NEW APPROACHES TO CIRCLE PACKING IN A SQUARE

With Program Codes

Optimization and Its Applications

VOLUME 6

Aims and Scope

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NEW APPROACHES TO CIRCLE PACKING IN A SQUARE

With Program Codes

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Preface

The problem of finding the densest packing of congruent circles in a square is obviously an interesting challenge of discrete and computational geometry with all its surprising structural forms and regularities. It is easy to understand what the problem is: to position a given number of equal circles in such a way that the circles fit fully in a square without overlapping.

With a large number of circles to be packed, the solution is very difficult to find. This difficulty is highlighted by many features. On one hand it is clear that an optimal solution can be rotated, reflected, or the circles can be reordered, and hence the number of equivalent optimal solutions blows up as the number of circles increases. On the other hand, in several cases there even exists a circle in an optimal packing that can be moved slightly while retaining the optimality. Such free circles (or rattles) mean that there exist not only a continuum of optimal solutions, but the measure of the set of optimal solutions is positive!

One nice aspect of circle packing is that the problem is clear without any further explanation, and a solution can be provided by a figure. In a certain sense, the number of circles for which we know the optimal packing with certainty indicates how sophisticated the available theoretical and computational tools are. One can think that it is a kind of measure of our technological and scientific capabilities.

Beyond the theoretically challenging character of the problem, there are several ways in which the solution methods can be applied to practical situations. Direct applications are related to cutting out congruent two-dimensional objects from an expensive material, or locating points within a square in such a way that the shortest distance between them is maximal. Circle packing problems are closely related to the 'obnoxious facility location' problems, to the Tammes problem (locate a given number of points on a sphere in such a way that the minimal distance among them is maximal), and less closely related to the Kissing Number Problem (determine how many unit spheres can touch a given unit sphere in an n-dimensional space). The emerging computational algorithms can also be well utilized in other hard-to-solve optimization problems like molecule conformation.

The involvement of the authors with circle packing began in 1993 when one of the authors, Tibor Csendes, visited the University of Karlsruhe in Germany. His colleague, Dietmar Ratz, showed him a paper written in the IBM Nachrichten (42(1992) 76–77) about the difficulties involved in packing ten circles in the unit square. The problem seemed to fit the interval arithmetic-based reliable optimization algorithms they were investigating.

However, the first tries were disappointing: not even the trivial three-circle problem instance could be solved with the hardware and software environments available at that time. It became clear that symmetric equivalent solutions made the problem hard to solve.

A couple of years later a PhD student, Péter Gábor Szabó, asked Tibor Csendes for an operations research subject. Among those offered was the circle packing problem, since Péter graduated as a mathematician (and not as a computer science expert, as do the majority of the PhD students at the Institute of Informatics of the University of Szeged). He began with an investigation of the structural questions of circle packing.

The Szeged team collaborated with the Department of Computer Architecture and Electronics at the University of Almería in Spain in optimization, first in the framework of a Tempus project, and then for some years within a European Erasmus / Socrates programme. The head of the Spanish department, Inmaculada García, sent two PhD students to Szeged for a few months. One of the PhD students, Leocadio González Casado, was interested in the circle packing problem and invested some time in a new approximating solution algorithm. Péter Gábor Szabó visited the Almería team, and the joint work resulted in the first double article reporting six world best packings at that time. These candidate optimal packings were all beaten later by the results of Eckard Specht at the University of Magdeburg. His collaboration with the Hungarian team brought other publications on the subject.

The last member joining the present team was Mihály Csaba Markót. He was a PhD student supervised by Tibor Csendes, but was working on another topic, improving the efficiency of interval arithmetic-based global optimization algorithms. His interest turned relatively late to circle packing, when the interval optimization methods seemed to be effective enough to tackle such tough problems. It was very interesting to see what kind of utilization of the problem specialities was necessary to build a numerically reliable computer procedure that cracked the next unsolved problem instances of 28, 29, and 30 circles. That produced a kind of happy end for this nearly 10-year story—and a return to the first hopeless-looking technique.

The authors had intended this volume to be a summary of results achieved in the past few years, providing the reader with a comprehensive view of the theoretical and computational achievements. One of the major aims was to publish all the programming codes used. The checking performed by the wider scientific community has helped in having the computational proofs accepted. The open source codes we used will enable the interested reader to improve on them and solve problem instances that still remain challenging, or to use them as a starting point for solving related application problems.

