Challenging Mathematics In and Beyond the Classroom

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Editors

Challenging Mathematics In and Beyond the Classroom

The 16th ICMI Study

International Commission on Mathematical Instruction

Springer

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Preface

In the mid 1980s, the International Commission on Mathematical Instruction (ICMI) inaugurated a series of studies in mathematics education by commissioning one on the influence of technology and informatics on mathematics and its teaching. These studies are designed to thoroughly explore topics of contemporary interest, by gathering together a group of experts who prepare a Study Volume that provides a considered assessment of the current state and a guide to further developments.

Studies have embraced a range of issues, some central, such as the teaching of algebra, some closely related, such as the impact of history and psychology, and some looking at mathematics education from a particular perspective, such as cultural differences between East and West.

These studies have been commissioned at the rate of about one per year. Once the ICMI Executive decides on the topic, one or two chairs are selected and then, in consultation with them, an International Program Committee (IPC) of about 12 experts is formed. The IPC then meets and prepares a Discussion Document that sets forth the issues and invites interested parties to submit papers.

These papers are the basis for invitations to a Study Conference, at which the various dimensions of the topic are explored and a book, the Study Volume, is sketched out. The book is then put together in collaboration, mainly using electronic communication. The entire process typically takes about six years.

The topic of the 16th Study was chosen in 2002, and we were appointed as Joint Chairs. Soon after, the IPC was selected and it met in Modena, Italy, in November 2003 to draft a Discussion Document. This was finalized at an IPC meeting in Copenhagen at ICME-10 in July 2004. The call for papers was then issued. About fifty people were invited to attend the Study Conference, which was held in Trondheim, Norway, at the beginning of July 2006.

Initially, the participants in the Study Conference were divided into two groups, examining challenges beyond the classroom, and in the classroom. About two-thirds wished to work on the second topic, so the second group was split in two, one focusing on the viewpoint of the student and the other on that of the teacher. Each group was convened by members of the International Program Committee.

Peter Kenderov convened Group 1: “Beyond the classroom”. Maria Bartolini Bussi, with assistance from Mark Saul, convened Group 2: “In the classroom: student perspective”, and Derek Holton convened Group 3: “In the classroom: teacher perspective”.

The groups refined the organization of the material, and then broke into teams to work on individual chapters. Chapters 1–3 are the result of the work of Group 1, Chapters 4 and 5 came from Group 2 and Chapters 6–8 from Group 3.

Each team selected a coordinating author to assemble the contributions and develop a coherent chapter. Communicating by email, each team was able to discuss the document as a whole before submitting it to the editors.

As a result, each chapter was written as a joint paper, even though it may be obvious at times that some sections originated from a particular author. In most cases, the first named author coordinated the writing.

During the editing phase participants were able to access a general working web site with the latest version of each chapter. Thus all were able to comment on any part of the volume throughout the editing process.

On very few occasions, writing teams co-opted work from outside the original group when it was felt to be particularly appropriate. The author list was then extended.

Each ICMI Study has its own evolution and character. In our case, the Study Volume consists of eight papers, each with its own reference list.

In a separate section, we acknowledge special individual contributions to the volume and some benefactors whose support has enabled authors to work on the Study.

References

As documents evolve in a modern technological age this volume contains many instant references to web sites. We have chosen to leave these in the text to avoid cross-referencing (in general). In many cases, normal references can be found on the Internet. They are shown with the last day of known access, in the normal way, at the end of a chapter. On other occasions a reference is not given, due to it being a relatively common term, which can easily be accessed by a direct Internet search.

School years

There is no international consistency in naming school years. In order to standardize our discussion we have generally adopted the K to 12 system used in a number of countries. In this system Year K is for students who have turned 5, Year 1 for students who have turned 6, up to Year 12 for students who have turned 17. In many countries this is the last year before normal university enrolment. We have generally adjusted all year level references to this system, but in places, to keep the original flavor we kept some French terminology. Where used, this means 6e is approximately Year 6, 5e is approximately Year 7, 2e is approximately Year 10, and Terminale is approximately Year 12.

June 2008

Edward J. Barbeau
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The Authors

A total of 45 authors have contributed to this Study Volume. The following is the list of authors together with the institutions to which they were attached at the time of proposing to participate, and their e-mail address at time of going to print.

Photographs and brief resumes of these authors can be found on the Study's web site. See www.amt.edu.au/icmis16participants.html for most participants and also www.amt.edu.au/icmis16ipc.html for members of the International Program Committee.

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Chapter 0
Introduction

Ed Barbeau

0.1 Challenging: a human activity

In modern society, mathematics is a prominent part of the school syllabus. It is praised for its utility and regarded as a foundation of our modern technological society. Yet school mathematics is the locus of much concern and criticism. Many leave school uncomfortable with, if not disdainful of, the subject.

Even those who get good grades may lack fluency and appreciation of its structure and significance. Mathematicians themselves may see little that reflects the character of mathematics as they experience it.

The blame for this situation is often laid at the door of the demands of a rigid syllabus and the imperatives of assessment.

We need to analyze these complaints. Mathematics is a highly structured subject. It is hard to see how one can proceed very far without orchestrating topics and assessing the mastery of its students from time to time to make sure they are prepared to make further progress. But does this mean that mathematical instruction should embrace so much mechanical learning and rely on recall and stock situations?

The issue is really one of ownership—who owns the mathematics?

Too often, the answer is "the system" or "the teacher". From the pupils' perspective, it seems imposed from without to achieve extrinsic goals. For many, it makes little sense. To be sure, mathematics can be difficult, but is it a difficulty one would want to surmount?

In the third book of the "Divine Comedy", Dante's pilgrim is advised by Beatrice that

> convienti ancor sedere un poco a mensa
> però che 'l cibo rigido c'hai preso,
> richiedi ancora aiuto a tua dispensa.

