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Foreword

The Romanian conferences in operator theory, as they are now commonly called, have started in the year 1976 as an annual workshop on operator theory held at the University of Timişoara, originally only with Romanian attendance. The meeting soon evolved into an international conference, with an increasingly larger participation. It has been organized jointly, initially by the Department of Mathematics of INCREST and by the Faculty of Sciences of the University of Timişoara, then (since 1990) by the Institute of Mathematics of the Romanian Academy and the Faculty of Mathematics of the West University of Timişoara. The venue was usually Timişoara (and, occasionally, Herculane, Bucharest or Predeal). Since 1986 the conference has been regularly held biannually at the beginning of the summer.

The *19th Conference on Operator Theory* (OT 19) took place between June 27th and July 2nd 2002, at the West University of Timişoara. It is a pleasure to acknowledge the considerable financial support received through the programme EURROMMAT of the European Community, under contract ICA1-CT-2000-70022. Partial support has also been provided by the Romanian Ministry of Education, Research and Youth, grants CERES 152/2001 and 153/2001.

The full programme of the conference is included in the sequel. It is worth mentioning also a special event that has taken place during the conference: professor Israel Gohberg has been awarded the title of Doctor Honoris Causa of the West University of Timişoara.

This volume is a careful selection of papers authored by participants at the 19th Conference on Operator Theory. Traditionally, these conferences are open to a broad range of contributions from operator theory, operator algebras and their applications. This feature is also shared by the proceedings volume, covering a large variety of topics, such as single operator theory, C^*-algebras, differential operators, integral transforms, stochastic processes and operators, quantum systems, special classes of operators, holomorphic operator functions, interpolation problems, and system theory.

Last but not least, special thanks are due to Barbara Ionescu, of the Theta Foundation, for her excellent work in preparing the final version of the manuscripts.

The Editors

Programme

Thursday, June 27

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10:30–11:30	F.-H. Vasilescu	*Existence of unitary dilations as a moment problem*
11:40–12:30	G. Weiss	*Traces, ideals and arithmetic means*
		Afternoon Session
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17:20–17:50	I. Chalendar	*Overcompleteness of sequences of reproducing kernels in the model space* K_θ
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10:50–11:30	G.F. Popescu	*Multivariable Nehari problem and interpolation*

Aula Magna

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List of Participants

H. Akça, *Dhahran*
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On Existence of Solutions of Semilinear Impulsive Functional Differential Equations with Nonlocal Conditions

Haydar Akça, Valéry Covachev and Eada Al-Zahrani

Abstract. The existence, uniqueness and continuous dependence of a mild solution of a semilinear impulsive functional-differential evolution nonlocal Cauchy problem in general Banach spaces are studied. Methods of fixed point theorems, of a C_0 semigroup of operators and the Banach contraction theorem are applied.

Mathematics Subject Classification (2000). 34A37, 34G20, 34K30, 34K99.

Keywords. Semilinear, Impulsive, Functional-differential equations, Nonlocal conditions, Mild solution.

1. Introduction

In this paper we study the existence, uniqueness and continuous dependence of a mild solution of a nonlocal Cauchy problem for a semilinear impulsive functional-differential evolution equation. Such problems arise in some physical applications as a natural generalization of the classical initial value problems. The results for a semilinear functional-differential evolution nonlocal problem ([2], [3], [4]) are extended for the case of impulse effect.

We consider a nonlocal Cauchy problem in the form:

$$\begin{cases} \dot{u}(t) + Au(t) = f(t, u(t), u(b_1(t)), \dots, u(b_m(t))), & t \in (t_0, t_0 + a],\, t \neq \tau_k, \\ u(\tau_k + 0) = Q_k u(\tau_k) \equiv u(\tau_k) + I_k u(\tau_k), & k = 1, 2, \dots, \kappa, \\ u(t_0) = u_0 - g(u), & \end{cases} \tag{1.1}$$

where $t_0 \geq 0,\ a > 0$ and $-A$ is the infinitesimal generator of a compact C_0 semigroup of operators on a Banach space E. I_k $(k = 1, 2, \dots, \kappa)$ are linear operators acting in the Banach space E. The functions $f,\ g,\ b_i$ $(i = 1, 2, \dots, m)$ are given functions satisfying some assumptions and u_0 is an element of the Banach space E. $I_k u(\tau_k) = u(\tau_k + 0) - u(\tau_k - 0)$ and the impulsive moments τ_k are such that $t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots < \tau_\kappa < t_0 + a,\ \kappa \in \mathbb{N}$.

Theorems about the existence, uniqueness and stability of solutions of differential and functional-differential abstract evolution Cauchy problems were studied in [2], [3], and [4]. The results presented in this paper are a generalization and a continuation of some results reported in publication [1]. We consider a classical semilinear impulsive functional-differential equation in the case of a nonlocal condition, reduced to the classical impulsive initial functional value problem. The nonlinearity f in problem (1.1) is of a more general type (involves more than one delay which may be variable) than the respective function in [1]. Also, in the present paper the *existence* of a mild solution (Theorem 2.1) is proved under less restrictive conditions using Schauder's fixed point theorem. The Lipschitz conditions are introduced later to prove the existence and *uniqueness* of the classical solution of problem (1.1), and the continuous dependence of the mild solution of problem (1.1) on the initial condition.