The present book can be recommended for those who are interested in discrete geometrical problems and their efficient solution techniques. The volume is also worth reading by operations research and optimization experts as a report or as a case study of how utilization of the problem structure and specialities enabled verified solutions of the previously hopeless high-dimensional nonlinear optimization problems with nonlinear constraints. The outlined history of the whole solution procedure provides a balanced picture of how theoretical results, like repeated patterns, lower and upper bounds on possible optimum values, and approximate stochastic optimization techniques, supported the final resolution of the original problem.

Acknowledgments. The authors would like to thank all their colleagues who participated in the research. This work was supported by the Grants OTKA T 016413, T 017241, T 034350, FKFP 0739/97, and by the Grants OMFB D-30/2000, OMFB E-24/2001. It was also supported by the SOCRATES-ERASMUS programme (25/ERMOB/1998-99), by the Spanish Ministry of Education (CICYT TIC96-1125-C03-03) and by the Consejería de Educación de la Junta de Andalucía (07/FSC/MDM). All grants obtained are gratefully acknowledged, and the authors hope they have been used in an efficient and effective way – as in part reflected by the present volume. Mihály Csaba Markót would like to thank the Advanced Concepts Team of the European Space Agency, Noordwijk, The Netherlands for the possibility to prepare the manuscript during his postdoctoral fellowship. The authors are grateful to Jose Antonio Bermejo (University of Almería, Spain) for preparation of the enclosed CD-ROM, and to David P. Curley for checking this book from a linguistic point of view.

Glossary of Symbols

n	the number of circles/points to be packed/arranged, page 1
$P(r_n, S)$	a circle packing, page 13
r_n	the common radius of a circle packing, page 13
S	the size of the enclosing square of a circle packing, page 13
$\overline{r}_n$	the optimum value of the circle packing problem, page 13
$A(m_n, \Sigma)$	a point arrangement, page 14
m_n	the minimal pairwise distance in a point arrangement, page 14
Σ	the size of the enclosing square for a point arrangement, page 14
$\overline{m}_n$	the optimum value of the point arrangement problem, page 14
$P'(R, s_n)$	an associated circle packing, page 14
s_n	the size of the enclosing square for an associated circle packing, page 14
R	the common radius for an associated circle packing, page 14
$\overline{s}_n$	the optimum value for an associated circle packing, page 14
$A'(M, \sigma_n)$	an associated point arrangement, page 14
σ_n	the size of the enclosing square for an associated point arrangement, page 14
M	the minimal pairwise distance for an associated point arrangement, page 14
$\overline{\sigma}_n$	the optimum value for an associated point arrangement problem, page 14
$\mathcal{P}_i^n$	Problem 2.i ($1 \leq i \leq 5$), page 14
$d_n(r_n, S)$	the density of a circle packing $P(r_n, S)$, page 17
T_h	threshold level in Threshold Accepting Algorithm, page 33
ξ	the variance of the perturbation size, page 35
d_{ij}	the squared distance between two points for a point arrangement, page 52
$d(x, y)$	the squared distance between $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$, page 52
$f_n(x, y)$	the objective function for Problem 6.2, page 52
$\mathbb{I}$	the set of real, compact intervals, page 53
$\mathrm{lb}(A)$	the lower bound of the interval A, page 53
$\mathrm{ub}(A)$	the upper bound of the interval A, page 53
$\mathbb{I}(S)$	the set of intervals in $S \subseteq \mathbb{R}$, and the set of boxes in $S \subseteq \mathbb{R}^n$, page 53
$w(A)$	the width of the interval A, page 53
$F(X)$	an interval inclusion function of a real function f over the box X, page 53

$f(X)$	the range of a real function f over the box X, page 53
D_{ij}	the natural interval extension of d_{ij}, page 56
$F_n(X, Y)$	an interval inclusion function of $f_n(x, y)$, page 56
$\tilde{f}$	the best known lower bound for the maximum of Problem 6.2, page 57
$\tilde{f}_0$	the best known lower bound for the maximum of Problem 6.1, page 57
$D(x, y)$	the natural interval extension of $d(x, y)$, page 79
$S^m_{s..f}$	input sets of tile combinations for the B&B method, page 89
$\bar{S}^m_{s..f}$	output sets of tile combinations for the B&B method, page 89
${}^{m_1}_{s_1..f_1}S^m_{s..f}$	input sets of tile combinations for the B&B method, page 89
${}^{m_1}_{s_1..f_1}\bar{S}^m_{s..f}$	output sets of tile combinations for the B&B method, page 89
PAT$(f(k))$	a pattern class, where $n = f(k)$, page 115
STR$(f(k))$	a structure class, where $n = f(k)$, page 123
$[[p, q]]$	a grid packing, page 132
GP	the set of grid packings, page 132
$p^I_n(x)$	a generalized minimal polynomial, page 145
$P_n(x)$	$p^1_n(x)$, page 145

