out however, some errors in the final solutions. This chapter rederives these equations, and finds a typographical error in Anthony and Sasaki's solutions. It is claimed that the solutions derived here are the final and exact form for the second-order corrections. Carrying out the second-order corrections, enhances the accuracy of our trajectory propagation scheme used in the orbit determination element of the navigation subsystem at a small cost in computing time and computer memory storage, but at a considerable effort in deriving these solutions.

1.3 The Second-Order Approximation to the Euler-Hill Equations of Relative Motion

Using Equations (1.28) through (1.30), London observed that the second-order approximation x_2, y_2, z_2, must satisfy

$$\ddot{x}_2 - 2n\dot{y}_2 = \frac{3n^2}{r_0}x_1y_1 \tag{1.46}$$

$$\ddot{y}_2 + 2n\dot{x}_2 - 3n^2 y_2 = -\frac{3n^2}{r_0}\left[y_1^2 - \frac{1}{2}\left(x_1^2 + z_1^2\right)\right] \tag{1.47}$$

$$\ddot{z}_2 + n^2 z_2 = 3n^2 \frac{y_1 z_1}{r_0} \tag{1.48}$$

where x_1, y_1, z_1, represent the first-order solutions given in Equations (1.34) through (1.36). London's approach in solving the preceding equations consists of guessing the form of the solution as

$$x_2 = \alpha_0 + \alpha_1 nt + \alpha_2\, s_{nt} + \alpha_3\, c_{nt} + \alpha_4\, s_{2nt} + \alpha_5\, c_{2nt} + \alpha_6\, nt\, s_{nt} + \alpha_7\, nt\, c_{nt} \tag{1.49}$$

$$y_2 = \beta_0 + \beta_1\, nt + \beta_2 (nt)^2 + \beta_3\, s_{nt} + \beta_4\, c_{nt} + \beta_5\, s_{2nt} + \beta_6\, c_{2nt} + \beta_7\, nt\, s_{nt} + \beta_8\, nt\, c_{nt} \tag{1.50}$$

$$z_2 = \gamma_0 + \gamma_1\, s_{nt} + \gamma_2\, c_{nt} + \gamma_3\, s_{2nt} + \gamma_4\, c_{2nt} + \gamma_5\, nt\, s_{nt} + \gamma_6\, nt\, c_{nt} \tag{1.51}$$

and then, substituting these expressions and their first and second time derivatives in Equations (1.46), (1.47) and (1.48), and comparing coefficients term by term, which results in a set of simultaneous equations which can be solved for the various α_i, β_i, and γ_i constants.

The method used in this chapter is the same as in References [1, 2], where the differential equations are solved directly. From Equation (1.46) and in view of Equations (1.34) and (1.35),

$$x_1 y_1 = A_1\, c_{nt}^2 + B_1\, s_{nt}^2 + C_1\, s_{nt}\, c_{nt} + D_1\, nt\, c_{nt} + E_1\, nt\, s_{nt} + G_1\, s_{nt} + F_1\, c_{nt} + I_1\, nt + J_1 \tag{1.52}$$

where

$$A_1 = k_2 k_5 = -\frac{4\dot{x}_0 \dot{y}_0}{n^2} + 6y_0 \frac{\dot{y}_0}{n} \tag{1.53}$$

$$B_1 = k_1 k_6 = \frac{4\dot{x}_0 \dot{y}_0}{n^2} - 6y_0 \frac{\dot{y}_0}{n} = -A_1 \tag{1.54}$$

$$C_1 = k_1 k_5 + k_2 k_6 = \frac{8\dot{x}_0^2}{n^2} - 24y_0 \frac{\dot{x}_0}{n} + 18y_0^2 - 2\frac{\dot{y}_0^2}{n^2} \tag{1.55}$$

$$D_1 = k_3 k_5 = 21\frac{\dot{x}_0}{n} y_0 - 18y_0^2 - \frac{6\dot{x}_0^2}{n^2} \tag{1.56}$$

$$E_1 = k_3 k_6 = 6y_0 \frac{\dot{y}_0}{n} - 3\dot{x}_0 \frac{\dot{y}_0}{n^2} \tag{1.57}$$

