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Jean-Paul Dufour
Nguyen Tien Zung

Poisson Structures and Their Normal Forms

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Authors:

Jean-Paul Dufour
Département de mathématique
Université de Montpellier 2
place Eugène Bataillon
34095 Montpellier
France
e-mail: dufourj@math.univ-montp2.fr

Nguyen Tien Zung
Laboratoire Émile Picard,
UMR 5580 CNRS
Institut de Mathématiques
Université Paul Sabatier
118 route de Narbonne
31062 Toulouse
France
e-mail: tienzung@picard.ups-tlse.fr

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à Hélène
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Preface

“Il ne semblait pas que cette importante théorie pût encore être perfectionnée, lorsque les deux géomètres qui ont le plus contribué à la rendre complète, en ont fait de nouveau le sujet de leurs méditations…”. By these words, Siméon Denis Poisson announced in 1809 [293] that he had found an improvement in the theory of Lagrangian mechanics, which was being developed by Joseph-Louis Lagrange and Pierre-Simon Laplace. In that pioneering paper, Poisson introduced (we slightly modernize his writing) the notation

$$(a, b) = \sum_{i=1}^{n} \left(\frac{\partial a}{\partial q_i} \frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q_i} \right), \tag{0.1}$$

where a and b are two functions of the coordinates q_i and the conjugate quantities $p_i = \frac{\partial R}{\partial \dot{q}_i}$ for a mechanical system with Lagrangian function R. He proved that, if a and b are first integrals of the system then (a, b) also is. This (a, b) is nowadays denoted by $\{a, b\}$ and called the Poisson bracket of a and b. Mathematicians of the 19th century already recognized the importance of this bracket. In particular, William Hamilton used it extensively to express his equations in an essay in 1835 [168] on what we now call Hamiltonian dynamics. Carl Jacobi in his “Vorlesungen über Dynamik” around 1842 (see [185]) showed that the Poisson bracket satisfies the famous Jacobi identity:

$$\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0. \tag{0.2}$$

This same identity is satisfied by Lie algebras, which are infinitesimal versions of Lie groups, first studied by Sophus Lie and his collaborators in the end of the 19th century [213].

In our modern language, a Poisson structure on a manifold M is a 2-vector field Π (Poisson tensor) on M, such that the corresponding bracket (Poisson bracket) on the space of functions on M, defined by

$$\{f, g\} := \langle \mathrm{d}f \wedge \mathrm{d}g, \Pi \rangle \ , \tag{0.3}$$

satisfies the Jacobi identity. (M, Π) is then called a Poisson manifold. This notion of Poisson manifolds generalizes both symplectic manifolds and Lie algebras. The

Poisson tensor of the original bracket of Poisson is

$$\Pi = \sum_{i=1}^{n} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}, \tag{0.4}$$

which is nondegenerate and corresponds to a symplectic 2-form, namely

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i \ . \tag{0.5}$$

On the other hand, each finite-dimensional Lie algebra gives rise to a linear Poisson tensor on its dual space and vice versa.

Poisson manifolds play a fundamental role in Hamiltonian dynamics, where they serve as phase spaces. They also arise naturally in other mathematical problems as well. In particular, they form a bridge from the "commutative world" to the "noncommutative world". For example, Lie groupoids give rise to noncommutative operator algebras, while their infinitesimal versions, called Lie algebroids, are nothing but "fiber-wise linear" Poisson structures. Poisson geometry, i.e., the geometry of Poisson structures, which began as an outgrowth of symplectic geometry, has seen rapid growth in the last three decades, and has now become a very large theory, with interactions with many other domains of mathematics, including Hamiltonian dynamics, integrable systems, representation theory, quantum groups, noncommutative geometry, singularity theory, and so on.

This book arises from its authors' efforts to study Poisson structures, and in particular their normal forms. As a result, the book aims to offer a quick introduction to Poisson geometry, and to give an extensive account on known results about the theory of normal forms of Poisson structures and related objects. This theory is relatively young. Though some earlier results may be traced back to V.I. Arnold, it really took off with a fundamental paper of Alan Weinstein in 1983 [346], in which he proved a formal linearization theorem for Poisson structures, a local symplectic realization theorem, and the following splitting theorem: locally any Poisson manifold can be written as the direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point. Since then, a large number of other results have emerged, many of them very recently.

Here is a brief summary of this book, which only highlights a few important points from each chapter. For a more detailed list of what the book has to offer, the reader may look at the table of contents.

The book consists of eight chapters and some appendices. Chapter 1 is based on lectures given by the authors in Montpellier and Toulouse for graduate students, and is a small self-contained introduction to Poisson geometry. Among other things, we show how Poisson manifolds can be viewed as singular foliations with symplectic leaves, and also as quotients of symplectic manifolds. The reader will also find in this chapter a section about the Schouten bracket of multi-vector fields, which was discovered by Schouten in 1940 [311], and whose importance goes beyond Poisson geometry.

Starting from Chapter 2, the book contains many recent results which have not been previously available in book form. A few results in this book are even original and not published elsewhere.

Chapter 2 is about Poisson cohomology, a natural and important invariant introduced by André Lichnerowicz in 1977 [211]. In particular, we show the role played by this cohomology in normal form problems, and its relations with de Rham cohomology of manifolds and Chevalley–Eilenberg cohomology of Lie algebras. Some known methods for computing Poisson cohomology are briefly discussed, including standard tools from algebraic topology such as the Mayer–Vietoris sequence and spectral sequences, and also tools from singularity theory. Many authors, including Viktor Ginzburg, Johannes Huebschmann, Mikhail Karasev, Jean-Louis Koszul, Izu Vaisman, Ping Xu, etc., contributed to the understanding of Poisson cohomology, and we discuss some of their results in this chapter. However, the computation of Poisson cohomology remains very difficult in general.

Chapter 3 is about a kind of normal form for Poisson structures, which are comparable to Poincaré–Birkhoff normal forms for vector fields, and which are called Levi decompositions because they are analogous to Levi–Malcev decompositions for finite-dimensional Lie algebras. The results of this chapter are due mainly to Aissa Wade [342] (the formal case), the second author and Monnier [369, 263] (the analytic and smooth cases). The proof of the formal case is purely algebraic and relatively simple. The analytic and smooth cases make use of the fast convergence methods of Kolmogorov and Nash–Moser.

Chapter 4 is about linearization of Poisson structures. The results of Chapter 3 are used in this chapter. In particular, Conn's linearization results for Poisson structures with a semi-simple linear part [80, 81] may be viewed as special cases of Levi decomposition. Among results discussed at length in this chapter, we will mention here Weinstein's theorem on the smooth degeneracy of real semisimple Lie algebras of real rank greater than or equal to 2 [348], and our result on the formal and analytic nondegeneracy of the Lie algebra $\mathfrak{aff}(n)$ [120].

In Chapter 5 we explain the links among quadratic Poisson structures, r-matrices, and the theory of Poisson–Lie groups introduced by Drinfeld [107]. So far, all quadratic Poisson structures known to us can be obtained from r-matrices, which have their origins in the theory of integrable systems. Some important contributions of Semenov–Tian–Shansky, Lu, Weinstein and other people can be found in this chapter. We then show how the curl vector field (also known as modular vector field) led the first author and other people to a classification of "nonresonant" quadratic Poisson structures, and quadratization results for Poisson structures which begin with a nonresonant quadratic part. Let us mention that Poisson–Lie groups are classical versions of *quantum groups*, a subject which is beyond the scope of this book.

Chapter 6 is devoted to n-ary generalizations of Poisson structures, which go under the name of Nambu structures. Though originally invented by physicists Nambu [275] and Takhtajan [328], these Nambu structures turn out to be dual to integrable differential forms and play an important role in the theory of singular foliations. A linearization theorem for Nambu structures [119] is given in this chapter. Its proof at one point makes use of Malgrange's "Frobenius with singularities" theorem [233, 234]. Malgrange's theorem is also discussed in this chapter, together with many other results on singular foliations and integrable differential forms. In particular, we present generalizations of Kupka's stability theorem [204], which are due to de Medeiros [244, 245], Camacho and Lins Neto [59], and ourselves.