1

Introduction and Problem History

1.1 Problem description and motivation

The general mathematical problem of finding the densest packing of equal objects in a bounded shape arises in many fields of the natural sciences, in engineering design, and also in everyday life. Some applications that involve the packing of *identical* circles are the following:

- *coverage*: place radio towers in a geographical region such that the coverage of the towers is maximal, with as little interference as possible;
- *storage*: place as many identical objects as possible (e.g. barrels) into a storage container;
- *packaging*: determine the smallest box into which one can pack a given number of bottles;
- *tree exploitation*: plant trees in a given region such that the forest is as dense as possible, but the trees allow each other to grow up to their maximal desired size;
- *cutting industry*: cut out as many identical disks as possible from a given (in the general case, irregular) piece of material.

All the above applications require the solution of the following general problem:

Place $n \geq 2$ identical circles (disks) in a given, bounded subset of the plane without overlapping, in such a way that the density of the packing is maximal. (The density of the packing is defined as the ratio of the filled and total available area.)

In this book we mainly deal with the problems of packing equal circles in the square. However, some of the results (especially, the stochastic algorithms of Chapters 4 and 5) can be applied or generalized to more general packing scenarios as well.

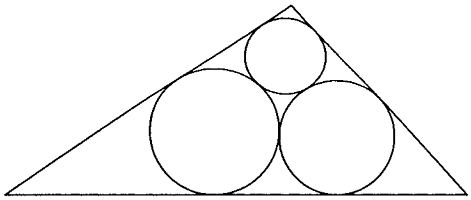

Fig. 1.1. The Malfatti circles.

1.2 The history of circle packing

The circle packing problem has a long history in mathematics, see e.g. [94, 118, 130]. In this section we will review some of the existing results, starting from the early investigations.

1.2.1 The problem of Malfatti

In European mathematics, one of the first examples of looking for some kind of optimal packing of circles is the famous *'Malfatti Problem'*. In 1803, the Italian mathematician Gianfrancesco Malfatti (1731–1807) posed the following question: *Consider a right prism with a right triangular base. How do we cut out three cylinders (perhaps of different sizes) from the prism, such that the total volume of the cylinders is maximal?*

Malfatti thought that the solution was to construct three circles in the triangle, in such a way that each one of them touched the other two circles and two sides of the triangle. In mathematical literature, the geometric construction of such circles in an *arbitrary* triangle is called the Malfatti Problem (Figure 1.1). However, it is worth mentioning that this latter problem was previously studied and solved earlier by the Japanese mathematician Chokuen Ajima (1732–1798) [30].

Interestingly, it turned out only about a hundred years later (according to our knowledge) that the Malfatti circles do not necessarily give the densest packing of three (non-identical) circles in a triangle: In 1930, H. Lob and H. W. Richmond [59] showed that in an equilateral triangle a different packing can give a greater density than those of the Malfatti circles (see Figure 1.2).

The problem of finding the densest possible configuration in an arbitrary triangle was solved in 1992 by V. A. Zalgaller and G. A. Los' [131].

1.2.2 The circle packing studies of Farkas Bolyai

It seems likely that the Hungarian mathematician Farkas Bolyai (1775–1856) was the first scientist who investigated the density of circle packing *sequences* in bounded regions and published the related results. In his main work – usually referred to as the 'Tentamen', 1832–33 [6] –, a dense packing of equal circles in an equilateral triangle was worked out, as depicted in Figure 1.3.

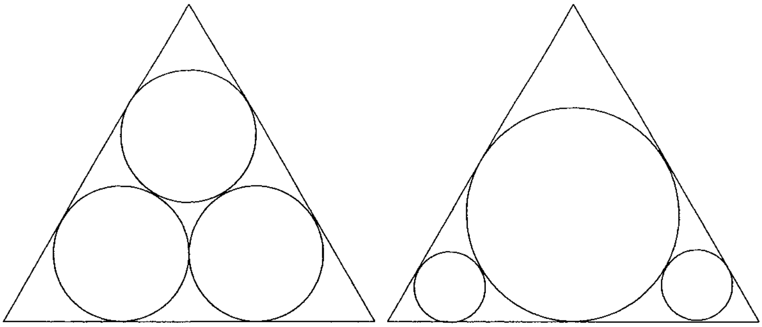

Fig. 1.2. The Malfatti circles and a denser packing in an equilateral triangle.