$$F_1 = k_2 k_7 + k_4 k_5 = -14y_0 \frac{\dot{y}_0}{n} + 8\dot{x}_0 \frac{\dot{y}_0}{n^2} + 2x_0 \dot{x}_0 - 3x_0 y_0 \tag{1.58}$$

$$G_1 = k_1 k_7 + k_4 k_6 = 28\frac{\dot{x}_0 y_0}{n} - 8\frac{\dot{x}_0^2}{n^2} - 24y_0^2 + 2\frac{\dot{y}_0^2}{n^2} + \frac{x_0 \dot{y}_0}{n} \tag{1.59}$$

$$I_1 = k_3 k_7 = 24y_0^2 - 24\frac{\dot{x}_0}{n} y_0 + \frac{6\dot{x}_0^2}{n^2} \tag{1.60}$$

$$J_1 = k_4 k_7 = 8y_0 \frac{\dot{y}_0}{n} - 4\frac{\dot{x}_0 \dot{y}_0}{n^2} + 4x_0 y_0 - 2x_0 \frac{\dot{x}_0}{n} \tag{1.61}$$

In a similar way, from Equations (1.47), (1.34), (1.35) and (1.36),

$$\begin{aligned} \ddot{y}_2 + 2n\dot{x}_2 - 3n^2 y_2 &= -\frac{3n^2}{r_0}\left[y_1^2 - \frac{1}{2}\left(x_1^2 + z_1^2\right)\right] \\ y_1^2 - \frac{1}{2}\left(x_1^2 + z_1^2\right) &= A_2\, c_{nt}^2 + B_2\, s_{nt}^2 + C_2\, s_{nt}\, c_{nt} + D_2\, nt\, c_{nt} + E_2\, nt\, s_{nt} \\ &\quad + F_2\, c_{nt} + G_2\, s_{nt} + H_2\, n^2 t^2 + I_2\, nt + J_2 \end{aligned} \tag{1.62}$$

where,

$$A_2 = k_5^2 - \frac{k_2^2}{2} - \frac{k_8^2}{2} = \frac{4\dot{x}_0^2}{n^2} + 9y_0^2 - 12y_0 \frac{\dot{x}_0}{n} - 2\frac{\dot{y}_0^2}{n^2} - \frac{z_0^2}{2} \tag{1.63}$$

$$B_2 = k_6^2 - \frac{k_1^2}{2} - \frac{k_9^2}{2} = \frac{\dot{y}_0^2}{n^2} - 8\frac{\dot{x}_0^2}{n^2} - 18y_0^2 + 24y_0 \frac{\dot{x}_0}{n} - \frac{\dot{z}_0^2}{2n^2} \tag{1.64}$$

$$C_2 = 2k_5 k_6 - k_1 k_2 - k_8 k_9 = 12\frac{\dot{x}_0 \dot{y}_0}{n^2} - 18y_0 \frac{\dot{y}_0}{n} - z_0 \frac{\dot{z}_0}{n} \tag{1.65}$$

$$D_2 = -k_2 k_3 = 12y_0 \frac{\dot{y}_0}{n} - 6\frac{\dot{x}_0 \dot{y}_0}{n^2} \tag{1.66}$$

$$E_2 = -k_1 k_3 = -42y_0 \frac{\dot{x}_0}{n} + 36y_0^2 + 12\frac{\dot{x}_0^2}{n^2} \tag{1.67}$$

$$F_2 = 2k_5 k_7 - k_2 k_4 = 28y_0 \frac{\dot{x}_0}{n} - 8\frac{\dot{x}_0^2}{n^2} - 24y_0^2 + 4\frac{\dot{y}_0^2}{n^2} + 24x_0 \frac{\dot{y}_0}{n} \tag{1.68}$$

$$G_2 = 2k_6 k_7 - k_1 k_4 = 20\frac{\dot{y}_0}{n} y_0 - 12\frac{\dot{x}_0 \dot{y}_0}{n^2} - 4x_0 \frac{\dot{x}_0}{n} + 6x_0 y_0 \tag{1.69}$$