As usual in the theory of impulsive differential equations, at the points of discontinuity τ_i of the solution $t \mapsto u(t)$ we assume that $u(\tau_i) \equiv u(\tau_i - 0)$. It is clear that, in general, the derivatives $\dot{u}(\tau_i)$ do not exist. On the other hand, according to the first equality of (1.1) there exist the limits $\dot{u}(\tau_i \mp 0)$. According to the above convention, we assume $\dot{u}(\tau_i) \equiv \dot{u}(\tau_i - 0)$.

Throughout the paper we assume that E is a Banach space with norm $\|\cdot\|$, $-A$ is the infinitesimal generator of a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on E, $D(A)$ is the domain of A. A C_0 semigroup $\{T(t)\}_{t\geq 0}$ is said to be a compact C_0 semigroup of operators on E if $T(t)$ is a compact operator for every $t > 0$. We denote $I := [t_0, t_0+a]$, $M := \sup\limits_{t\in[0,a]} \left\{\|T(t)\|_{BL(E,E)}\right\}$ and X is the space of piecewise continuous functions $I \to E$ with discontinuities of the first kind at $\tau_1, \tau_2, \ldots, \tau_\kappa$. Let $f : I \times E^{m+1} \to E,\ g : X \to E$ (for instance, we can have $g(u) = \tilde{g}(u(t_1), u(t_2), \ldots, u(t_p))$, where $\tilde{g} : E^p \to E,\ t_0 < t_1 < t_2 < \cdots < t_p < t_0 + a,\ p \in \mathbb{N}$), $b_i : I \to I$ $(i = 1, 2, \ldots, m)$ and $u_0 \in E$. In the sequel, the operator norm $\|\cdot\|_{BL(E,E)}$ will be denoted by $\|\cdot\|$. We need the following sets:

$$E_\rho := \{z \in E,\ \|z\| \leq \rho\} \quad \text{and} \quad X_\rho := \{w \in X,\ \|w\|_X \leq \rho\}, \quad \rho > 0.$$

Introduce the following assumptions:

A1: $f \in C(I \times E^{m+1}, E)$, $g \in C(X, E)$ and $b_i \in C(I, I)$, $i = 1, 2, \ldots, m$ and there are constants $C_i > 0$, $i = 1, 2, 3$, such that

$$\begin{cases} \|f(s, z_0, z_1, \ldots, z_m)\| \leq C_1 \text{ for } s\in I,\ z_i\in E_r,\ i = 0, 1, \ldots, m, \\ \|g(w)\| \leq C_2 \text{ and } \max\limits_{k=1,2,\ldots,\kappa} \|I_k w\| \leq C_3 \text{ for } w \in X_r, \end{cases} \tag{1.2}$$

where $r := M(aC_1 + \|u_0\| + C_2 + \kappa C_3)$.

A2: $g(\lambda w_1 + (1-\lambda)w_2) = \lambda g(w_1) + (1-\lambda)g(w_2)$ for $w_i \in X_r$, $i = 1, 2$, and $\lambda \in (0, 1)$ and r is given by (1.2).

A3: The set

$$\{w(t_0) = u_0 - g(w) : w \in X_r\},$$

where r is given by (1.2), is precompact in E.

Example 1.1. Consider the scalar problem

$$\begin{cases} \dot{u}(t) + Au(t) = \frac{1}{4a}\Big(u^2(t) + \sum\limits_{i=1}^{m} 2^{-i}u^2\big(b_i(t)\big)\Big), & t \in (t_0, t_0 + a],\ t \neq \tau_k, \\ u(\tau_k + 0) = (1 + c_k)u(\tau_k), & k = 1, 2, \ldots, \kappa, \\ u(t_0) = \frac{1}{6} - \frac{1}{2}\sum\limits_{j=1}^{p} 2^{-j}u(t_j), \end{cases}$$

where $E = \mathbb{R}$, $A > 0$, the constants c_k satisfy $|c_k| \le \frac{1}{4\kappa}$, $k = 1, 2, \ldots, \kappa$.

In this case $T(t) = e^{-At}$, $M = 1$ and it is easy to see that condition **A1** is satisfied with $r = 1$. Assumptions **A2** and **A3** are obviously satisfied, thus Theorem 2.1 can be applied to this problem.

Consider the initial value problem (see [3])

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), & t \in (t_0, t_0 + a], \\ u(t_0) = x, \end{cases} \tag{1.3}$$

where $f : I \to E$, $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \ge 0$, and $x \in E$.

Definition 1.2. A function u is said to be a **strong solution** of problem (1.3) on I if u is differentiable almost everywhere on I, so that $(du/dt) \in L^1((t_0, t_0 + a); E)$, $u(t_0) = x$ and $\dot{u}(t) + Au(t) = f(t)$ a.e. on I.

The unique strong solution u on I is given by the formula

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)f(s)\, ds, \quad t \in I. \tag{1.4}$$

Definition 1.3. A function $u : I \to E$ is said to be a **classical solution** of the problem (1.3) on I if u is continuous on I and continuously differentiable on $(t_0, t_0 + a]$, such that $u(t) \in D(A)$ for $t_0 < t \le t_0 + a$ and the problem (1.3) is satisfied on I.

If E is a Banach space and $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \ge 0$, $f : I \to E$ is continuous on I and $x \in D(A)$, then the problem (1.3) has a classical solution u on I given by (1.4).