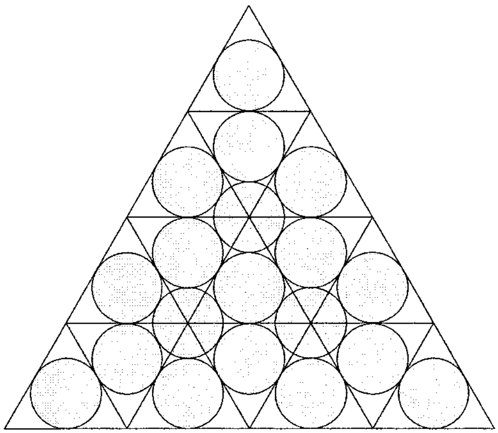

Fig. 1.3. The example of Bolyai for packing 19 equal circles in an equilateral triangle.

This particular packing is obtained by dividing the triangle into congruent equilateral triangles and placing the circles in these smaller triangles. Then three additional circles with the same radius can be placed in the positions which are surrounded by six circles.

Bolyai defined an infinite packing sequence based on the refinement of the subdivision of the large triangle, and investigated the limit of the *'vacuitas'* (in Latin, the area not covered by the circles). It is easy to demonstrate that the limit density of the resulting packing sequence is $\pi/\sqrt{12}$.

An interesting historical question is the motivation behind the studies of Bolyai in this subject. According to research of mathematical history, he was about to apply for a position in the forestry commission, and this led him to the investigation of problems like planting trees in given regions such that 'they share the same amount of light and air'. Figures 1.4 and 1.5 show some of his other packing examples in this

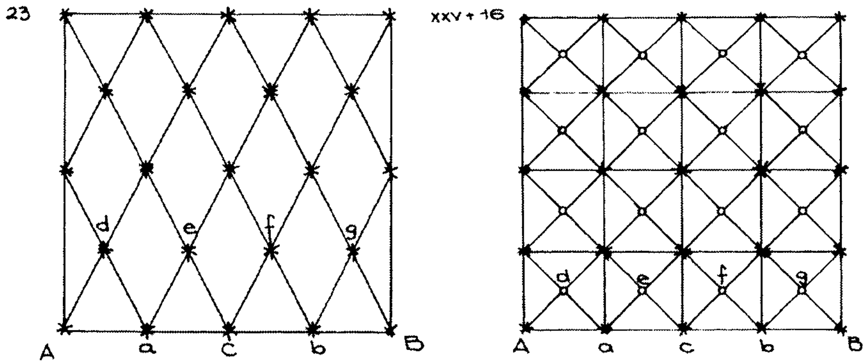

Fig. 1.4. Two examples by Farkas Bolyai for planting trees in a square-shaped region.

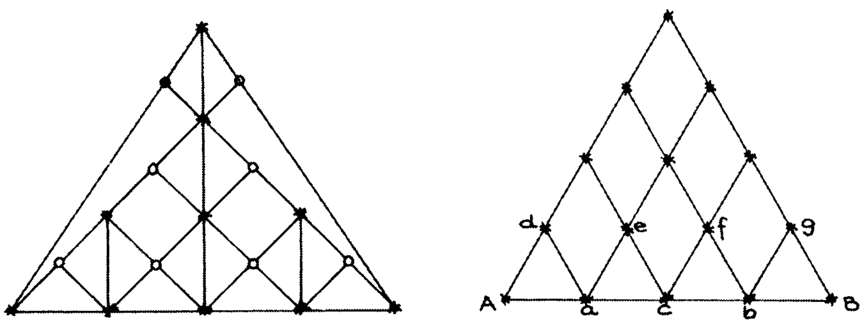

Fig. 1.5. Two examples by Farkas Bolyai for planting trees in a triangular region.

subject. Reference [113] gives more detailed information on this topic with further examples.

It should be mentioned here that the above packings of Farkas Bolyai are not necessarily optimal in terms of density. For instance, the packing of Figure 1.3 has been recently shown to be suboptimal: if the sides of the triangle are chosen to have unit length, the common radius of the circles in the packing of Bolyai is approximately 0.072169. However, in 1995, R. L. Graham and B. D. Lubachevsky discovered the packing configurations portrayed in Figure 1.6 with the approximate radii of 0.074360, 0.074326, and 0.074323, respectively [35]. (Recall that Bolyai did not intend to look for the densest packings, but actually investigated the convergence of the sequence of densities.)

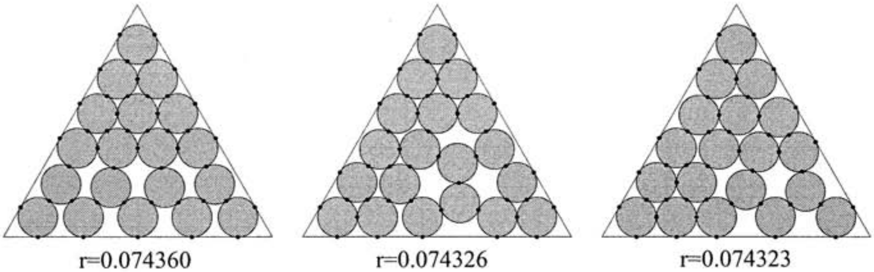

Fig. 1.6. The packings of 19 circles with densities larger than that of the example by Bolyai.