$$H_2 = \frac{-k_3^2}{2} = -18y_0^2 - \frac{9}{2}\frac{\dot{x}_0^2}{n^2} + 18y_0 \frac{\dot{x}_0}{n} \tag{1.70}$$

$$I_2 = -k_3 k_4 = -12y_0 \frac{\dot{y}_0}{n} - 6x_0 y_0 + 6\frac{\dot{x}_0 \dot{y}_0}{n^2} + 3x_0 \frac{\dot{x}_0}{n} \tag{1.71}$$

$$J_2 = \frac{-k_4^2}{2} + k_7^2 = -2\frac{\dot{y}_0^2}{n^2} - \frac{x_0^2}{2} - 2x_0 \frac{\dot{y}_0}{n} + 16y_0^2 + 4\frac{\dot{x}_0^2}{n^2} - 16y_0 \frac{\dot{x}_0}{n} \tag{1.72}$$

and finally for the z equation,

$$\begin{gathered}\ddot{z}_2 + n^2 z_2 = 3n^2 \frac{y_1 z_1}{r_0} \\ y_1 z_1 = A_3\ c_{nt}^2 + C_3\ s_{nt}\ c_{nt} + B_3\ s_{nt}^2 + F_3\ c_{nt} + G_3\ s_{nt}\end{gathered} \tag{1.73}$$

where,

$$A_3 = k_5 k_8 = 2\frac{\dot{x}_0}{n} z_0 - 3y_0 z_0 \tag{1.74}$$

$$B_3 = k_6 k_9 = \frac{\dot{y}_0 \dot{z}_0}{n^2} \tag{1.75}$$

$$C_3 = k_5 k_9 + k_6 k_8 = 2\frac{\dot{x}_0 \dot{z}_0}{n^2} - 3y_0 \frac{\dot{z}_0}{n} + \frac{\dot{y}_0}{n} z_0 \tag{1.76}$$

$$F_3 = k_7 k_8 = 4y_0 z_0 - 2\frac{\dot{x}_0}{n} z_0 \tag{1.77}$$

$$G_3 = k_7 k_9 = 4y_0 \frac{\dot{z}_0}{n} - 2\frac{\dot{x}_0 \dot{z}_0}{n^2} \tag{1.78}$$

The z equation takes the following form:

$$\ddot{z}_2 + n^2 z_2 = \frac{3n^2}{r_0}\left(A_3\ c_{nt}^2 + B_3\ s_{nt}^2 + C_3\ s_{nt}\ c_{nt} + F_3\ c_{nt} + G_3\ s_{nt}\right) \tag{1.79}$$

whose general solution is given by

$$\begin{aligned} z_2 = {} & c_1'\ s_{nt} + c_2'\ c_{nt} \\ & + \frac{3n}{r_0}\int_0^t \left(A_3\ c_{nv}^2 + B_3\ s_{nv}^2 + \frac{C_3}{2}\ s_{2nv} + F_3\ c_{nv} + G_3\ s_{nv}\right) \\ & \times\ \sin n(t-v) dv \end{aligned} \tag{1.80}$$

and because the initial conditions on z_0 and $\dot{z}_0$ are already absorbed in the first-order solutions, we let $c_1' = c_2' = 0$, such that, after carrying out the various integrations,

$$\begin{aligned} z_2 = \frac{3}{r_0}\Bigg[& \frac{(A_3 + B_3)}{2} - \frac{(A_3 + 2B_3)}{3} c_{nt} + \left(\frac{C_3}{3} + \frac{G_3}{2}\right) s_{nt} - \frac{C_3}{6} s_{2nt} \\ & + \frac{(B_3 - A_3)}{6} c_{2nt} + \frac{F_3}{2} nt\ s_{nt} - \frac{G_3}{2} nt\ c_{nt}\Bigg] \end{aligned} \tag{1.81}$$

which is of the form given in Equation (1.51), such that, by direct comparison

$$\gamma_0 = \frac{3}{2r_0}(A_3 + B_3) = \frac{3}{2r_0}\left(2\frac{\dot{x}_0 z_0}{n} - 3y_0 z_0 + \frac{\dot{y}_0 \dot{z}_0}{n^2}\right) \tag{1.82}$$