Chapter 7 deals with Lie groupoids. Among other things, it contains a recent slice theorem due to Weinstein [354] and the second author [370]. This slice theorem is a normal form theorem for proper Lie groupoids near an orbit, and generalizes the classical Koszul–Palais slice theorem for proper Lie group actions. We also discuss symplectic groupoids, an important object of Poisson geometry introduced independently by Karasev [189], Weinstein [349], and Zakrzewski [364] in the 1980s. A local normal form theorem for proper symplectic groupoids is also given.

Chapter 8 is about Lie algebroids, introduced by Pradines [294] in 1967 as infinitesimal versions of Lie groupoids. They correspond to fiber-wise linear Poisson structures, and many results about general Poisson structures, including the splitting theorem and the Levi decomposition, apply to them. Our emphasis is again on their local normal forms, though we also discuss cohomology of Lie algebroids, and the problem of integrability of Lie algebroids, including a recent strong theorem of Crainic and Fernandes [86].

Finally, Appendix A is a collection of discussions which help make the book more self-contained or which point to closely related subjects. It contains, among other things, Vorobjev's description of a neighborhood of a symplectic leaf [340], toric characterization of Poincaré–Birkhoff normal forms of vector fields, a brief introduction to deformation quantization, including a famous theorem of Kontsevich [195] on the existence of deformation quantization for an arbitrary Poisson structure, etc.

The book is biased towards what we know best, i.e., local normal forms. May the specialists in Poisson geometry forgive us for not giving more discussions on other topics, due to our lack of competence. Familiarity with symplectic manifolds is not required, though it will be helpful for reading this book. There are many nice books readily available on symplectic geometry. On the other hand, books on Poisson geometry are relatively rare. The only general introductory reference to date is Vaisman [333]. Some other references are Cannas da Silva and Weinstein [60] (a nice book about geometric models for noncommutative algebras, where Poisson geometry plays a key role), Karasev and Maslov [190] (a book on Poisson manifolds with an emphasis on quantization), Mackenzie [228] (a general reference on Lie groupoids and Lie algebroids), Ortega and Ratiu [288] (a comprehensive book on symmetry and reduction in Poisson geometry), and a book in preparation by Xu [362] (with an emphasis on Poisson groupoids). We hope that our book is complementary to the above books, and will be useful for students and researchers interested in the subject.

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Chapter 1

Generalities on Poisson Structures

1.1 Poisson brackets

Definition 1.1.1. A C^∞-smooth *Poisson structure* on a C^∞-smooth finite-dimensional manifold M is an $\mathbb{R}$-bilinear antisymmetric operation

$$\mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M),\ (f,g) \longmapsto \{f,g\} \tag{1.1}$$

on the space $\mathcal{C}^\infty(M)$ of real-valued C^∞-smooth functions on M, which verifies the Jacobi identity

$$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0 \tag{1.2}$$

and the Leibniz identity

$$\{f,gh\} = \{f,g\}h + g\{f,h\}, \quad \forall f,g,h \in \mathcal{C}^\infty(M). \tag{1.3}$$

In other words, $\mathcal{C}^\infty(M)$, equipped with $\{,\}$, is a Lie algebra whose Lie bracket satisfies the Leibniz identity. This bracket $\{,\}$ is called a *Poisson bracket*. A manifold equipped with such a bracket is called a *Poisson manifold*.

Similarly, one can define real analytic, holomorphic, and formal Poisson manifolds, if one replaces $\mathcal{C}^\infty(M)$ by the corresponding sheaf of local analytic (respectively, holomorphic, formal) functions. In order to define C^k-smooth Poisson structures ($k \in \mathbb{N}$), we will have to express them in terms of 2-vector fields. This will be done in the next section.

Remark 1.1.2. In this book, when we say that something is smooth without making precise its smoothness class, we usually mean that it is C^∞-smooth. However, most of the time, being C^1-smooth or C^2-smooth will also be good enough, though we don't want to go into these details. Analytic means either real analytic or holomorphic. Though we will consider only finite-dimensional Poisson structures in this book, let us mention that infinite-dimensional Poisson structures also appear naturally (especially in problems of mathematical physics), see, e.g., [281, 285] and references therein.

Example 1.1.3. One can define a trivial Poisson structure on any manifold by putting $\{f, g\} = 0$ for all functions f and g.

Example 1.1.4. Take $M = \mathbb{R}^2$ with coordinates (x, y) and let $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be an arbitrary smooth function. One can define a smooth Poisson structure on $\mathbb{R}^2$ by putting

$$\{f, g\} = \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) p \ . \tag{1.4}$$

Exercise 1.1.5. Verify the Jacobi identity and the Leibniz identity for the above bracket. Show that any smooth Poisson structure of $\mathbb{R}^2$ has the above form.

Definition 1.1.6. A *symplectic manifold* (M, ω) is a manifold M equipped with a nondegenerate closed differential 2-form ω, called the *symplectic form*.

The nondegeneracy of a differential 2-form ω means that the corresponding homomorphism $\omega^\flat : TM \to T^*M$ from the tangent space of M to its cotangent space, which associates to each vector X the covector $i_X\omega$, is an isomorphism. Here $i_X\omega = X \lrcorner \omega$ is the contraction of ω by X and is defined by $i_X\omega(Y) = \omega(X, Y)$.

If $f : M \to \mathbb{R}$ is a function on a symplectic manifold (M, ω), then we can define its *Hamiltonian vector field*, denoted by X_f, as follows:

$$i_{X_f}\omega = -\mathrm{d}f \ . \tag{1.5}$$

We can also define on (M, ω) a natural bracket, called the Poisson bracket of ω, as follows:

$$\{f, g\} = \omega(X_f, X_g) = -\langle \mathrm{d}f, X_g\rangle = -X_g(f) = X_f(g). \tag{1.6}$$

Proposition 1.1.7. *If (M, ω) is a smooth symplectic manifold, then the bracket $\{f, g\} = \omega(X_f, X_g)$ is a smooth Poisson structure on M.*

Proof. The Leibniz identity is obvious. Let us show the Jacobi identity. Recall the following *Cartan's formula* for the differential of a k-form η (see, e.g., [41]):

$$\begin{aligned} \mathrm{d}\eta(X_1, \dots, X_{k+1}) &= \sum_{i=1}^{k+1} (-1)^{i-1} X_i \left(\eta(X_1, \dots, \widehat{X_i}, \dots, X_{k+1})\right) \\ &+ \sum_{1 \le i < j \le k+1} (-1)^{i+j} \eta\left([X_i, X_j], X_1, \dots, \widehat{X_i}, \dots, \widehat{X_j}, \dots, X_{k+1}\right), \end{aligned} \tag{1.7}$$

where $X_1, \ldots, X_{k+1}$ are vector fields, and the hat means that the corresponding entry is omitted. Applying Cartan's formula to ω and X_f, X_g, X_h, we get:

$$\begin{aligned}
0 &= \mathrm{d}\omega(X_f, X_g, X_h) \\
&= X_f(\omega(X_g, X_h)) + X_g(\omega(X_h, X_f)) + X_h(\omega(X_f, X_g)) \\
&\quad - \omega([X_f, X_g], X_h) - \omega([X_g, X_h], X_f) - \omega([X_h, X_f], X_g) \\
&= X_f\{g, h\} + X_g\{h, f\} + X_h\{f, g\} \\
&\quad + [X_f, X_g](h) + [X_g, X_h](f) + [X_h, X_f](g) \\
&= \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} + X_f(X_g(h)) - X_g(X_f(h)) \\
&\quad + X_g(X_h(f)) - X_h(X_g(f)) + X_h(X_f(g)) - X_f(X_h(g)) \\
&= 3(\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\}). \qquad \square
\end{aligned}$$

Thus, any symplectic manifold is also a Poisson manifold, though the inverse is not true.