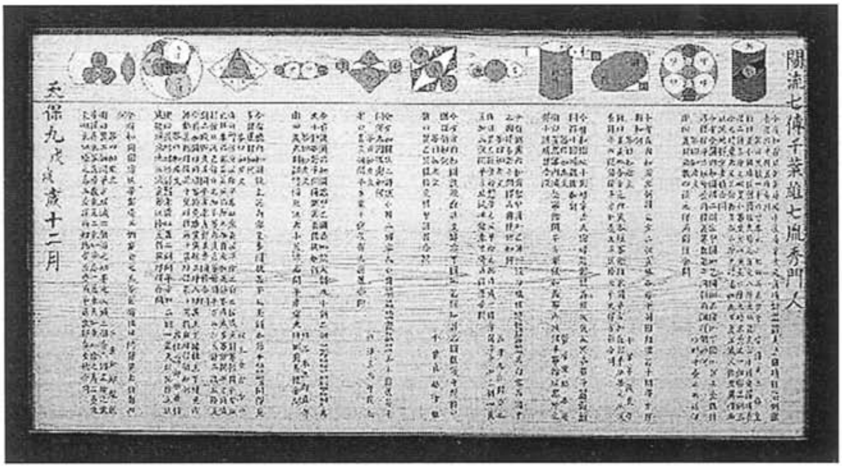

Fig. 1.7. A Japanese Sangaku.

1.2.3 Packing problems from ancient Japan

Historically, the work of Bolyai was not the very first on the packing of circles. There are other interesting early packings in fine arts, relics of religions, in nature [121], and also outside Europe. For instance, many of the Japanese 'Sangaku's of the Edo period (1603–1867) deal with various problems of locating circles in various context. The name Sangaku means mathematical tablet in English. These wooden tablets usually contain geometrical problems. They were displayed in temples and Shinto shrines, probably for meditation purposes (Figure 1.7).

Figure 1.8 shows an example where six equal circles have been packed in a rectangle, also from Japan. For more information on this interesting area of the history of mathematics, see [30], say.

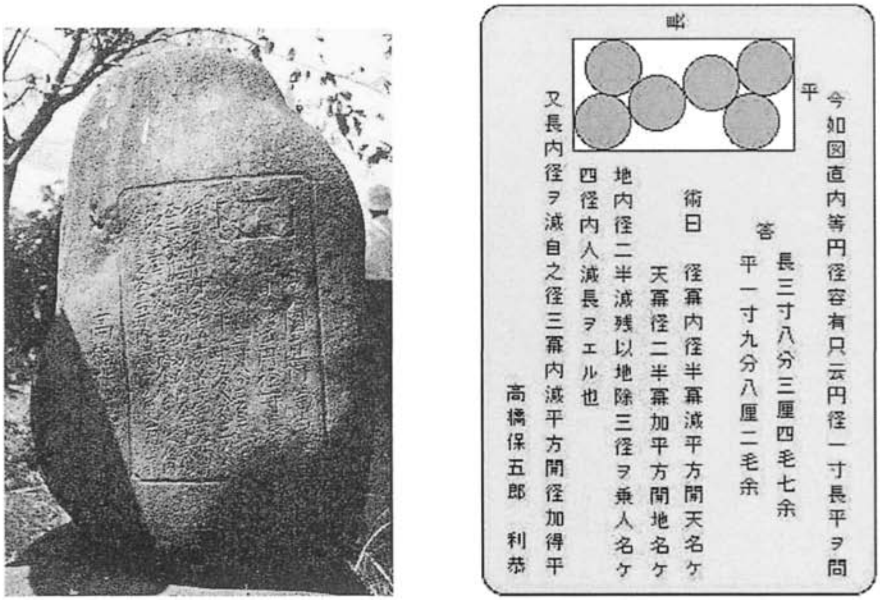

Fig. 1.8. The packing of six equal circles in a rectangle, carved on a rock in Japan.

1.2.4 A circle packing from Hieronymus Bosch

Perhaps the most extraordinary example depicting circle packings is the famous picture by the Dutch painter Hieronymus Bosch (c. 1450–1516): On the left wing of his triptych 'The Garden of Earthly Delights' (c. 1500), an arrangement of circles is displayed on the surface of a sphere. Interestingly, the packing is very hard to discover even in high-quality catalogues; the details become visible only after magnification (Figure 1.9).