$$\gamma_1 = -2\gamma_3 - \gamma_6 = \frac{C_3}{r_0} + \frac{3}{2r_0} G_3 = \left(-\frac{\dot{x}_0 \dot{z}_0}{n^2} + 3\frac{y_0 \dot{z}_0}{n} + \frac{\dot{y}_0 z_0}{n} \right) / r_0 \tag{1.83}$$

$$\gamma_2 = -\gamma_0 - \gamma_4 = -(A_3 + 2B_3)/r_0 = \left(-2\frac{\dot{x}_0 z_0}{n} + 3y_0 z_0 - 2\frac{\dot{y}_0 \dot{z}_0}{n^2} \right) / r_0 \tag{1.84}$$

$$\gamma_3 = -\frac{C_3}{2r_0} = \left(-\frac{\dot{x}_0 \dot{z}_0}{n^2} + \frac{3}{2}\frac{y_0 \dot{z}_0}{n} - \frac{\dot{y}_0 z_0}{2n} \right) / r_0 \tag{1.85}$$

$$\gamma_4 = -\frac{(A_3 - B_3)}{2r_0} = \left(-\frac{\dot{x}_0 z_0}{n} + \frac{3}{2} y_0 z_0 + \frac{\dot{y}_0 \dot{z}_0}{2n^2} \right) / r_0 \tag{1.86}$$

$$\gamma_5 = \frac{3}{2r_0} F_3 = \left(6y_0 z_0 - 3\frac{z_0 \dot{x}_0}{n} \right) / r_0 \tag{1.87}$$

$$\gamma_6 = -\frac{3}{2r_0} G_3 = \left(-6y_0 \frac{\dot{z}_0}{n} + 3\frac{\dot{x}_0 \dot{z}_0}{n^2} \right) / r_0 \tag{1.88}$$

Now, the differential equation in x_2 can be integrated once, such that, from,

$$\begin{aligned}
\ddot{x}_2 - 2n\dot{y}_2 &= \frac{3n^2}{r_0} x_1 y_1 = \frac{3n^2}{r_0} \big(A_1\, c_{nt}^2 + B_1\, s_{nt}^2 + C_1\, s_{nt}\, c_{nt} + D_1\, nt\, c_{nt} + E_1\, nt\, s_{nt} \\
&\quad + F_1\, c_{nt} + G_1\, s_{nt} + I_1\, nt + J_1 \big) \\
\dot{x}_2 &= 2ny_2 + \frac{3n^2}{r_0} \Big[(A_1 + B_1)\frac{t}{2} + (E_1 + F_1)\frac{s_{nt}}{n} + (D_1 - G_1)\frac{c_{nt}}{n} \Big] \\
&\quad + (A_1 - B_1)\frac{s_{2nt}}{4n} - C_1 \frac{c_{2nt}}{4n} + D_1\, ts_{nt} - E_1\, tc_{nt} + I_1 \frac{nt^2}{2} + J_1 t \Big] + K_1
\end{aligned} \tag{1.89}$$

where K_1 is determined from $(\dot{x}_2)_0 = 0$, with $(y_2)_0 = 0$ too, so that

$$K_1 = -\frac{3n}{r_0} \left(D_1 - G_1 - \frac{C_1}{4} \right) \tag{1.90}$$

Using the preceding expression for $\dot{x}_2$ in the differential equation for y_2, and regrouping terms, yields

$$\begin{aligned}
\ddot{y}_2 + 2n\dot{x}_2 - 3n^2 y_2 &= -\frac{3n^2}{r_0} \Big[y_1^2 - \frac{1}{2}\big(x_1^2 + z_1^2\big) \Big] \\
&= -\frac{3n^2}{r_0} \big(A_2\, c_{nt}^2 + B_2\, s_{nt}^2 + C_2\, s_{nt}\, c_{nt} + D_2\, nt\, c_{nt} \big) \\
&\quad + E_2\, nt\, s_{nt} + F_2\, c_{nt} + G_2\, s_{nt} + H_2\, n^2 t^2 + I_2\, nt + J_2 \big) \\
\ddot{y}_2 + n^2 y_2 &= -2nK_1 + a_1\, s_{nt} + a_2\, c_{nt} + a_3\, s_{2nt} + a_4\, c_{2nt} + a_5\, nt \\
&\quad + a_6 nt\, s_{nt} + a_7\, nt\, c_{nt} + a_8\, n^2 t^2 + a_9
\end{aligned}$$