Next consider the initial value problem for the impulsive linear system

$$\begin{cases} \dot{u}(t) + Au(t) = f(t), & t \in (t_0, t_0 + a],\ t \neq \tau_k, \\ u(\tau_k + 0) = u(\tau_k) + I_k u(\tau_k), & k = 1, 2, \ldots, \kappa, \\ u(t_0) = x, \end{cases} \tag{1.5}$$

where A, f and x are as in problem (1.3), and τ_k and I_k are as in problem (1.1).

Definition 1.4. A function $u : I \to E$ is said to be a **classical solution** of the problem (1.5) on I if u is piecewise continuous on I with discontinuities of the first kind at $\tau_1, \tau_2, \ldots, \tau_\kappa$ and continuously differentiable on $(t_0, t_0 + a] \setminus \{\tau_k\}_{k=1}^{\kappa}$, such that $u(t) \in D(A)$ for $t_0 < t \le t_0 + a$ and the problem (1.5) is satisfied on I.

If A, f and x are as above and $I_k : D(A) \to D(A)$, then the problem (1.5) has a classical solution u on I given by the formula

$$u(t) = T(t - t_0)x + \int_{t_0}^{t} T(t - s)f(s)\,ds + \sum_{t_0 \le \tau_k < t} T(t - \tau_k)I_k u(\tau_k). \tag{1.6}$$

Formula (1.6) motivates us to give the following definition.

Definition 1.5. A function $u \in X$ satisfying the following integro-summary equation

$$\begin{aligned} u(t) = {} & T(t - t_0)u_0 - T(t - t_0)g(u) \\ & + \int_{t_0}^{t} T(t - s)f(s, u(s), u(b_1(s)), \ldots, u(b_m(s)))\,ds \\ & + \sum_{t_0 \le \tau_k < t} T(t - \tau_k)I_k u(\tau_k), \qquad t \in [t_0, t_0 + a] \end{aligned}$$

is said to be a **mild solution** of the nonlocal Cauchy problem (1.1).

2. Existence and uniqueness theorems

Theorem 2.1. *Suppose that assumptions* **A1**–**A3** *are satisfied, then the impulsive nonlocal Cauchy problem* (1.1) *has a mild solution.*

Proof. The mild solution of the impulsive system (1.1) with nonlocal condition satisfies the operator equation

$$u(t) = (Fu)(t),$$

where

$$\begin{aligned} (Fw)(t) := {} & T(t - t_0)u_0 - T(t - t_0)g(w) \\ & + \int_{t_0}^{t} T(t - s)f(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\,ds \\ & + \sum_{t_0 \le \tau_k < t} T(t - \tau_k)I_k w(\tau_k), \qquad t \in [t_0, t_0 + a], \end{aligned} \tag{2.1}$$

so that

$$\|(Fw)(t)\| \le M\|u_0\| + MC_2 + aMC_1 + \kappa MC_3 = r, \tag{2.2}$$

where the impulsive moments τ_k are such that $t_0 < \tau_1 < \tau_2 < \cdots < \tau_k < \cdots < \tau_\kappa < t_0 + a$, $\kappa \in \mathbb{N}$.

Let $Y := \{w \in X_r : u_0 = w(t_0) + g(w)\}$. According to assumption **A2**, Y is a convex subset of X_r. We have $F : Y \to Y$. Moreover, **A1** implies that $F \in C(Y, Y)$. Now we will show that $F(Y) = \{F(w)(t) : w \in Y,\, t \in I\}$ is precompact in E.

Observe that

$$Y(t_0) = \{(Fw)(t_0) : w \in Y,\, t \in I\} = \{u_0 - g(w) : w \in X_r\} = \{w(t_0) : w \in X_r\}\,.$$

Thus according to the assumption **A3**, $Y(t_0)$ is precompact in E. Let $t > t_0$ be fixed. For an arbitrary $\varepsilon \in (t_0, t)$, define a mapping F_ε on Y by the expression

$$\begin{aligned}(F_\varepsilon w)(t) := {} & T(t-t_0)u_0 - T(t-t_0)g(w) \\ & + \int_{t_0}^{t-\varepsilon} T(t-s) f(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\, ds \\ & + \sum_{t_0 \le \tau_k < t-\varepsilon} T(t-\tau_k) I_k w(\tau_k) = T(t-t_0)u_0 - T(t-t_0)g(w) \\ & + T(\varepsilon) \int_{t_0}^{t-\varepsilon} T(t-s-\varepsilon) f(s, w(s) w(b_1(s)), \ldots, w(b_m(s)))\, ds \\ & + T(\varepsilon) \sum_{t_0 \le \tau_k < t-\varepsilon} T(t-\varepsilon-\tau_k) I_k w(\tau_k).\end{aligned} \tag{2.3}$$

Since $T(t)$ is compact for every $t > t_0$, then the set

$$Y_\varepsilon := \{(F_\varepsilon w)(t) : w \in Y\}$$

is precompact in E for every $\varepsilon \in (t_0, t)$. Moreover, from the formulae (1.2), (2.1) and (2.3) we have

$$\begin{aligned}&\|(Fw)(t) - (F_\varepsilon w)(t)\| \\ &\quad \le \left\| \int_{t-\varepsilon}^{t} T(t-s) f(s, w(s), w(b_1(s)), \ldots, w(b_m(s)))\, ds \right\| \\ &\quad\quad + \left\| \sum_{t-\varepsilon \le \tau_k < t} T(t-\tau_k) I_k w(\tau_k) \right\| \le \varepsilon M C_1 + M C_2 i(t-\varepsilon, t),\end{aligned} \tag{2.4}$$