The classical *Darboux theorem* says that in the neighborhood of every point of (M, ω) there is a local system of coordinates $(p_1, q_1, \ldots, p_n, q_n)$, where $2n = \dim M$, called *Darboux coordinates* or *canonical coordinates*, such that

$$\omega = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i \ . \tag{1.8}$$

A proof of Darboux's theorem will be given in Section 1.4. In such a Darboux coordinate system one has the following expressions for the Poisson bracket and the Hamiltonian vector fields:

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right), \tag{1.9}$$

$$X_h = \sum_{i=1}^{n} \frac{\partial h}{\partial p_i} \frac{\partial}{\partial q_i} - \sum_{i=1}^{n} \frac{\partial h}{\partial q_i} \frac{\partial}{\partial p_i}. \tag{1.10}$$

The *Hamiltonian equation* of h (also called the *Hamiltonian system* of h), i.e., the ordinary differential equation for the integral curves of X_h, has the following form, which can be found in most textbooks on analytical mechanics:

$$\dot{q}_i = \frac{\partial h}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial h}{\partial q_i}. \tag{1.11}$$

In fact, to define the Hamiltonian vector field of a function, what one really needs is not a symplectic structure, but a Poisson structure: The Leibniz identity means that, for a given function f on a Poisson manifold M, the map $g \longmapsto \{f, g\}$ is a *derivation*. Thus, there is a unique vector field X_f on M, called the *Hamiltonian vector field* of f, such that for any $g \in C^\infty(M)$ we have

$$X_f(g) = \{f, g\} \ . \tag{1.12}$$

Exercise 1.1.8. Show that, in the case of a symplectic manifold, Equation (1.5) and Equation (1.12) give the same vector field.

Example 1.1.9. If N is a manifold, then its cotangent bundle T^*N has a unique natural symplectic structure, hence T^*N is a Poisson manifold with a natural Poisson bracket. The symplectic form on T^*N can be constructed as follows. Denote by $\pi : T^*N \to N$ the projection which assigns to each covector $p \in T^*_q N$ its base point q. Define the so-called *Liouville 1-form* θ on T^*N by

$$\langle \theta, X \rangle = \langle p, \pi_* X \rangle \quad \forall\, X \in T_p(T^*N).$$

In other words, $\theta(p) = \pi^*(p)$, where on the left-hand side p is considered as a point of T^*N and on the right-hand side it is considered as a cotangent vector to N. Then $\omega = \mathrm{d}\theta$ is a symplectic form on N: ω is obviously closed; to see that it is nondegenerate take a local coordinate system $(p_1, \dots, p_n, q_1, \dots, q_n)$ on T^*N, where $(q_1, \dots, q_n)$ is a local coordinate system on N and $(p_1, \dots, p_n)$ are the coefficients of covectors $\sum p_i \mathrm{d}q_i(q)$ in this coordinate system. Then $\theta = \sum p_i \mathrm{d}q_i$ and $\omega = \mathrm{d}\theta = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$, i.e., $(p_1, \dots, p_n, q_1, \dots, q_n)$ is a Darboux coordinate system for ω. In classical mechanics, one often deals with Hamiltonian equations on a cotangent bundle T^*N equipped with the natural symplectic structure, where N is the *configuration space*, i.e., the space of all possible configurations or positions; T^*N is called the *phase space*.

A function g is called a *first integral* of a vector field X if g is constant with respect to X: $X(g) = 0$. Finding first integrals is an important step in the study of dynamical systems. Equation (1.12) means that a function g is a first integral of a Hamiltonian vector field X_f if and only if $\{f, g\} = 0$. In particular, every function h is a first integral of its own Hamiltonian vector field: $X_h(h) = \{h, h\} = 0$ due to the anti-symmetricity of the Poisson bracket. This fact is known in physics as the principle of *conservation of energy* (here h is the energy function).

The following classical theorem of Poisson [293] allows one sometimes to find new first integrals from old ones:

Theorem 1.1.10 (Poisson). *If g and h are first integrals of a Hamiltonian vector field X_f on a Poisson manifold M, then $\{g, h\}$ also is.*

Proof. Another way to formulate this theorem is

$$\left.\begin{array}{ll} \{g, f\} & = 0 \\ \{h, f\} & = 0 \end{array}\right\} \Rightarrow \{\{g, h\}, f\} = 0. \tag{1.13}$$

But this is a corollary of the Jacobi identity. □

Another immediate consequence of the definition of Poisson brackets is the following lemma:

Lemma 1.1.11. *Given a smooth Poisson manifold $(M, \{,\})$, the map $f \mapsto X_f$ is a homomorphism from the Lie algebra $\mathcal{C}^\infty(M)$ of smooth functions under the*

Poisson bracket to the Lie algebra of smooth vector fields under the usual Lie bracket. In other words, we have the following formula:

$$[X_f, X_g] = X_{\{f,g\}}. \tag{1.14}$$

Proof. For any $f, g, h \in \mathcal{C}^\infty(M)$ we have $[X_f, X_g]\, h = X_f\,(X_g h) - X_g\,(X_f h) = \{f, \{g, h\}\} - \{g, \{f, h\}\} = \{\{f, g\}, h\} = X_{\{f,g\}} h$. Since h is arbitrary, it means that $[X_f, X_g] = X_{\{f,g\}}$. □

1.2 Poisson tensors

In this section, we will express Poisson structures in terms of 2-vector fields which satisfy some special conditions.

Let M be a smooth manifold and q a positive integer. We denote by $\Lambda^q TM$ the space of tangent *q-vectors* of M: it is a vector bundle over M, whose fiber over each point $x \in M$ is the space $\Lambda^q T_x M = \Lambda^q(T_x M)$, which is the exterior (antisymmetric) product of q copies of the tangent space $T_x M$. In particular, $\Lambda^1 TM = TM$. If $(x_1, \ldots, x_n)$ is a local system of coordinates at x, then $\Lambda^q T_x M$ admits a linear basis consisting of the elements $\dfrac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \dfrac{\partial}{\partial x_{i_q}}(x)$ with $i_1 < i_2 < \cdots < i_q$. A smooth *q-vector field* Π on M is, by definition, a smooth section of $\Lambda^q TV$, i.e., a map Π from V to $\Lambda^q TM$, which associates to each point x of M a q-vector $\Pi(x) \in \Lambda^q T_x M$, in a smooth way. In local coordinates, Π will have a local expression

$$\Pi(x) = \sum_{i_1 < \cdots < i_q} \Pi_{i_1 \ldots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}} = \frac{1}{q!} \sum_{i_1 \ldots i_q} \Pi_{i_1 \ldots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}}, \tag{1.15}$$

where the components $\Pi_{i_1 \ldots i_q}$, called the *coefficients* of Π, are smooth functions. The coefficients $\Pi_{i_1 \ldots i_q}$ are antisymmetric with respect to the indices, i.e., if we permute two indices then the coefficient is multiplied by -1. For example, $\Pi_{i_1 i_2 \ldots} = -\Pi_{i_2 i_1 \ldots}$. If $\Pi_{i_1 \ldots i_q}$ are C^k-smooth, then we say that Π is C^k-smooth, and so on.

Smooth q-vector fields are dual objects to differential q-forms in a natural way. If Π is a q-vector field and α is a differential q-form, which in some local system of coordinates are written as $\Pi(x) = \sum_{i_1 < \cdots < i_q} \Pi_{i_1 \ldots i_q} \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_q}}$ and $\alpha = \sum_{i_1 < \cdots < i_q} a_{i_1 \ldots i_q} \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_q}$, then their *pairing* $\langle \alpha, \Pi \rangle$ is a function defined by

$$\langle \alpha, \Pi \rangle = \sum_{i_1 < \cdots < i_q} \Pi_{i_1 \ldots i_q} a_{i_1 \ldots i_q} \ . \tag{1.16}$$

Exercise 1.2.1. Show that the above definition of $\langle \alpha, \Pi \rangle$ does not depend on the choice of local coordinates.

In particular, smooth q-vector fields on a smooth manifold M can be considered as $\mathcal{C}^\infty(M)$-linear operators from the space of smooth differential q-forms on M to $\mathcal{C}^\infty(M)$, and vice versa.