whose general solution can be written as

$$
\begin{aligned}
y_2 = k_1' \, s_{nt} + k_2' \, c_{nt} + \Bigg[\frac{1}{n} \int_0^t (& -2nK_1 + a_1 \, s_{nv} + a_2 \, c_{nv} + a_3 \, s_{2nv} + a_4 \, c_{2nv} \\
& + a_5 nv + a_6 nv \, s_{nv} + a_7 nv \, c_{nv} + a_8 n^2 v^2 + a_9) \sin n(t - v) dv \Bigg]
\end{aligned}
\tag{1.91}
$$

Once again the constants k_1' and k_2' are zero because the initial conditions are already absorbed in the first-order solutions. The various coefficients a_i are given by:

$$a_1 = -\frac{3n^2}{r_0}(2E_1 + 2F_1 + G_2) \tag{1.92}$$

$$a_2 = -\frac{3n^2}{r_0}(2D_1 - 2G_1 + F_2) \tag{1.93}$$

$$a_3 = -\frac{3n^2}{2r_0}(A_1 - B_1 + C_2) \tag{1.94}$$

$$a_4 = -\frac{3n^2}{2r_0}(A_2 - B_2 - C_1) \tag{1.95}$$

$$a_5 = -\frac{3n^2}{r_0}(A_1 + B_1 + 2J_1 + I_2) \tag{1.96}$$

$$a_6 = -\frac{3n^2}{r_0}(2D_1 + E_2) \tag{1.97}$$

$$a_7 = -\frac{3n^2}{r_0}(D_2 - 2E_1) \tag{1.98}$$

$$a_8 = -\frac{3n^2}{r_0}(I_1 + H_2) \tag{1.99}$$

$$a_9 = -\frac{3n^2}{r_0}\left(\frac{A_2 + B_2}{2} + J_2\right) \tag{1.100}$$

After carrying out the various integrations in Equation (1.91), and regrouping identical terms, the y_2 solution takes the form

$$
\begin{aligned}
y_2 = & \left(\frac{a_9}{n^2} - \frac{2}{n^2} a_8 - \frac{2K_1}{n}\right) + \left(\frac{a_1}{2n^2} + \frac{2a_3}{3n^2} - \frac{a_5}{n^2} - \frac{a_7}{4n^2}\right) s_{nt} \\
& + \left(\frac{a_4}{3n^2} + \frac{2a_8}{n^2} - \frac{a_9}{n^2} + \frac{2K_1}{n}\right) c_{nt} - \frac{a_4}{3n^2} c_{2nt} - \frac{a_3}{3n^2} s_{2nt} \\
& + \left(\frac{a_2}{2n^2} + \frac{a_6}{4n^2}\right) nt \, s_{2nt} + \left(\frac{a_7}{4n^2} - \frac{a_1}{2n^2}\right) nt \, c_{nt} + \frac{a_5}{n^2} nt \\
& + \frac{a_7}{4n^2} n^2 t^2 \, s_{nt} - \frac{a_6}{4n^2} n^2 t^2 \, c_{nt} + \frac{a_8}{n^2} n^2 t^2
\end{aligned}
\tag{1.101}
$$

which is of the same form as Equation (1.50), such that,

$$\beta_0 = \frac{a_9}{n^2} - \frac{2}{n^2} a_8 - 2\frac{K_1}{n} \tag{1.102}$$

$$\beta_1 = a_5/n^2 \tag{1.103}$$

$$\beta_2 = a_8/n^2 \tag{1.104}$$

$$\beta_3 = \frac{a_1}{2n^2} + \frac{2a_3}{3n^2} - \frac{a_5}{n^2} - \frac{a_7}{4n^2} \tag{1.105}$$