where $i(t-\varepsilon, t)$ is the number of impulses on the interval $(t-\varepsilon, t)$ and $i(t-\varepsilon, t) \to 0$ as $\varepsilon \to 0$, so $\|(Fw)(t) - (F_\varepsilon w)(t)\| \to 0$ as $\varepsilon \to 0$, and consequently the set $F(Y)$ is a uniformly bounded on each interval of continuity family of functions. From formulae (1.2) and (2.1) we observe that

$$
\begin{aligned}
&\|(Fw)(t_1) - (Fw)(t_2)\| \\
&\quad \le \|(T(t_1 - t_0) - T(t_2 - t_0))u_0\| + \|(T(t_1 - t_0) - T(t_2 - t_0))g(w)\| \\
&\qquad + \left\| \int_{t_0}^{t_1} (T(t_1 - s) - T(t_2 - s)) f(s, w(s), w(b_1(s)), \dots, w(b_m(s)))\, ds \right\| \\
&\qquad + \left\| \int_{t_1}^{t_2} T(t_2 - s) f(s, w(s), w(b_1(s)), \dots, w(b_m(s)))\, ds \right\| \\
&\qquad + \left\| \sum_{t_0 \le \tau_k < t_1} T(t_1 - \tau_k) I_k w(\tau_k) - \sum_{t_0 \le \tau_k < t_2} T(t_2 - \tau_k) I_k w(\tau_k) \right\| \\
&\quad \le \|T(t_1 - t_0) - T(t_2 - t_0)\| (\|u_0\| + C_2) \\
&\qquad + C_1 \int_{t_0}^{t_1} \|T(t_1 - s) - T(t_2 - s)\|\, ds + M C_1 (t_2 - t_1) \\
&\qquad + C_3 \sum_{t_0 \le \tau_k < t_1} \|T(t_1 - \tau_k) - T(t_2 - \tau_k)\| + M C_3 i(t_1, t_2).
\end{aligned}
\tag{2.5}
$$

The coefficients of $\|u_0\|$, C_1, C_2 and C_3 in (2.5) are independent of $w \in Y$ and those terms tend to zero when $t_2 \to t_1$, except for the case $t_1 = \tau_k$ for some $k = 1, 2, \dots, \kappa$ and $t_2 \to \tau_k + 0$. As a consequence of the continuity of $T(t)$ in the uniform operator topology for $t > 0$, which follows from the compactness of $T(t)$ for $t > 0$, $F(Y)$ is an equicontinuous on each interval of continuity family of functions. Since all the assumptions of Arzela–Ascoli's theorem are satisfied on each interval of continuity, then $F(Y)$ is a precompact subset of Y. Finally, applying Schauder's fixed point theorem to X, Y and F, it follows that F has a fixed point in Y and any fixed point of F is a mild solution of the nonlocal Cauchy problem (1.1). This completes the proof of the theorem. □

Theorem 2.2. *Suppose that the functions f, g and b_i $(i = 1, 2, \dots, m)$ satisfy assumptions* **A1** *and* **A2**, *where $u_0 \in E$. Then in the class of all the functions w, for which assumption* **A3** *holds, the nonlocal Cauchy problem* (1.1) *has a mild solution u. If in addition:*

(i) *E is a reflexive Banach space,*

(ii) *there exists a constant $L > 0$ such that*

$$
\begin{cases}
\|f(s, u_0, u_1, \dots, u_m) - f(\tilde{s}, \tilde{u_0}, \tilde{u_1}, \dots, \tilde{u_m})\| \le L_1 \Big(|s - \tilde{s}| + \sum\limits_{k=0}^{m} \|u_k - \tilde{u_k}\| \Big), \\
\|I_k \nu\|_E \le L_2 \|\nu\|_E \quad \text{for } \nu \in E,\ k = 1, 2, \dots, \kappa,
\end{cases}
$$

where $s, \tilde{s} \in I$, $u_i, \tilde{u_i} \in E_r$ $(i = 0, 1, 2, \dots, m)$ and $L = \max\{L_1, L_2\}$,

(iii) *u is the unique mild solution of the problem* (1.1) *and there is a constant $K > 0$ such that*

$$\|u(b_i(s)) - u(b_i(\tilde{s}))\| \leq K\|u(s) - u(\tilde{s})\| \quad \textit{for } \ s, \tilde{s} \in I,$$

(iv) *the element $u_0 \in D(A)$ and $g(u) \in D(A)$,*

then u is the unique classical solution of the impulsive nonlocal Cauchy problem (1.1).

Proof. Since all the assumptions of Theorem 2.1 are satisfied, then the nonlocal impulsive Cauchy problem (1.1) possesses a mild solution u which, according to assumption (iii), is the unique mild solution of the problem (1.1). Now, we will show that u is the unique classical solution of semilinear, nonlocal and impulsive, Cauchy problem (1.1). Therefore observe that