A C^k-smooth q-vector field Π will define an $\mathbb{R}$-multilinear skewsymmetric map from $\mathcal{C}^\infty(M) \times \cdots \times \mathcal{C}^\infty(M)$ (q times) to $\mathcal{C}^\infty(M)$ by the following formula:

$$\Pi(f_1, \ldots, f_q) := \langle \Pi, \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_q \rangle \ . \tag{1.17}$$

Conversely, we have:

Lemma 1.2.2. *An $\mathbb{R}$-multilinear map $\Pi : \mathcal{C}^\infty(M) \times \cdots \times \mathcal{C}^\infty(M) \to \mathcal{C}^k(M)$ arises from a C^k-smooth q-vector field by Formula* (1.17) *if and only if Π is skewsymmetric and satisfies the Leibniz rule (or condition):*

$$\Pi(fg, f_2, \ldots, f_q) = f\Pi(g, f_2, \ldots, f_q) + g\Pi(f, f_2, \ldots, f_q). \tag{1.18}$$

A map Π which satisfies the above conditions is called a *multi-derivation*, and the above lemma says that multi-derivations can be identified with multi-vector fields.

Proof (sketch). The "only if" part is straightforward. For the "if" part, we have to check that the value of $\Pi(f_1, \ldots, f_q)$ at a point x depends only on the value of $\mathrm{d}f_1, \ldots, \mathrm{d}f_q$ at x. Equivalently, we have to check that if $\mathrm{d}f_1(x) = 0$ then $\Pi(f_1, \ldots, f_q)(x) = 0$. If $\mathrm{d}f_1(x) = 0$ then we can write $f_1 = c + \sum_i x_i g_i$ where c is a constant and x_i and g_i are smooth functions which vanish at x. According to the Leibniz rule we have $\Pi(1 \times 1, f_2, \ldots, f_q) = 1 \times \Pi(1, f_2, \ldots, f_q) + 1 \times \Pi(1, f_2, \ldots, f_q) = 2\Pi(1, f_2, \ldots, f_q)$, hence $\Pi(1, f_2, \ldots, f_q) = 0$. Now according to the linearity and the Leibniz rule we have $\Pi(f_1, \ldots, f_q)(x) = c\Pi(1, f_2, \ldots, f_q)(x) + \sum x_i(x)\Pi(g_i, f_2, \ldots, f_q)(x) + \sum g_i(x)\Pi(x_i, f_2, \ldots, f_q)(x) = 0$. □

In particular, if Π is a Poisson structure, then it is skewsymmetric and satisfies the Leibniz condition, hence it arises from a 2-vector field, which we will also denote by Π:

$$\{f, g\} = \Pi(f, g) = \langle \Pi, \mathrm{d}f \wedge \mathrm{d}g \rangle \ . \tag{1.19}$$

A 2-vector field Π, such that the bracket $\{f, g\} := \langle \Pi, \mathrm{d}f \wedge \mathrm{d}g \rangle$ is a Poisson bracket (i.e., satisfies the Jacobi identity $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$ for any smooth functions f, g, h), is called a *Poisson tensor*, or also a *Poisson structure*. The corresponding Poisson bracket is often denoted by $\{,\}_\Pi$. If the Poisson tensor Π is a C^k-smooth 2-vector field, then we say that we have a C^k-smooth Poisson structure, and so on.

In a local system of coordinates $(x_1, \ldots, x_n)$ we have

$$\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{1}{2} \sum_{i,j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \ , \tag{1.20}$$

where $\Pi_{ij} = \langle \Pi, \mathrm{d}x_i \wedge \mathrm{d}x_j \rangle = \{x_i, x_j\}$, and

$$\{f, g\} = \langle \sum_{i<j} \{x_i, x_j\} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}, \sum_{i,j} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \mathrm{d}x_i \wedge \mathrm{d}x_j \rangle = \sum_{i,j} \Pi_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \ . \tag{1.21}$$

Example 1.2.3. The Poisson tensor corresponding to the standard symplectic structure $\omega = \sum_{j=1}^{n} \mathrm{d}x_j \wedge \mathrm{d}y_j$ on $\mathbb{R}^{2n}$ is $\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial y_j}$.

Notation 1.2.4. In this book, if functions $f_1, \dots, f_p$ depend on variables $x_1, \dots, x_p$, and maybe other variables, then we will denote by

$$\frac{\partial(f_1, \dots, f_p)}{\partial(x_1, \dots, x_p)} := \det \left(\frac{\partial f_i}{\partial x_j} \right)_{i,j=1}^{p} \tag{1.22}$$

the *Jacobian determinant* of $(f_1, \dots, f_p)$ with respect to $(x_1, \dots, x_p)$. For example,

$$\frac{\partial(f, g)}{\partial(x_i, x_j)} := \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_i} \, . \tag{1.23}$$

With the above notation, we have the following local expression for Poisson brackets:

$$\{f, g\} = \sum_{i,j} \{x_i, x_j\} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j} = \sum_{i<j} \{x_i, x_j\} \frac{\partial(f, g)}{\partial(x_i, x_j)} \, . \tag{1.24}$$

Due to the Jacobi condition, not every 2-vector field will be a Poisson tensor.

Exercise 1.2.5. Show that the 2-vector field $\frac{\partial}{\partial x} \wedge (\frac{\partial}{\partial y} + x\frac{\partial}{\partial z})$ in $\mathbb{R}^3$ is *not* a Poisson tensor.

Exercise 1.2.6. Show that if $X_1, \dots, X_m$ are pairwise commuting vector fields and a_{ij} are constants, then $\sum_{ij} a_{ij} X_i \wedge X_j$ is a Poisson tensor.

To study the Jacobi identity, we will use the following lemma:

Lemma 1.2.7. *For any C^1-smooth 2-vector field Π, one can associate to it a 3-vector field Λ defined by*

$$\Lambda(f, g, h) = \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} \tag{1.25}$$

where $\{k, l\}$ denotes $\langle \Pi, \mathrm{d}k \wedge \mathrm{d}l \rangle$ (i.e., the bracket of Π).

Proof. It is clear that the right-hand side of Formula (1.25) is $\mathbb{R}$-multilinear and antisymmetric. To show that it corresponds to a 3-vector field, one has to verify that it satisfies the Leibniz rule with respect to f, i.e.,

$$\begin{aligned} &\{\{f_1 f_2, g\}, h\} + \{\{g, h\}, f_1 f_2\} + \{\{h, f_1 f_2\}, g\} \\ &\qquad = f_1(\{\{f_2, g\}, h\} + \{\{g, h\}, f_2\} + \{\{h, f_2\}, g\}) \\ &\qquad\qquad + f_2(\{\{f_1, g\}, h\} + \{\{g, h\}, f_1\} + \{\{h, f_1\}, g\}). \end{aligned}$$

This is a simple direct verification, based on the Leibniz rule $\{ab, c\} = a\{b, c\} + b\{a, c\}$ for the bracket of the 2-vector field Π. It will be left to the reader as an exercise. □

Direct calculations in local coordinates show that

$$\Lambda(f,g,h) = \sum_{ijk} \left(\oint_{ijk} \sum_s \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} \right) \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \frac{\partial h}{\partial x_k} \ , \tag{1.26}$$

where $\oint_{ijk} a_{ijk}$ means the cyclic sum $a_{ijk} + a_{jki} + a_{kij}$. In other words,

$$\Lambda = \sum_{i<j<k} \left(\oint_{ijk} \sum_s \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} \right) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} \wedge \frac{\partial}{\partial x_k} \ . \tag{1.27}$$

Clearly, the Jacobi identity for Π is equivalent to the condition that $\Lambda = 0$. Thus we have:

Proposition 1.2.8. *A 2-vector field* $\Pi = \sum_{i<j} \Pi_{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ *expressed in terms of a given system of coordinates* $(x_1, \ldots, x_n)$ *is a Poisson tensor if and only if it satisfies the following system of equations:*

$$\oint_{ijk} \sum_s \frac{\partial \Pi_{ij}}{\partial x_s} \Pi_{sk} = 0 \quad (\forall\ i,j,k) \ . \tag{1.28}$$

□

An obvious consequence of the above proposition is that the condition for a 2-vector field to be a Poisson structure is a local condition. In particular, the restriction of a Poisson structure to an open subset of the manifold is again a Poisson structure.