1.2.5 The densest packing of circles in the plane

The first solution to the problem of determining the densest packing of equal circles in the plane was given by the Norwegian mathematician Axel Thue (1863–1922)—see [123, 124]. One can prove that the densest packing is the one coming from intuition: the hexagonal structure, where each circle is surrounded by six others (Figure 1.10). In 1773, J. L. Lagrange (1736–1813) proved that this structure results in the densest *lattice* packing on the plane [58]. The density of the hexagonal packing is $\pi/\sqrt{12}$, equal to the limit of the earlier mentioned packing sequence of Farkas Bolyai.

Note that the first optimality proof by Thue was not very convincing; it was completed by the Hungarian mathematician László Fejes Tóth (1915–2005) in 1940 [21].

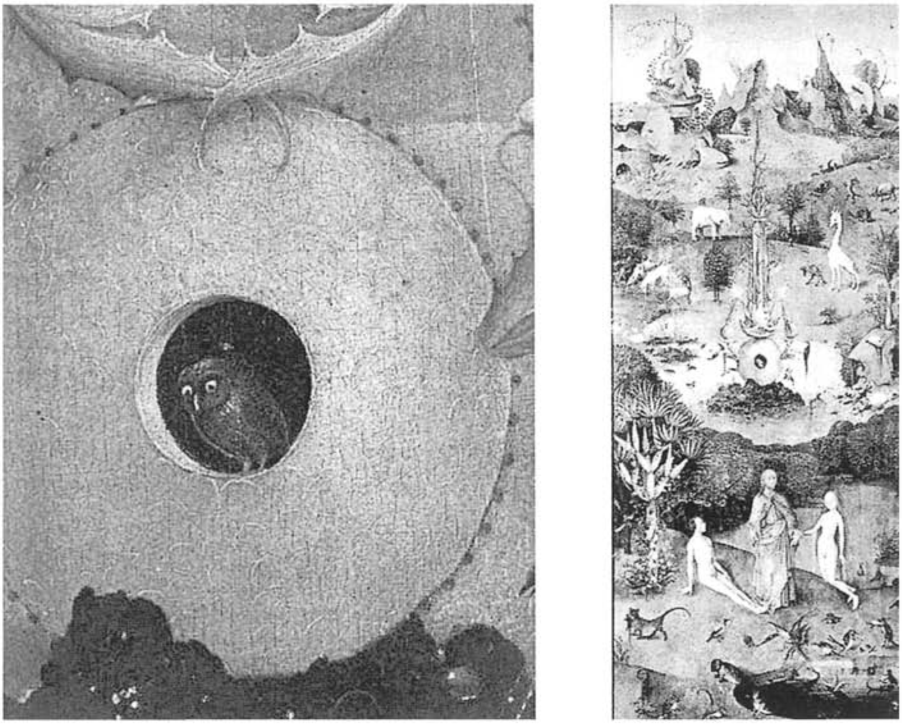

Fig. 1.9. Circles on a sphere in 'The Garden of Earthly Delights' by Hieronymus Bosch.

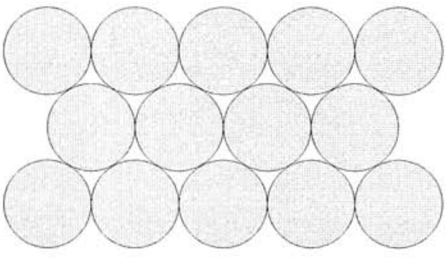

Fig. 1.10. The densest packing of identical circles in the plane.

1.2.6 Circle packings in bounded shapes

The literature of the studies related to the densest packing of identical circles in bounded regions is very rich. Besides the problem class of packing in a square, there are notable results for the packing of circles in a circle [26, 27, 28, 33, 37, 57, 76, 96, 101], in an equilateral triangle [35, 74, 77, 78, 91, 92], in an isosceles right triangle [45], and in a rectangle [103].

1.2.7 Generalizations and related problems

One can formulate the circle packing problems in such a way that the packing is carried out in some non-Euclidean geometry (e.g. in Bolyai-Lobachevsky geometry). However, in such cases even the proper way of defining the concept of density raises some difficulties [7]. Furthermore, the problem can be modified so that we can use a metric other than the usual Euclidean one [4, 22, 25]. Of course, one can consider packings in higher dimensions; see, for example the famous *Kepler Problem* [42], and the *Kissing Number* problem class [12].