$$\begin{aligned}
u(t+h) - u(t) &= [T(t+h-t_0)u_0 - T(t-t_0)u_0] \\
&\quad - [T(t+h-t_0)g(u) - T(t-t_0)g(u)] \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s) f(s, u(s), u(b_1(s)), u(b_2(s)), \ldots, u(b_m(s)))\, ds \\
&\quad + \int_{t_0+h}^{t+h} T(t+h-s) f(s, u(s), u(b_1(s)), u(b_2(s)), \ldots, u(b_m(s)))\, ds \\
&\quad - \int_{t_0}^{t} T(t-s) f(s, u(s), u(b_1(s)), u(b_2(s)), \ldots, u(b_m(s)))\, ds \\
&\quad + \sum_{t_0 \leq \tau_k < t+h} T(t+h-\tau_k) I_k(u(\tau_k)) - \sum_{t_0 \leq \tau_k < t} T(t-\tau_k) I_k(u(\tau_k)) \\
&= T(t-t_0)[T(h) - I]u_0 \ - \ T(t-t_0)[T(h)-I]g(u) \\
&\quad + \int_{t_0}^{t_0+h} T(t+h-s) f(s, u(s), u(b_1(s)), u(b_2(s)), \ldots, u(b_m(s)))\, ds \\
&\quad + \int_{t_0}^{t} T(t-s)[f(s+h, u(s+h), u(b_1(s+h)), \ldots, u(b_m(s+h))) \\
&\quad - f(s, u(s), u(b_1(s)), u(b_2(s)), \ldots, u(b_m(s)))]\, ds \\
&\quad + \sum_{t \leq \tau_k < t+h} T(t+h-\tau_k) I_k(u(\tau_k)) \\
&\quad + \sum_{t_0 \leq \tau_k < t} \big(T(t+h-\tau_k) - T(t-\tau_k)\big) I_k(u(\tau_k))
\end{aligned}$$

for $t \in [t_0, t_0 + a)$, $h > 0$ and $t + h \in (t_0, t_0 + a]$.

Consequently we have

$$\begin{aligned}
&\|u(t+h)-u(t)\| \\
&\quad \le hM\|Au_0\| + hM\|Ag(u)\| + hMC_1 + MLah \\
&\qquad + ML\int_{t_0}^{t}\Big(\|u(s+h)-u(s)\| + \sum_{k=1}^{m}\|u(b_k(s+h))-u(b_k(s))\|\Big)ds \\
&\qquad + \sum_{t\le\tau_k<t+h}\|T(t+h-\tau_k)I_k(u(\tau_k))\| \qquad\qquad (2.6)\\
&\qquad + \sum_{t_0\le\tau_k<t}\|\big(T(t+h-\tau_k)-T(t-\tau_k)\big)I_k(u(\tau_k))\| \\
&\quad \le C_*h + ML(1+mK)\int_{t_0}^{t}\|u(s+h)-u(s)\|\,ds + MC_3 i(t,t+h),
\end{aligned}$$

where

$$C_* := M\big(\|Au_0\| + \|Ag(u)\| + C_1 + aL + \|AI_k(u(\tau_k))\| i(t_0,t)\big).$$

By applying Gronwall's inequality and from (2.6) we have

$$\|u(t+h)-u(t)\| \le \big(C_*h + MC_3 i(t,t+h)\big)\exp(aML(1+mK)).$$

Thus $\|u(t+h)-u(t)\| \to 0$ as $h\to 0$, and u is Lipschitz continuous on each interval of continuity in I. The Lipschitz continuity of u on each interval of continuityin I combined with the Lipschitz continuity of f on $I\times E^{m+1}$ imply that $t\mapsto f(t,u(t),u(b_1(t)),\dots,u(b_m(t)))$ is Lipschitz continuous on each interval of continuity in I. This property of $t\mapsto f(t,u(t),u(b_1(t)),\dots,u(b_m(t)))$, together with the assumptions of Theorem 2.2, implies that the linear Cauchy problem

$$\begin{aligned}
\dot v(t) + Av(t) &= f(t,u(t),u(b_1(t)),\dots,u(b_m(t))), && t\in I,\ t\ne\tau_k,\\
v(\tau_k+0) &\equiv u(\tau_k) + I_k(u(\tau_k)), && k=1,2,\dots,\kappa,\\
v(t_0) &= u_0 - g(u),
\end{aligned}$$

has a unique classical solution v such that

$$\begin{aligned}
v(t) &= T(t-t_0)u_0 - T(t-t_0)g(u)\\
&\quad + \int_{t_0}^{t} T(t-s)f(s,u(s),u(b_1(s)),\dots,u(b_m(s)))\,ds\\
&\quad + \sum_{t_0\le\tau_k<t} T(t-\tau_k)I_k(u(\tau_k)), \quad t\in I.
\end{aligned}$$

Consequently, u is the unique classical solution of the nonlocal impulsive Cauchy problem (1.1), and the proof is complete. □

3. Continuous dependence of a mild solution on the initial condition

Theorem 3.1. *Suppose that the functions f, g and $I(u)$ satisfy the assumptions* **A1**–**A3** *and there exist constants μ_1, μ_2, μ_3 such that*

(i) $\|g(u) - g(\tilde{u})\| \le \mu_1 \|u - \tilde{u}\|$,

(ii) $\|f(s, u(s), \dots, u(b_m(s))) - f(s, \tilde{u}(s), \dots, \widetilde{u(b_m(s))})\| \le \mu_2 \|u - \tilde{u}\|$,

(iii) $\|I_k(u(\tau_k)) - I_k(\tilde{u}(\tau_k))\| \le \mu_3 \|u(\tau_k) - \tilde{u}(\tau_k)\|$,

where $u, \tilde{u} \in C(I, E)$. If u and $\tilde{u}$ are mild solutions of the problem (1.1) *with the respective initial values $u_0, \tilde{u}_0$ and the constants μ_1 and $\mu = \max\{\mu_2, \mu_3\}$ satisfy the inequality*

$$\mu_1 < \frac{\exp(-(t_0 + a)M\mu)(1 + M\mu)^{-\kappa}}{M}, \tag{3.1}$$

then the following inequality holds:

$$\|u(t) - \tilde{u}(t)\| \le \frac{M \exp((t_0 + a)M\mu)(1 + M\mu)^{\kappa}}{1 - M\mu_1 \exp((t_0 + a)M\mu)(1 + M\mu)^{\kappa}} \|u_0 - \tilde{u}_0\|. \tag{3.2}$$