Example 1.2.9. Constant Poisson structures on $\mathbb{R}^n$: Choose arbitrary constants Π_{ij}. Then Equation (1.28) is obviously satisfied. The canonical Poisson structure on $\mathbb{R}^{2n}$, associated to the canonical symplectic form $\omega = \sum dq_i \wedge dp_i$, is of this type.

Example 1.2.10. Any 2-vector field on a two-dimensional manifold is a Poisson tensor. Indeed, the 3-vector field Λ in Lemma 1.2.7 is identically zero because there are no nontrivial 3-vectors on a two-dimensional manifold. Thus the Jacobi identity is nontrivial only starting from dimension 3.

Example 1.2.11. Let V be a finite-dimensional vector space over $\mathbb{R}$ (or $\mathbb{C}$). A *linear Poisson structure* on V is a Poisson structure on V for which the Poisson bracket of two linear functions is again a linear function. Equivalently, in linear coordinates, the components of the corresponding Poisson tensor are linear functions. In this case, by restriction to linear functions, the operation $(f,g) \mapsto \{f,g\}$ gives rise to an operation $[\ ,\ ] : V^* \times V^* \longrightarrow V^*$, which is a Lie algebra structure on V^*, where V^* is the dual linear space of V.

Conversely, any Lie algebra structure on V^* determines a linear Poisson structure on V. Indeed, consider a finite-dimensional Lie algebra $(\mathfrak{g}, [\ ,\ ])$. For each linear function $f : \mathfrak{g}^* \longrightarrow \mathbb{R}$ we denote by $\tilde{f}$ the element of $\mathfrak{g}$ corresponding to it. If f and g are two linear functions on $\mathfrak{g}^*$, then we put $\{f,g\}(\alpha) = \langle \alpha, [\tilde{f}, \tilde{g}] \rangle$ for every α in $\mathfrak{g}^*$. If we choose a linear basis $e_1, \ldots, e_n$ of $\mathfrak{g}$, with $[e_i, e_j] = \sum c_{ij}^k e_k$, then we have $\{x_i, x_j\} = \sum c_{ij}^k x_k$ where x_l is the function such that $\tilde{x}_l = e_l$. By taking $(x_1, \ldots, x_n)$ as a linear system of coordinates on $\mathfrak{g}^*$, it follows from the Jacobi identity for $[\ ,\ ]$ that the functions $\Pi_{ij} = \{x_i, x_j\}$ verify Equation (1.28). Thus we get a Poisson structure on $\mathfrak{g}^*$. This Poisson structure can be defined intrinsically by the following formula:

$$\{f,g\}(\alpha) = \langle \alpha, [\mathrm{d}f(\alpha), \mathrm{d}g(\alpha)] \rangle \ , \tag{1.29}$$

where $\mathrm{d}f(\alpha)$ and $\mathrm{d}g(\alpha)$ are considered as elements of $\mathfrak{g}$ via the identification $(\mathfrak{g}^*)^* = \mathfrak{g}$. Thus, there is a natural bijection between finite-dimensional linear Poisson structures and finite-dimensional Lie algebras. One can even try to study Lie algebras by viewing them as linear Poisson structures (see, e.g., [61]).

Remark 1.2.12. Multi-vector fields are also known as antisymmetric *contravariant tensors*, because their coefficients change contravariantly under a change of local coordinates. In particular, the local expression of a Poisson bracket will change contravariantly under a change of local coordinates: Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two local coordinate systems on the same open subset of a Poisson manifold $(M, \{,\})$. Viewing y_i as functions of $(x_1, \ldots, x_n)$, we have

$$\{y_i, y_j\} = \sum_{r<s} \frac{\partial(y_i, y_j)}{\partial(x_r, x_s)} \{x_r, x_s\} \ . \tag{1.30}$$

Denote $\Pi_{rs}(x) = \{x_r, x_s\}(x), \Pi'_{ij}(y) = \{y_i, y_j\}(y)$. Then the above equation can be rewritten as

$$\Pi'_{ij}(y(x)) = \sum_{r<s} \frac{\partial(y_i, y_j)}{\partial(x_r, x_s)}(x) \Pi_{rs}(x). \tag{1.31}$$

Exercise 1.2.13. Consider the Poisson structure on $\mathbb{R}^2$ defined by $\{x, y\} = e^x$. Show that in the new coordinates $(u, v) = (x, ye^{-x})$ the Poisson tensor will have the standard form $\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}$.

Exercise 1.2.14. Let $\Pi = \sum \Pi_{ij} \partial/\partial x_i \wedge \partial/\partial x_j$ be a constant Poisson structure on $\mathbb{R}^n$, i.e., the coefficients Π_{ij} are constants. Show that there is a number $p \geq 0$ and a linear coordinate system $(y_1, \ldots, y_n)$ in which the Poisson bracket has the form

$$\{f, g\} = \frac{\partial(f,g)}{\partial(y_1, y_2)} + \frac{\partial(f,g)}{\partial(y_3, y_4)} + \cdots + \frac{\partial(f,g)}{\partial(y_{2p-1}, y_{2p})} \ . \tag{1.32}$$

1.3 Poisson morphisms

Definition 1.3.1. If $(M_1, \{,\}_1)$ and $(M_2, \{,\}_2)$ are two smooth Poisson manifolds, then a smooth map ϕ from M_1 to M_2 is called a smooth *Poisson morphism* or *Poisson map* if the associated pull-back map $\phi^* : \mathcal{C}^\infty(M_2) \to \mathcal{C}^\infty(M_1)$ is a Lie algebra homomorphism with respect to the corresponding Poisson brackets.

In other words, $\phi : (M_1, \{,\}_1) \to (M_2, \{,\}_2)$ is a Poisson morphism if

$$\{\phi^* f, \phi^* g\}_1 = \phi^* \{f, g\}_2 \ \ \forall\ f, g \in \mathcal{C}^\infty(M_2)\ . \tag{1.33}$$

Of course, Poisson manifolds together with Poisson morphisms form a category: the composition of two Poisson morphisms is again a Poisson morphism, and so on. Notice that a Poisson morphism which is a diffeomorphism will automatically be a *Poisson isomorphism*: the inverse map is also a Poisson map.

Similarly, a map $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is called a *symplectic morphism* if $\phi^*\omega_2 = \omega_1$. Clearly, a symplectic isomorphism is also a Poisson isomorphism. However, a symplectic morphism is *not* a Poisson morphism in general. For example, if M_1 is a point with a trivial symplectic form, and M_2 is a symplectic manifold of positive dimension, then any map $\phi : M_1 \to M_2$ is a symplectic morphism but not a Poisson morphism.

Example 1.3.2. If $\phi : \mathfrak{h} \to \mathfrak{g}$ is a Lie algebra homomorphism, then the linear dual map $\phi^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is a Poisson map, where $\mathfrak{g}^*$ and $\mathfrak{h}^*$ are equipped with their respective linear Poisson structures. The proof of this fact will be left to the reader as an exercise. In particular, if $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then the canonical projection $\mathfrak{g}^* \to \mathfrak{h}^*$ is Poisson.

Example 1.3.3. If ϕ is a diffeomorphism of a manifold N, then it can be lifted naturally to a diffeomorphism $\phi_* : T^*N \to T^*N$ covering ϕ. By definition, ϕ_* preserves the Liouville 1-form θ (see Example 1.1.9), hence it preserves the symplectic form $\mathrm{d}\theta$. Thus, ϕ_* is a Poisson isomorphism.