1.2.8 Packing of equal circles in a unit square

The first appearance of this particular problem in the mathematical literature is the publication by Leo Moser [82] from 1960, in which he raised the following conjecture: *"eight points in or on a unit square determine a distance less than or equal to* $\frac{1}{2}\sec 15^o$*"*. (For historical completeness, however, note that from the memoirs of L. Fejes Tóth it turns out that he and his compatriot Dezső Lázár (1913–c. 1943) had already investigated the problem before 1940 [112, 117].)

The above conjecture of Moser was proved by J. Schaer and A. Meir [105] in 1965. In the latter publication the authors mentioned that for $n = 6$ there already exists a proof by R. L. Graham, and they were also aware of a proof for $n = 7$. In the end, neither of these claimed proofs was published. (The optimal arrangements are obvious up to $n = 5$.) Schaer sent his early proof for $n = 7$ to P. G. Szabó (one of the authors of this volume), adding a comment that he thought it was an 'ugly proof', so he had decided not to publish it. To the best of our knowledge, no optimality proof for the case $n = 7$ has yet been published (with purely mathematical tools, i.e. without using computers). The $n = 6$ case was later solved by B. L. Schwartz [108] in 1970 and by H. Melissen [75] in 1994, independently of each other.

In 1965, Schaer solved the $n = 9$ case as well [103]. In 1970, M. Goldberg reviewed the previous results [32] and gave conjecturally optimal packings up to $n = 27$ and for some further number of circles. Between 1970 and 1990, at least ten papers reported solutions for $n = 10$ [32, 38, 39, 40, 79, 80, 95, 104, 107, 126]. Table 1.1 gives an overview of the improvements of the best-known packings until 1990, when C. de Groot, R. Peikert, and D. Würtz found the densest packing and proved its optimality with a computer-assisted method [39]. In the table, r_{10} denotes the common radius of the circles (given to six decimal places) packed in the unit square.

A conventional proof for the optimality for ten circles still does not exist, although there are many promising results in this direction as well – see, for example, the work of M. Hujter [49].

Based on their computer technique, in 1992 R. Peikert, D. Würtz, M. Monagan, and C. de Groot published the densest circle packings up to $n = 20$ [94].

In 1983, G. Wengerodt reported an optimality proof for $n = 16$ [127], using only conventional mathematical tools. In 1987, he extended his results for the optimality of $n = 14$ [128], $n = 25$ [129], and (together with K. Kirchner) for $n = 36$ [53]. However, recently there have been some doubts raised concerning the correctness of

Table 1.1. Results for the packing of 10 circles.

Year	Author(s)	r_{10}	Reference
1970	M. Goldberg	0.147060	[32]
1971	J. Schaer	0.147770	[104]
1979	K. Schlüter	0.148204	[107]
1987	R. Milano	0.147920	[79]
1989	G. Valette	0.148180	[126]
1990	B. Grünbaum	0.148197	[40]
1990	M. Grannell	0.148204	[38]
1990	M. Mollard and C. Payan	0.148204	[80]
1990	C. de Groot, R. Peikert, and D. Würtz	0.148204	[39]

the proofs for $n = 25, 36$; for details, see the notes of R. Blind in the Mathematical Reviews (MR1453444).

In 1995, C. D. Maranas, C. A. Floudas, and P. M. Pardalos reported packings for up to 30 circles [66] (without proving their optimality), using the nonlinear programming solver MINOS. Later it turned out that the published result for $n = 21$ was incorrect: namely, the reported function value was better than that of the proven optimum obtained a couple of years later. This also indicates that special care must be taken (that is, numerically reliable calculations must be used) when computer techniques are applied to find approximate or optimal packings.

In 1996, T. Tarnai and Zs. Gáspár [122] improved the packing configuration by M. Goldberg [32] for $n = 19$ (independently of the results of Peikert et al.) and reported upper bounds for the packing densities.

In 1997, K. J. Nurmela and P. R. J. Östergård published approximately optimal packings up to $n = 50$ [86]. In their work, some new classes of packing patterns were introduced. These classes were extended by R. L. Graham and B. D. Lubachevsky [36] with the aid of new results obtained by a method called *billiard simulation*.

In 1999, K. J. Nurmela and P. R. J. Östergård gave computer-assisted optimality proofs for the cases $n \leq 27$ [87]. Together with R. aus dem Spring, they published new theoretical lower bounds for the optimum values, and determined a conjecturally optimal packing sequence [89].

In 2000, D. W. Boll, J. Donovan, R. L. Graham, and B. D. Lubachevsky improved the best-known packings for $n = 32, 37, 48, 50$ [5]. In the same year, P. Ament and G. Blind claimed an improved packing for $n = 34$ (although at that time a better solution was already known for that case) [3].