Proof. Assume that $u, \tilde{u}$ are the mild solutions of problem (1.1). Then

$$\begin{aligned} u(t) - \tilde{u}(t) = {} & [T(t - t_0)(u_0 - \tilde{u}_0) - T(t - t_0)(g(u) - g(\tilde{u}))] \\ & + \int_{t_0}^{t} T(t - s)\big(f(s, u(s), u(b_1(s)), u(b_2(s)), \dots, u(b_m(s))) \\ & - f(s, \tilde{u}(s), \tilde{u}(b_1(s)), \tilde{u}(b_2(s)), \dots, \tilde{u}(b_m(s)))\big)\, ds \\ & + \sum_{t_0 \le \tau_k < t+h} T(t - \tau_k)[I_k(u(\tau_k)) - I_k(\tilde{u}(\tau_k))], \end{aligned}$$

where $t \in [t_0, t_0 + a]$. From **A1**–**A3** and the hypotheses of the theorem, we have

$$\begin{aligned} \|u(\zeta) - \tilde{u}(\zeta)\| \le {} & M\|u_0 - \tilde{u}_0\| + M\mu_1 \|u - \tilde{u}\| \\ & + M\mu_2 \int_{t_0}^{\zeta} \|u - \tilde{u}\|\, ds + M\mu_3 \sum_{t_0 \le \tau_k < \zeta} \|u(\tau_k) - \tilde{u}(\tau_k)\| \end{aligned}$$

for $t_0 \le \zeta \le t_0 + a$. Using this result, it follows that

$$\begin{aligned} \sup_{\zeta \in I} \|u(\zeta) - \tilde{u}(\zeta)\| \le {} & M\|u_0 - \tilde{u}_0\| + M\mu_1 \|u - \tilde{u}\| \\ & + M\mu_2 \int_{t_0}^{t} \|u - \tilde{u}\|\, ds + M\mu_3 \sum_{t_0 \le \tau_k < t} \|u(\tau_k) - \tilde{u}(\tau_k)\|. \end{aligned}$$

Thus

$$\|u(t)-\tilde u(t)\| \le M\|u_0-\tilde u_0\| + M\mu_1\|u-\tilde u\| + M\mu\Big\{\int_{t_0}^{t}\|u-\tilde u\|\,ds + \sum_{t_0\le\tau_k<t}\|u(\tau_k)-\tilde u(\tau_k)\|\Big\}.$$

Applying Gronwall's inequality for discontinuous functions (see [5]), it follows that

$$\|u(t)-\tilde u(t)\| \le \{\|u_0-\tilde u_0\| + \mu_1\|u-\tilde u\|\}\, M\exp((t_0+a)M\mu)(1+M\mu)^{\kappa}.$$

We can also write this inequality in the form

$$\begin{aligned}&[1-M\mu_1\exp\big((t_0+a)M\mu\big)(1+M\mu)^{\kappa}]\|u(t)-\tilde u(t)\|\\ &\qquad \le M\exp\big((t_0+a)M\mu\big)(1+M\mu)^{\kappa}\|u_0-\tilde u_0\|.\end{aligned} \tag{3.3}$$

Additionally, if equality (3.1) is valid, then inequality (3.3) is equivalent to inequality (3.2). This completes the proof of Theorem 3.1. □

Remark 3.2. If $\mu_1 = \kappa = 0$, then inequality (3.2) is reduced to the classical inequality

$$\|u(t)-\tilde u(t)\| \le M\exp((t_0+a)M\mu)\|u_0-\tilde u_0\|,$$

which is characteristic for the continuous dependence of the semilinear functional-differential evolution Cauchy problem with the classical initial condition.

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On Banach-Lie Algebras, Spectral Decompositions and Complex Polarizations

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Abstract. Complex Kähler polarizations are constructed for a class of real Banach-Lie algebras that are not necessarily L^*-algebras but include all the real compact L^*-algebras. The approach is based on the theory of spectral decompositions of Banach space operators, and particularly on Dunford scalar operators. The main results are illustrated by means of a family of examples that are constructed starting from the Schatten-von Neumann classes of Hilbert space operators $\mathcal{C}_p$ with $p \geq 2$.

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Introduction

Our aim in the present paper is to extend to a Banach setting certain ideas that arose in [20] in the study of a special kind of infinite-dimensional Lie groups modeled over Hilbert spaces. That local study was motivated by the orbit method in infinite dimensions, and what we are doing here is essentially to explore some spectral theoretic aspects of that method. To explain this in more detail, let us begin with the case of finite-dimensional groups.

In the representation theory of Lie groups, an important problem related to the orbit method is to construct complex structures compatible with the natural symplectic structures of the coadjoint orbits of a finite-dimensional Lie group G, and to this end one has to make use of complex polarizations (see, e.g., [15] and [19]). As it is well known, these are complex subalgebras of the complexified Lie algebra of G, and are closely related to root-space decompositions (see, e.g., Proposition IX.2.9 in [19]), thus essentially leaning on *spectral theory of matrices*.