Example 1.3.4. Direct product of Poisson manifolds. Let $(M_1, \{,\}_1)$ and $(M_2, \{,\}_2)$ be two Poisson manifolds. Then their direct product $M_1 \times M_2$ can be equipped with the following natural bracket:

$$\{f(x_1, x_2), g(x_1, x_2)\} = \{f_{x_2}, g_{x_2}\}_1 (x_1) + \{f_{x_1}, g_{x_1}\}_2 (x_2) \tag{1.34}$$

where we use the notation $h_{x_1}(x_2) = h_{x_2}(x_1) = h(x_1, x_2)$ for any function h on $M_1 \times M_2$, $x_1 \in M_1$ and $x_2 \in M_2$. Using Equation (1.28), one can verify easily that this bracket is indeed a Poisson bracket on $M_1 \times M_2$. It is called the *product Poisson structure*. With respect to this product Poisson structure, the projection maps $M_1 \times M_2 \to M_1$ and $M_1 \times M_2 \to M_2$ are Poisson maps.

Exercise 1.3.5. Let $M_1 = M_2 = \mathbb{R}^n$ with trivial Poisson structure. Find a nontrivial Poisson structure on $M_1 \times M_2 = \mathbb{R}^{2n}$ for which the two projections $M_1 \times M_2 \to M_1$ and $M_1 \times M_2 \to M_2$ are Poisson maps.

Exercise 1.3.6. Show that any Poisson map from a Poisson manifold to a symplectic manifold is a submersion.

A vector field X on a Poisson manifold (M, Π), is called a *Poisson vector field* if it is an *infinitesimal automorphism* of the Poisson structure, i.e., the Lie derivative of Π with respect to X vanishes:

$$\mathcal{L}_X \Pi = 0\ . \tag{1.35}$$

Equivalently, the local flow (φ_X^t) of X, i.e., the one-dimensional pseudo-group of local diffeomorphisms of M generated by X, preserves the Poisson structure: $\forall t \in \mathbb{R}$, (φ_X^t) is a Poisson morphism wherever it is well defined.

By the Leibniz rule we have $\mathcal{L}_X(\{f,g\}) = \mathcal{L}_X(\langle \Pi, \mathrm{d}f \wedge \mathrm{d}g\rangle) = \langle \mathcal{L}_X \Pi, \mathrm{d}f \wedge \mathrm{d}g\rangle + \langle \Pi, \mathrm{d}\mathcal{L}_X f \wedge \mathrm{d}g\rangle + \langle \Pi, \mathrm{d}f \wedge \mathrm{d}\mathcal{L}_X g\rangle = \langle \mathcal{L}_X \Pi, \mathrm{d}f \wedge \mathrm{d}g\rangle + \{X(f), g\} + \{f, X(g)\}$. So another equivalent condition for X to be a Poisson vector field is the following:

$$\{Xf, g\} + \{f, Xg\} = X\{f, g\} \ . \tag{1.36}$$

When $X = X_h$ is a Hamiltonian vector field, then Equation (1.36) is nothing but the Jacobi identity. Thus any Hamiltonian vector field is a Poisson vector field. The inverse is not true in general, even locally. For example, if the Poisson structure is trivial, then any vector field is a Poisson vector field, while the only Hamiltonian vector field is the trivial one.

Exercise 1.3.7. Show that on $\mathbb{R}^{2n}$ with the standard Poisson structure $\sum \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$ any Poisson vector field is also Hamiltonian.

Example 1.3.8. Infinitesimal version of Example 1.3.3. If X is a vector field on a manifold N, then X admits a unique natural lifting to a vector field $\hat{X}$ on T^*N which preserves the Liouville 1-form. In a local coordinate system $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ on T^*N, where $(q_1, \ldots, q_n)$ is a local coordinate system on N and the Liouville 1-form is $\theta = \sum_i p_i \mathrm{d}q_i$ (see Example 1.1.9), we have the following expression for $\hat{X}$:

$$\text{If } X = \sum_i \alpha_i(q) \frac{\partial}{\partial q_i} \text{ then } \hat{X} = \sum_i \alpha_i(q) \frac{\partial}{\partial q_i} - \sum_{i,j} \frac{\partial \alpha_i(q)}{\partial q_j} p_i \frac{\partial}{\partial p_j}.$$

The vector field $\hat{X}$ is in fact the Hamiltonian vector field of the function

$$\mathcal{X}(p_1, \ldots, p_n, q_1, \ldots, q_n) = \sum_i \alpha_i(q) p_i$$

on T^*N. This function $\mathcal{X}$ is nothing else than X itself, considered as a fiber-wise linear function on T^*N.

Example 1.3.9. Let G be a connected Lie group, and denote by $\mathfrak{g}$ the Lie algebra of G. By definition, $\mathfrak{g}$ is isomorphic to the Lie algebra of left-invariant tangent vector fields of G (i.e., vector fields which are invariant under left translations $L_g : h \mapsto gh$ on G). Denote by e the neutral element of G. For each $X_e \in T_eG$, there is a unique left-invariant vector field X on G whose value at e is X_e (X obtained from X_e by left translations), so we may identify T_eG with $\mathfrak{g}$ via this association $X_e \mapsto X$. We will write $T_eG = \mathfrak{g}$, and $T_e^*G = \mathfrak{g}^*$ by duality. Consider the left translation map

$$L : T^*G \to \mathfrak{g}^* = T_e^*G, \quad L(p) = (L_g)^* p = L_{g^{-1}} p \ \ \forall\, p \in T_g^*G, \tag{1.37}$$

where $L_{g^{-1}} p$ means the push-forward $(L_{g^{-1}})_* p$ of p by $L_{g^{-1}}$ (we will often omit the subscript asterisk when writing push-forwards to simplify the notation).

Theorem 1.3.10. *The above left translation map $L : T^*G \to \mathfrak{g}^*$ is a Poisson map, where T^*G is equipped with the standard symplectic structure, and $\mathfrak{g}^*$ is equipped with the standard linear Poisson structure (induced from the Lie algebra structure of $\mathfrak{g}$).*

Proof (sketch). It is enough to verify that, if x, y are two elements of $\mathfrak{g}$, considered as linear functions on $\mathfrak{g}^*$, then we have

$$\{L^*x, L^*y\} = L^*([x, y]).$$

Notice that L^*x is nothing else than x itself, considered as a left-invariant vector field on G and then as a left-invariant fiber-wise linear function on T^*G. By the formulas given in Example 1.3.8, the Hamiltonian vector field X_{L^*x} of L^*x is the natural lifting to T^*G of x, considered as a left-invariant vector field on G. Since the process of lifting of vector fields from N to T^*N preserves the Lie bracket for any manifold N, we have

$$[X_{L^*x}, X_{L^*y}] = X_{L^*[x,y]}.$$

It follows from the above equation and Lemma 1.1.11 that $\{L^*x, L^*y\}$ and $L^*([x, y])$ have the same Hamiltonian vector field on T^*G. Hence these two functions differ by a function which vanishes on the zero section of T^*G and whose Hamiltonian vector field is trivial on T^*G. The only such function is 0, so $\{L^*x, L^*y\} = L^*([x, y])$. □

Exercise 1.3.11. Show that the right translation map $R : T^*G \to \mathfrak{g}^* = T^*_eG$, defined by $L(p) = (R_g)_*p \; \forall \, p \in T^*_gG$, is an anti-Poisson map. A map $\phi : (M, \Pi) \to (N, \Lambda)$ is called an *anti-Poisson map* if $\phi : (M, \Pi) \to (N, -\Lambda)$ is a Poisson map.