$$\beta_4 = \frac{a_4}{3n^2} + \frac{2a_8}{n^2} - \frac{a_9}{n^2} + 2\frac{K_1}{n} \tag{1.106}$$

$$\beta_5 = -\frac{a_3}{3n^2} \tag{1.107}$$

$$\beta_6 = -\frac{a_4}{3n^2} \tag{1.108}$$

$$\beta_7 = \frac{a_2}{2n^2} + \frac{a_6}{4n^2} \tag{1.109}$$

$$\beta_8 = \frac{a_7}{4n^2} - \frac{a_1}{2n^2} \tag{1.110}$$

The coefficients a_6 and a_7 are zero because $E_2 = -2D_1$ and $D_2 = 2E_1$. Now, another identify is $I_1 = -\frac{4}{3}H_2$, such that,

$$\beta_0 = -\frac{3}{2r_0}(A_2 + B_2) - \frac{3J_2}{r_0} - \frac{2H_2}{r_0} - \frac{6G_1}{r_0} + \frac{6D_1}{r_0} - \frac{3C_1}{2r_0} \tag{1.111}$$

Using the identify $I_2 = -\frac{3}{2}J_1$,

$$\beta_1 = -\frac{3}{2r_0} J_1 \tag{1.112}$$

and using the identify $I_1 = -\frac{4}{3}H_2$,

$$\beta_2 = -\frac{3}{4r_0} I_1 \tag{1.113}$$

Also by inspection

$$\beta_3 = -\beta_1 - 2\beta_5 - \beta_8 \tag{1.114}$$

$$\beta_4 = -\beta_0 - \beta_6 \tag{1.115}$$

$$\beta_5 = \frac{C_2}{2r_0} + \frac{1}{2r_0}(A_1 - B_1) \tag{1.116}$$

$$\beta_6 = \frac{1}{2r_0}(A_2 - B_2 - C_1) \tag{1.117}$$

and using the identifies $E_2 = -2D_1$ and $D_2 = 2E_1$,

$$\beta_7 = \frac{3}{2r_0}E_2 + \frac{3G_1}{r_0} - \frac{3F_2}{2r_0} \tag{1.118}$$

$$\beta_8 = \frac{3E_1}{r_0} + \frac{3F_1}{r_0} + \frac{3G_2}{2r_0} \tag{1.119}$$

and in terms of the initial conditions,

$$\beta_0 = \frac{3}{r_0}\left(\frac{x_0^2}{2} + \frac{\dot{x}_0^2}{n^2} + \frac{7}{2}y_0^2 - \frac{\dot{y}_0^2}{2n^2} - \frac{4\dot{x}_0 y_0}{n} + \frac{z_0^2}{4} + \frac{\dot{z}_0^2}{4n^2}\right) \tag{1.120}$$

$$\beta_1 = -\frac{3}{r_0}\left(2x_0y_0 - x_0\frac{\dot{x}_0}{n} + 4y_0\frac{\dot{y}_0}{n} - 2\frac{\dot{x}_0\dot{y}_0}{n^2}\right) \tag{1.121}$$

$$\beta_2 = -\frac{9}{2r_0}\left(4y_0^2 - 4y_0\frac{\dot{x}_0}{n} + \frac{\dot{x}_0^2}{n^2}\right) \tag{1.122}$$

$$\beta_3 = \frac{1}{r_0}\left(12y_0\frac{\dot{y}_0}{n} + 6x_0y_0 - 7\frac{\dot{x}_0\dot{y}_0}{n^2} - 3x_0\frac{\dot{x}_0}{n} + z_0\frac{\dot{z}_0}{n}\right) \tag{1.123}$$

$$\beta_4 = \frac{1}{r_0}\left(-\frac{3}{2}x_0^2 - 5\frac{\dot{x}_0^2}{n^2} - 15y_0^2 + 2\frac{\dot{y}_0^2}{n^2} + 18y_0\frac{\dot{x}_0}{n} - \frac{\dot{z}_0^2}{n^2} - \frac{z_0^2}{2}\right) \tag{1.124}$$