On the other hand, in connection with the investigation of highest weight representations of certain Hilbert-Lie groups (see [7], [8] and [20]), complex polarizations of the corresponding Hilbert-Lie algebras are required again, and a central role in their construction is played by the *spectral theory of self-adjoint operators*. And the latter theory is just a Hilbert space extension of the diagonalization theory of Hermitian matrices.

In view of some recent successful studies in representation theory of Banach-Lie groups modeled on certain ideals of compact operators (see [16] and [18]), the problem to relate that representation theory to complex polarizations of the corresponding Lie algebras thus arises naturally. It is just the *aim* of the present paper to do a first step in that direction, by providing a method to construct complex polarizations for a class of Banach-Lie algebras which are more general than the L^*-algebras dealt with in [20] (see Theorem 2.10 below). To this end, we make use of the *theory of spectral decompositions* – also called local spectral theory – for bounded linear operators on Banach spaces (see, e.g., [9], [10], [26], [12]).

Before proceeding to a more detailed description of the contents of the present paper, let us briefly explain why we are working here only on the level of Lie algebras, without making any attempt to draw consequences which our results can have for the corresponding Lie groups. The motivation for this fact is two-fold. Firstly, the spectral theoretic flavor of the methods we use fits the best to the world of "linear objects", i.e., of Lie algebras. Secondly, it is one of the basic principles in Lie theory that things should be understood on the Lie algebraic level first, and the corresponding Lie group problems should be approached afterwards, starting from the already available Lie algebraic information. Accordingly, it seems more appropriate to postpone to another paper ([3]) the consequences of the Lie algebraic results in the present note. The paper [3] (see also [4]) includes in particular a method to provide invariant complex structures for homogeneous spaces of certain Banach-Lie groups, heavily leaning on the complex polarizations constructed in the present note.

In Section 1 we are concerned with spectral decompositions for bounded derivations of (not necessarily associative) Banach algebras. The motivation for considering spectral decompositions in this special instance stems from the essential role played by the spectral theory of derivations in order to understand certain most interesting objects arising in the infinite-dimensional Lie theory (see, e.g., [20] and [21]). More specifically, the main result of Section 1 is Theorem 1.4, which essentially asserts that each normal derivation with closed range of a complex Banach-Lie algebra leads to a "triangular decomposition" of that algebra. A version of this result without the closed-range hypothesis is contained in Remark 1.6.

The core of the present paper is Section 2, where we prove our main result concerning the construction of weak Kähler polarizations by means of the spectral decompositions (Theorem 2.10). The class of real Banach-Lie algebras to which this result applies is described by the set of hypotheses $1^\circ - 5^\circ$ that are stated before Lemma 2.8. An algebra $\mathfrak{g}(D)$ of this class is constructed by means of a bounded derivation D of a real Banach-Lie algebra $\mathfrak{g}$ for which certain conditions are satisfied. In particular, we require that the coadjoint representation of $\mathfrak{g}$ should be embedded into the adjoint one (see hypothesis 2°). The idea of construction of $\mathfrak{g}(D)$ comes from the construction of restricted Lie algebras (cf. Definition III.1 in [22]). The last result of Section 2 (namely Proposition 2.12) shows when the Lie algebra $\mathfrak{g}$ we are working with is a compact L^*-algebra. In this special case, our construction of polarizations agrees with the one described in Lemma VII.4 in [20]. Though, we note that, in our more general situation, we can obtain only *weak* Kähler polarizations. (See the precise definitions in the following.)

In Section 3 we take into consideration as illustrating examples the family of Banach-Lie algebras $\{\mathcal{C}_p(\mathcal{H})\}_{1\le p\le\infty}$ (the Schatten-von Neumann classes of operators on the complex Hilbert space $\mathcal{H}$). As we previously mentioned, the present note is a preliminary step of the attempt to extend the representation theory of the Hilbert-Lie groups in [7] and [20] to the more general Banach-Lie groups investigated in [16] and [18]. The latter groups are modeled on the Schatten-von Neumann classes, and that is why we apply the results of Section 1 and Section 2 only to Lie algebras derived from these classical Banach-Lie algebras (in the terminology of [13]). In particular, we conclude Section 3 by working out the details of a family of examples which fall under the hypotheses 1°–5° of Section 2 but are not L^*-algebras (see Example 3.5).

Preliminaries

Throughout the paper we denote by $\mathrm{Der}(\mathfrak{A})$ the Banach-Lie algebra of all bounded derivations of a Banach (not necessarily associative) algebra $\mathfrak{A}$. (We say that $\mathfrak{A}$ is a Banach algebra if $\mathfrak{A}$ is a Banach space endowed with a bounded bilinear map $\mathfrak{A}\times\mathfrak{A}\to\mathfrak{A}$, $(a,b)\mapsto a\cdot b$.) For a real or complex Banach space $\mathfrak{X}$ we denote by $\mathrm{id}_{\mathfrak{X}}$ the identical map on $\mathfrak{X}$, by $\mathfrak{X}^*$ the topological dual of $\mathfrak{X}$, by $\mathcal{B}(\mathfrak{X})$ the algebra of all bounded linear operators on $\mathfrak{X}$ and, when $\mathfrak{X}$ is complex, for $D\in\mathcal{B}(\mathfrak{X})$ we denote by $\sigma(D)$ the spectrum of D. In this case, for every $x\in\mathfrak{X}$ we denote by $\sigma_D(x)$ the *local spectrum* of x with respect to D. We recall that $\sigma_D(x)$ is a closed subset of $\sigma(D)$ and by definition, a complex number w belongs to $\mathbb{C}\setminus\sigma_D(x)$ if and only if there exists an open neighborhood W of w and a holomorphic function $\xi: W\to\mathfrak{X}$ such that

$$(z\,\mathrm{id}_{\mathfrak{X}} - D)\xi(z) = x \quad \text{for every } z\in W.$$