Given a subspace $V \in T_xM$ of a tangent space T_xM of a symplectic manifold (M, ω), we will denote by $V^\perp$ the *symplectic orthogonal* to V: $V^\perp = \{X \in T_xM \mid \omega(X, Y) = 0 \; \forall \; Y \in V\}$. Clearly, $V = (V^\perp)^\perp$. V is called *Lagrangian* (resp. *isotropic*, *coisotropic*, *symplectic*) if $V = V^\perp$ (resp. $V \subset V^\perp$, $V \supset V^\perp$, $V \cap V^\perp = 0$). A submanifold of a symplectic manifold is called *Lagrangian* (resp. *isotropic*, *coisotropic*, resp. *symplectic*) if its tangent spaces are so. Lagrangian submanifolds play a central role in symplectic geometry, see, e.g., [345, 243]. In particular, we have the following characterization of symplectic isomorphisms in terms of Lagrangian submanifolds:

Proposition 1.3.12. *A diffeomorphism $\phi : (M, \omega_1) \to (M_2, \omega_2)$ is a symplectic isomorphism if and only if its graph $\Delta = \{(x, \phi(x))\} \subset M_1 \times \overline{M_2}$ is a Lagrangian manifold of $M_1 \times \overline{M_2}$, where $\overline{M_2}$ means M_2 together with the opposite symplectic form $-\omega_2$.*

The proof is almost obvious and is left as an exercise. □

A subspace $V \subset T_xM$ of a Poisson manifold (M, Π) is called *coisotropic* if for any $\alpha, \beta \in T^*_xM$ such that $\langle \alpha, X\rangle = \langle \beta, X\rangle = 0 \; \forall \; X \in V$ we have $\langle \Pi, \alpha \wedge \beta\rangle = 0$.

In other words, $V^\circ \subset (V^\circ)^\perp$, where $V^\circ = \{\alpha \in T^*_xM \mid \langle \alpha, X\rangle = 0 \;\forall\; X \in V\}$ is the annulator of V and $(V^\circ)^\perp = \{\beta \in T^*_xM \mid \langle \Pi, \alpha\wedge\beta\rangle = 0 \;\forall\; \alpha \in V^\circ\}$ is the "Poisson orthogonal" of V°. A submanifold N of a Poisson manifold is called *coisotropic* if its tangent spaces are coisotropic.

Proposition 1.3.13. *A map $\phi : (M_1, \Pi_1) \to (M_2, \Pi_2)$ between two Poisson manifolds is a Poisson map if and only if its graph $\Gamma(\phi) := \{(x, y) \in M_1 \times M_2;\; y = \phi(x)\}$ is a coisotropic submanifold of $(M_1, \Pi_1) \times (M_2, -\Pi_2)$.*

Again, the proof will be left as an exercise. □

1.4 Local canonical coordinates

In this section, we will prove the *splitting theorem* of Alan Weinstein [346], which says that locally a Poisson manifold is a direct product of a symplectic manifold with another Poisson manifold whose Poisson tensor vanishes at a point. This splitting theorem, together with the Darboux theorem which will be proved at the same time, will give us local canonical coordinates for Poisson manifolds.

Given a Poisson structure Π (or more generally, an arbitrary 2-vector field) on a manifold M, we can associate to it a natural homomorphism

$$\sharp = \sharp_\Pi : T^*M \longrightarrow TM, \tag{1.38}$$

which maps each covector $\alpha \in T^*_xM$ over a point x to a unique vector $\sharp(\alpha) \in T_xM$ such that

$$\langle \alpha \wedge \beta, \Pi\rangle = \langle \beta, \sharp(\alpha)\rangle \tag{1.39}$$

for any covector $\beta \in T^*_xM$. We will call $\sharp = \sharp_\Pi$ the *anchor map* of Π.

The same notations $\sharp$ (or $\sharp_\Pi$) will be used to denote the operator which associates to each differential 1-form α the vector field $\sharp(\alpha)$ defined by $(\sharp(\alpha))(x) = \sharp(\alpha(x))$. For example, if f is a function, then $\sharp(\mathrm{d}f) = X_f$ is the Hamiltonian vector field of f.

The restriction of $\sharp_\Pi$ to a cotangent space T^*_xM will be denoted by $\sharp_x$ or $\sharp_{\Pi(x)}$. In a local system of coordinates $(x_1, \dots, x_n)$ we have

$$\sharp\Big(\sum_{i=1}^n a_i dx_i\Big) = \sum_{ij} \{x_i, x_j\}\, a_i \frac{\partial}{\partial x_j} = \sum_{ij} \Pi_{ij} a_i \frac{\partial}{\partial x_j}.$$

Thus $\sharp_x$ is a linear operator, given by the matrix $[\Pi_{ij}(x)]$ in the linear bases $(dx_1, \dots, dx_n)$ and $\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$.

Definition 1.4.1. Let (M, Π) be a Poisson manifold and x a point of M. Then the image $\mathcal{C}_x := \mathrm{Im}\sharp_x$ of $\sharp_x$ is called the *characteristic space* at x of the Poisson structure Π. The dimension $\dim \mathcal{C}_x$ of $\mathcal{C}_x$ is called the *rank* of Π at x, and $\max_{x\in M} \dim \mathcal{C}_x$ is called the *rank* of Π. When $\operatorname{rank} \Pi_x = \dim M$ we say that Π

is *nondegenerate* at x. If $\operatorname{rank}\Pi_x$ is a constant on M, i.e., does not depend on x, then Π is called a *regular Poisson structure*.

Example 1.4.2. The constant Poisson structure $\sum_{i=1}^{s} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_{i+s}}$ on $\mathbb{R}^m$ $(m \geq 2s)$ is a regular Poisson structure of rank $2s$.

Exercise 1.4.3. Show that $\operatorname{rank}\Pi_x$ is always an even number, and that Π is nondegenerate everywhere if and only if it is the associated Poisson structure of a symplectic structure.

The characteristic space $\mathcal{C}_x$ admits a unique natural antisymmetric nondegenerate bilinear scalar product, called the *induced symplectic form*: if X and Y are two vectors of $\mathcal{C}_x$, then we put

$$(X,Y) := \langle \beta, X\rangle = \langle \Pi, \alpha \wedge \beta\rangle = -\langle \Pi, \beta \wedge \alpha\rangle = -\langle \alpha, Y\rangle = -(Y,X) \tag{1.40}$$

where $\alpha, \beta \in T^*_x M$ are two covectors such that $X = \sharp\alpha$ and $Y = \sharp\beta$.

Exercise 1.4.4. Verify that the above scalar product is anti-symmetric nondegenerate and is well defined (i.e., does not depend on the choice of α and β). When Π is nondegenerate then the above formula defines the corresponding symplectic structure on M.

Theorem 1.4.5 (Splitting theorem [346]). *Let x be a point of rank $2s$ of a Poisson m-dimensional manifold (M, Π): $\dim \mathcal{C}_x = 2s$ where $\mathcal{C}_x$ is the characteristic space at x. Let N be an arbitrary $(m-2s)$-dimensional submanifold of M which contains x and is transversal to $\mathcal{C}_x$ at x. Then there is a local system of coordinates $(p_1, \ldots, p_s, q_1, \ldots, q_s, z_1, \ldots, z_{m-2s})$ in a neighborhood of x, which satisfy the following conditions:*

a) $p_i(N_x) = q_i(N_x) = 0$ *where N_x is a small neighborhood of x in N.*
b) $\{q_i, q_j\} = \{p_i, p_j\} = 0\ \forall\, i,j$; $\{p_i, q_j\} = 0$ *if* $i \neq j$ *and* $\{p_i, q_i\} = 1\ \forall\, i$.
c) $\{z_i, p_j\} = \{z_i, q_j\} = 0\ \forall\, i,j$.
d) $\{z_i, z_j\}(x) = 0\ \forall\, i,j$.

A local coordinate system which satisfies the conditions of the above theorem is called a system of local *canonical coordinates*. In such canonical coordinates we have

$$\{f,g\} = \sum_{i,j} \{z_i, z_j\} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} + \sum_{i=1}^{s} \frac{\partial(f,g)}{\partial(p_i, q_i)} = \{f,g\}_N + \{f,g\}_S\ , \tag{1.41}$$

where

$$\{f,g\}_S = \sum_{i=1}^{s} \frac{\partial(f,g)}{\partial(p_i, q_i)} \tag{1.42}$$

defines the nondegenerate Poisson structure $\sum \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i}$ on the local submanifold $S = \{z_1 = \cdots = z_{m-2s} = 0\}$, and

$$\{f,g\}_N = \sum_{u,v} \{z_i, z_j\} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} \tag{1.43}$$

defines a Poisson structure on a neighborhood of x in N. Notice that, since $\{z_i, p_j\} = \{z_i, q_j\} = 0 \ \forall \ i, j$, the functions $\{z_i, z_j\}$ do not depend on the variables $(p_1, \dots, p_s, q_1, \dots, q_s)$. The equality $\{z_i, z_j\}(x) = 0 \ \forall \ i, j$ means that the Poisson tensor of $\{,\}_N$ vanishes at x.