$$\beta_5 = \frac{1}{r_0}\left(2\frac{\dot{x}_0\dot{y}_0}{n^2} - 3y_0\frac{\dot{y}_0}{n} - z_0\frac{\dot{z}_0}{2n}\right) \tag{1.125}$$

$$\beta_6 = \frac{1}{r_0}\left(\frac{9}{2}y_0^2 + 2\frac{\dot{x}_0^2}{n^2} - \frac{1}{2}\frac{\dot{y}_0^2}{n^2} + \frac{\dot{z}_0^2}{4n^2} - \frac{z_0^2}{4} - 6y_0\frac{\dot{x}_0}{n}\right) \tag{1.126}$$

$$\beta_7 = -\frac{3}{r_0}\left(7y_0\frac{\dot{x}_0}{n} - 2\frac{\dot{x}_0^2}{n^2} - 6y_0^2\right) \tag{1.127}$$

$$\beta_8 = \frac{3}{r_0}\left(-\frac{\dot{x}_0\dot{y}_0}{n^2} + 2\frac{y_0\dot{y}_0}{n}\right) \tag{1.128}$$

with the solution given by

$$\begin{aligned} y_2 = \beta_0 + \beta_1\, nt + \beta_2\, n^2t^2 + \beta_3\, s_{nt} + \beta_4\, c_{nt} + \beta_5\, s_{2nt} + \beta_6\, c_{2nt} + \beta_7\, nt\, s_{nt} \\ + \beta_8\, nt\, c_{nt} \end{aligned} \tag{1.129}$$

Using this solution in Equation (1.89) for $\dot{x}_2$, and integrating once more, yields the x_2 solution

$$x_2 = (2n\beta_0 + K_1)t + \frac{b_1}{2n}n^2t^2 + \frac{b_2}{3n}n^3t^3 + (b_8 - b_3)\frac{c_{nt}}{n} + (b_4 + b_7)\frac{s_{nt}}{n} + \frac{b_6}{2n}s_{2nt} - \frac{b_5}{2n}c_{2nt} + b_8\, t\, s_{nt} - b_7\, t\, c_{nt} + K_1' \tag{1.130}$$

The constant K_1' is determined from $(x_2)_0 = 0$ at time zero,

$$K_1' = \frac{b_3}{n} + \frac{b_5}{2n} - \frac{b_8}{n} \tag{1.131}$$

The other constants appearing in Equation (1.130) are given by

$$b_1 = 2n\beta_1 + \frac{3n}{r_0}J_1 \tag{1.132}$$

$$b_2 = 2n\beta_2 + \frac{3n}{2r_0}I_1 \tag{1.133}$$

$$b_3 = 2n\beta_3 + \frac{3n}{r_0}(E_1 + F_1) \tag{1.134}$$

$$b_4 = 2n\beta_4 + \frac{3n}{r_0}(D_1 - G_1) \tag{1.135}$$

$$b_5 = 2n\beta_5 + \frac{3n}{2r_0}A_1 \tag{1.136}$$

$$b_6 = 2n\beta_6 - \frac{3n}{4r_0}C_1 \tag{1.137}$$

$$b_7 = 2n\beta_7 + \frac{3n}{r_0}D_1 \tag{1.138}$$

$$b_8 = 2n\beta_8 - \frac{3n}{r_0}E_1 \tag{1.139}$$

Using the β_i expressions developed earlier, the first two coefficients reduce to $b_1 = 0$, $b_2 = 0$, which, then, gives to x_2 the form in Equation (1.49), such that

$$\alpha_0 = -\alpha_3 - \alpha_5 \tag{1.140}$$

$$\alpha_1 = -\frac{3}{r_0}(A_2 + B_2) - \frac{6J_2}{r_0} - 4\frac{H_2}{r_0} - \frac{9}{r_0}\left(G_1 - D_1 + \frac{C_1}{4}\right) \tag{1.141}$$

$$\alpha_2 = 2\beta_4 + \frac{3G_1}{r_0} - \frac{3F_2}{r_0} \tag{1.142}$$

$$\alpha_3 = -2\beta_3 + \frac{3F_1}{r_0} + \frac{3G_2}{r_0} \tag{1.143}$$