If F is a subset of $\mathbb{C}$ we further denote

$$\mathfrak{X}_D(F) = \{x\in\mathfrak{X} \mid \sigma_D(x)\subseteq F\}.$$

We note that, in the case when $\mathfrak{X}$ has *finite dimension* m, we have

$$\mathfrak{X}_D(F) = \bigoplus_{\lambda\in F\cap\sigma(D)} \mathrm{Ker}\,((D-\lambda\,\mathrm{id}_{\mathfrak{X}})^m) \quad \text{for every } F\subseteq\mathbb{C},$$

while in the case when $\mathfrak{X}$ is a *Hilbert space* and D is a *normal operator with the spectral measure* $E_D(\cdot)$ we have

$$\mathfrak{X}_D(F) = \mathrm{Ran}\,E_D(F) \quad \text{whenever } F \text{ is a closed subset of } \mathbb{C}.$$

We refer to Section 12 in [5] for a review of the few elements of local spectral theory we need. (For more details see the Notes of Chapter I in [5].)

For the sake of completeness, we now recall some elementary facts on complex polarizations (see, e.g., Section VI in [20]). Let $\mathfrak{g}$ be a real Banach-Lie algebra and ω a *continuous* 2*-cocycle* of $\mathfrak{g}$, that is $\omega : \mathfrak{g}\times\mathfrak{g} \to \mathbb{R}$ is a continuous skew-symmetric bilinear form such that

$$\omega([X,Y],Z) + \omega([Y,Z],X) + \omega([Z,X],Y) = 0$$

for every $X, Y, Z \in \mathfrak{g}$. In this case,

$$\mathfrak{h} := \{X \in \mathfrak{g} \mid \omega(X,\mathfrak{g}) = \{0\}\}$$

is a closed subalgebra of $\mathfrak{g}$. A *complex polarization* of $\mathfrak{g}$ in ω is a closed subspace $\mathfrak{p}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ such that there exists a closed subspace of $\mathfrak{g}_{\mathbb{C}}$ complementary to $\mathfrak{p}$ and such that the following properties hold:

(C1) $\mathfrak{p}$ is a (complex closed) subalgebra of the complex Banach-Lie algebra $\mathfrak{g}_{\mathbb{C}}$ such that $[\mathfrak{h},\mathfrak{p}] \subseteq \mathfrak{p}$,
(C2) $\mathfrak{p} \cap \overline{\mathfrak{p}} = \mathfrak{h}_{\mathbb{C}}$,
(C3) $\mathfrak{p} + \overline{\mathfrak{p}} = \mathfrak{g}_{\mathbb{C}}$,
(C4) $\omega(\mathfrak{p},\mathfrak{p}) = \{0\}$,

where $Z \mapsto \overline{Z}$ denotes the complex conjugation on $\mathfrak{g}_{\mathbb{C}}$ whose set of fixed points is just $\mathfrak{g}$. In this case, the natural inclusion map $\mathfrak{g} \to \mathfrak{g}_{\mathbb{C}}$ induces an isomorphism $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{p}$ of real Banach spaces (see Section VI in [20]), thus $\mathfrak{g}/\mathfrak{h}$ is actually a complex Banach space. On the other hand, note that ω induces a symplectic form on $\mathfrak{g}/\mathfrak{h}$ by

$$\mathfrak{g}/\mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \to \mathbb{R}, \quad (X+\mathfrak{h}, Y+\mathfrak{h}) \mapsto \omega(X,Y).$$

This symplectic form is the imaginary part of the continuous Hermitian sesquilinear form $(\cdot \mid \cdot)$ which is defined on the complex Banach space $\mathfrak{g}/\mathfrak{h} \cong \mathfrak{g}_{\mathbb{C}}/\mathfrak{p} = (\overline{\mathfrak{p}}+\mathfrak{p})/\mathfrak{p}$ by

$$(Z_1+\mathfrak{p} \mid Z_2+\mathfrak{p}) = \mathrm{i}\omega(Z_1,\overline{Z_2}) \quad \text{for } Z_1, Z_2 \in \overline{\mathfrak{p}}.$$

If the following condition holds,

(C5) $\mathrm{i}\omega(Z,\overline{Z}) > 0$ for every $Z \in \mathfrak{p} \setminus \mathfrak{h}_{\mathbb{C}}$,

then $(\cdot \mid \cdot)$ is even a scalar product (i.e., it is positively definite) and $\mathfrak{p}$ is called a *weak Kähler polarization.* In the case when the map

$$\mathfrak{g}/\mathfrak{h} \to (\mathfrak{g}/\mathfrak{h})^*, \quad X+\mathfrak{h} \mapsto \omega(X,\cdot),$$

is moreover invertible, one says that $\mathfrak{p}$ is a *strong Kähler polarization.* (An equivalent condition is that $(\cdot \mid \cdot)$ defines the topology of $\mathfrak{g}/\mathfrak{h}$, i.e., $\mathfrak{g}/\mathfrak{h}$ is actually a complex Hilbert space; see stage I in the proof of Proposition 2.12 below.)