Formula (1.41) means that the Poisson manifold (M, Π) is locally isomorphic (in a neighborhood of x) to the direct product of a symplectic manifold $(S, \sum_1^s \mathrm{d}p_i \wedge \mathrm{d}q_i)$ with a Poisson manifold $(N_x, \{,\}_N)$ whose Poisson tensor vanishes at x. That's why Theorem 1.4.5 is called the splitting theorem for Poisson manifolds: locally, we can split a Poisson structure in two parts – a regular part and a singular part which vanishes at a point.

Proof of Theorem 1.4.5. If $\Pi(x) = 0$ then $s = 0$ and there is nothing to prove. Suppose that $\Pi(x) \neq 0$. Let p_1 be a local function (defined in a small neighborhood of x in M) which vanishes on N and such that $\mathrm{d}p_1(x) \neq 0$. Since $\mathcal{C}_x$ is transversal to N, there is a vector $X_g(x) \in \mathcal{C}_x$ such that $\langle X_g(x), \mathrm{d}p_1(x)\rangle \neq 0$, or equivalently, $X_{p_1}(g)(x) \neq 0$, where X_{p_1} denotes the Hamiltonian vector field of p_1 as usual. Therefore $X_{p_1}(x) \neq 0$. Since $\mathcal{C}_x \ni \sharp(\mathrm{d}p_1)(x) = X_{p_1}(x) \neq 0$ and is not tangent to N, there is a local function q_1 such that $q_1(N) = 0$ and $X_{p_1}(q_1) = 1$ in a neighborhood of x, or

$$X_{p_1} q_1 = \{p_1, q_1\} = 1 \ . \tag{1.44}$$

Moreover, X_{p_1} and X_{q_1} are linearly independent ($X_{q_1} = \lambda X_{p_1}$ would imply that $\{p_1, q_1\} = -\lambda X_{p_1}(p_1) = 0$), and we have

$$[X_{p_1}, X_{q_1}] = X_{\{p_1, q_1\}} = 0 \ . \tag{1.45}$$

Thus X_{p_1} and X_{q_1} are two linearly independent vector fields which commute. Hence they generate a locally free infinitesimal $\mathbb{R}^2$-action in a neighborhood of x, which gives rise to a local regular two-dimensional foliation. As a consequence, we can find a local system of coordinates $(y_1, \dots, y_m)$ such that

$$X_{q_1} = \frac{\partial}{\partial y_1}, \qquad X_{p_1} = \frac{\partial}{\partial y_2} \ . \tag{1.46}$$

With these coordinates we have $\{q_1, y_i\} = X_{q_1}(y_i) = 0$ and $\{p_1, y_i\} = X_{p_1}(y_i) = 0$, for $i = 3, \dots, m$. Poisson's Theorem 1.1.10 then implies that $\{p_1, \{y_i, y_j\}\} = \{q_1, \{y_i, y_j\}\} = 0$ for $i, j \geq 3$, whence

$$\begin{aligned} &\{y_i, y_j\} = \varphi_{ij}(y_3, \dots, y_n) \quad \forall \ i, j \geq 3 \ , \\ &\{p_1, q_1\} = 1 \ , \\ &\{p_1, y_j\} = \{q_1, y_j\} = 0 \quad \forall \ j \geq 3 \ . \end{aligned} \tag{1.47}$$

We can take $(p_1, q_1, y_3, \dots, y_n)$ as a new local system of coordinates. In fact, the Jacobian matrix of the map $\varphi : (y_1, y_2, y_3, \dots, y_m) \mapsto (p_1, q_1, y_3, \dots, y_m)$ is of the form

$$\begin{pmatrix} 0 & 1 & \\ -1 & 0 & * \\ & 0 & \mathrm{Id} \end{pmatrix} \tag{1.48}$$

(because $\frac{\partial q_1}{\partial y_1} = X_{q_1} q_1 = 0$, $\frac{\partial q_1}{\partial y_2} = X_{p_1} q_1 = \{q_1, p_1\} = 1, \ldots$), which has a non-zero determinant (equal to 1). In the coordinates $(q_1, p_1, y_3, \ldots, y_m)$, we have

$$\Pi = \frac{\partial}{\partial p_1} \wedge \frac{\partial}{\partial q_1} + \frac{1}{2} \sum_{i,j \geq 3} \Pi'_{ij}(y_3, \ldots, y_m) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j} . \tag{1.49}$$

The above formula implies that our Poisson structure is locally the product of a standard symplectic structure on a plane $\{(p_1, q_1)\}$ with a Poisson structure on a $(m-2)$-dimensional manifold $\{(y_3, \ldots, y_m)\}$. In this product, N is also the direct product of a point (= the origin) of the plane $\{(p_1, q_1)\}$ with a local submanifold in the Poisson manifold $\{(y_3, \ldots, y_m)\}$. The splitting theorem now follows by induction on the rank of Π at x. □

Remark 1.4.6. In the above theorem, when $m = 2s$, we recover Darboux's theorem which gives local canonical coordinates for symplectic manifolds. If (M, Π) is a regular Poisson structure, then the Poisson structure of N_x in the above theorem must be trivial, and we get the following generalization of Darboux's theorem: any regular Poisson structure is locally isomorphic to a standard constant Poisson structure.

Exercise 1.4.7. Prove the following generalization of Theorem 1.4.5. Let N be a submanifold of a Poisson manifold (M, Π), and x be a point of N such that $T_xN + \mathcal{C}_x = T_xM$ and $T_xN \cap \mathcal{C}_x$ is a symplectic subspace of $\mathcal{C}_x$, i.e., the restriction of the symplectic form on the characteristic space $\mathcal{C}_x$ to $T_zN \cap \mathcal{C}_x$ is nondegenerate. (Such a submanifold N is sometimes called *cosymplectic*.) Then there is a coordinate system in a neighborhood of x which satisfies the conditions a), b), c) of Theorem 1.4.5, where $2s = \dim M - \dim N = \dim C_x - \dim(T_xN \cap \mathcal{C}_x)$.

1.5 Singular symplectic foliations

A smooth *singular foliation* in the sense of Stefan–Sussmann [320, 327] on a smooth manifold M is by definition a partition $\mathcal{F} = \{\mathcal{F}_\alpha\}$ of M into a disjoint union of smooth immersed connected submanifolds $\mathcal{F}_\alpha$, called *leaves*, which satisfies the following *local foliation property* at each point $x \in M$: Denote the leaf that contains x by $\mathcal{F}_x$, the dimension of $\mathcal{F}_x$ by d and the dimension of M by m. Then there is a smooth local chart of M with coordinates $y_1, \ldots, y_m$ in a neighborhood U of x, $U = \{-\varepsilon < y_1 < \varepsilon, \ldots, -\varepsilon < y_m < \varepsilon\}$, such that the d-dimensional disk $\{y_{d+1} = \cdots = y_m = 0\}$ coincides with the path-connected component of the intersection of $\mathcal{F}_x$ with U which contains x, and each d-dimensional disk $\{y_{d+1} = c_{d+1}, \ldots, y_m = c_m\}$, where $c_{d+1}, \ldots, c_m$ are constants, is wholly contained in some leaf $\mathcal{F}_\alpha$ of $\mathcal{F}$. If all the leaves $\mathcal{F}_\alpha$ of a singular foliation $\mathcal{F}$ have the same dimension, then one says that $\mathcal{F}$ is a *regular foliation*.

A *singular distribution* on a manifold M is the assignment to each point x of M a vector subspace D_x of the tangent space T_xM. The dimension of D_x