In 2002, M. Locatelli and U. Raber developed a deterministic computer method, which enabled them to improve the existing best packings for $n = 32$ and 37, and to prove the approximate optimality of the best existing packings for $n = 10 - 35, 38$ and 39 [61].

In 2005, B. Addis, M. Locatelli, and F. Schoen further improved the previously known best packings for $n = 53, 59, 66, 68, 73, 77, 85, 86$ [1].

Overall, then, the currently available results for the problem class are the following:

Table 1.2. The authors of the known optimal packings.

Year	Authors	Results for n
1965	J. Schaer and A. Meir [103, 105]	$8, 9$
1970	B. L. Schwartz [108]	6
1983	G. Wengerodt [127]	16
1987	G. Wengerodt [128]	14
1992	R. Peikert et al. [94]	$10 - 20$
1999	K. J. Nurmela and P. R. J. Östergård [87]	$7, 21 - 27$
2004	M. Cs. Markót [68]	28
2005	M. Cs. Markót and T. Csendes [70]	$29 - 30$

- *Proven (optimal) packings* are known up to $n = 30$. The cases $n = 2, \ldots, 6, 8, 9$, 14 and 16 are solved by conventional mathematical tools, while the others are results of computer-assisted methods. Table 1.2 summarizes the known optimal packings and the corresponding publications.

- *Approximate packings* (i.e. packings determined by various methods without proof of optimality) are reported in the literature for up to 200 circles. Table 1.3 contains the most important improvements of the last decade. These numerical results have been obtained via several different strategies; for instance, using billiard simulation [36, 62, 65], the minimization of an energy function [86], standard BFGS quasi-Newton algorithm [94], a nonlinear programming solver (MINOS 5.3) [66], simulated annealing and threshold accepting techniques [11], and the Cabri-Géomètre software [80]. Although it is not known whether these solutions are optimal, in some cases the numerical results can help us find better solutions when they are used as lower bounds of the optimal (maximal) solutions. For instance, several strategies described in the packing literature, like that of R. Peikert et al. in [94], are based on the knowledge of a good lower bound of the optimum value.

- Finally, numerous other *related results* (concerning e.g. patterns, bounds, and various properties of the optimal solutions) have been published as well [36, 61, 89, 114, 118].

Some other details on the history of the problem of packing equal circles in a square (PECS) can be found in [8, 9, 66, 94, 118, 120, 130].

Table 1.3. The authors of approximate packings between 1995 and 2006.

Year	Authors	Results for n
1995	C. D. Maranas et al. [66]	up to 30
1996	R. L. Graham and B. D. Lubachevsky [36]	up to 61
1997	K. J. Nurmela and P. R. J. Östergård [86]	up to 50
2000	D. W. Boll et al. [5]	$32, 37, 48, 50$
2001	L. G. Casado et al. [11]	up to 100
2002	M. Locatelli and U. Raber [61]	up to 39
2005	B. Addis et al. [1]	$50 - 100$
2006	P. G. Szabó and E. Specht [120]	up to 200

2

Problem Definitions and Formulations

In this chapter we will specify the densest packing of equal circles in a square problem, and discuss some equivalent problem settings. Since, besides the geometric investigations, we also consider the problem from a global optimization point of view, some possible mathematical programming models will be included here.

2.1 Geometrical models

Informally speaking, the packing circles in a square and its related problems can be described in the following ways:

Problem 2.1. Place $n \geq 2$ equal and non-overlapping circles in a square, such that the common radius of the circles is maximal.

Problem 2.2. Place $n \geq 2$ points in a square, such that the minimum of the pairwise distances is maximal.

Problem 2.3. Place $n \geq 2$ equal and non-overlapping circles with the common radius in the smallest possible square.

Problem 2.4. Place $n \geq 2$ points with pairwise distances of at least a given positive value in the smallest possible square.

Of course, in order to investigate these problems and their relations in detail, we need their formal definitions and a consistent system of notation.

Formal description of Problem 2.1:

Definition 2.5. $P(r_n, S) \in P_{r_n}$ *is a* circle packing *with the common radius* r_n *in the square* $[0, S]^2$, *where* $P_{r_n} = \{((x_1, y_1), \dots, (x_n, y_n)) \in [0, S]^{2n} \mid (x_i - x_j)^2 + (y_i - y_j)^2 \geq 4r_n^2; x_i, y_i \in [r_n, S - r_n]\ (1 \leq i < j \leq n)\}$. $P(r_n, S) \in P_{\overline{r}_n}$ *is an* optimal circle packing, *if* $\overline{r}_n = \max_{P_{r_n} \neq \emptyset} r_n$.