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> Apri la merte a quel ch'io ti paleso
> a fermalvi entro; ché non fa scienza
> sanza lo ritenere, avere inteso.

Or, in English

> you must stay longer at the table
> because the food you have eaten is tough
> and needs time for digestion.
>
> Open your mind to what I shall reveal
> and keep it there; whoever has heard
> and not retained, knows nothing.
>
> Paradise, Canto V, 37–42

This could be applied to the student of mathematics. She must learn to be patient, open the mind, and seek resilient rather than transient understanding.

These are qualities that require the learner to draw on internal resources. If the student is permitted to be passive, she may become alienated and resist accepting this responsibility.

The task of the teacher and expositor is to present the subject in such a way as to awaken these resources.

Danesi (2000) suggests the key to this conundrum. He notes that "puzzles have been around since the dawn of history" and that people have "been so fascinated by seemingly trivial posers, which nonetheless require substantial time and mental effort to solve, for no apparent reward other than the simple satisfaction of solving them". He asks,

> Is there a *puzzle instinct* in the human species,
> developed and refined by the forces of natural selection
> for some survival function? Or is this instinctual love
> of puzzles the product of some metaphysical force buried
> deep within the psyche, impelling people to behave in ways
> that defy rational explanation?

Danesi sees this propensity for puzzles as tightly bound up in human culture. There is a natural curiosity to be found in every human civilization; the environment poses wonders, threats and opportunities that must be understood and exploited by beings lacking many of the physical advantages of the beasts but endowed with a powerful mind. However, it is not just necessity but choice that leads people to accept challenges, as is borne out by the popularity of such recreations as Sudoku, topological puzzles and games of all sorts.

We should not overlook the social aspects of challenge that involve both cooperation and competition. In sports and the arts, the participant is motivated by a goal—a game or concert—and collaborates with others, a coach or conductor for expert guidance and colleagues for support and inspiration for the achievement of excellence.

The pursuit of the goal leads to the acquisition of new skills.

However, the goal must be commensurate with the abilities and characteristics of the group; one plays with teams of comparable skill or performs musical pieces whose technical requirements can be mastered with reasonable effort and expenditure of time.

Thus, growth is promoted through the presence of an appropriate challenge.

On the one hand, there is an aversion to the aridity of much school mathematics, and on the other a natural attraction to challenge. This Study is predicated on the possibility that the first can be counteracted by an exploitation of the second.

First, we discuss the fundamental question, "what is challenge?"

We shall examine sources of challenge and identify contexts in which expositors can introduce challenge. We will address the fundamental question as to whether challenges enhance learning.

This Study places the issue on the international agenda and has as its purpose to press for an answer to this question and to develop our understanding of the role of challenge.

0.2 Challenges and education

The tendency to see education in terms of formal institutions designed to meet societal goals has become very pronounced in many countries. We are told that children must be educated in order to fit them for various careers, in order that the nation becomes competitive and in order that economic success is achieved.

This is too narrow an outlook. Every society has provided some kind of initiation to prepare its young to accept the demands, responsibilities and privileges of adulthood.

In less sophisticated societies, this has tended also to provide children with a succession of socially useful tasks, so that they maintain a feeling of cohesion with the larger community. This is described in a very graphic way in the early chapters of Haley's novel *Roots* (1976) which recounts in considerable detail the first seventeen years of the life of Kunta Kinte as he is progressively integrated into the Madinka tribe in Juffure (now in Gambia) prior to his enslavement in the eighteenth century.

However, modern education often can separate children from the larger society and insulate children in their own world. So we should think of education in terms of three time frames.

Modern citizens may as a matter of course spend a third of their lives in some kind of school. Accordingly, we have to think of education for the present, a schooling that is immediately rewarding for the student in and of itself, a schooling in which the pupil experiences the joy of learning and growth as well as integration into the adult world through a broadening of interests and points of contact.

Properly handled, mathematics can be part of this. In many mathematical situations, children with their curiosity and mental agility are in a position of equality with adults. In particular, mathematical challenges become not only a way in which they can feel intellectually alive and productive, but also something that can be shared outside of their own age group.

To be sure, schooling must allow students to function as citizens and employees, and to provide opportunities for work that is appropriate and productive (in whatever sense we want to use this word) for the individual. This is education for the near future that will open doors and prepare for the next stage of life. It can be claimed that exposure to mathematical challenges can support the resourceful flexibility of thought and deeper mathematical understanding that will make the preparation of students more successful.

But there is more to life than a job; many seek fulfillment through other activities and jobs. This is often movingly portrayed in descriptions in the press of rich lives of some quite ordinary people who have suffered some disaster.

In particular, through schooling we produce new generations whose interest and curiosity will see that the arts and sciences and all such tokens of human excellence do not vanish, but transcend our individual mortality. This is education for the far future.

This dichotomy between the transitory nature of human existence and the nobility of human achievement is strikingly expressed in the two answers that the Old Testament provides for the question, "What is Man?"

> Man is like to vanity: his days are as a shadow that passeth away.
>
> Ps. 144:4

> Thou has made him a little lower than the angels, and has crowned him with glory and honor. Thou madest him to have dominion over the works of Thy hands; Thou has put all things under his feet.
>
> Ps. 8:5–6

The lesson of the first is to redeem each moment in education, and of the second, to plunge students into the broad stream of civilized achievement and development.

As we shall see in this volume, mathematical challenges have a place in all aspects of education and intellectual growth, whether in a formal school setting or through informal means such as clubs, museums, books, magazines, games and puzzles, or the Internet.

The history of mathematics is a long saga of great thinkers pushing the bounds of their knowledge by formulating and solving problems. It seems clear that they exulted in their growing mastery of ideas as step by step they progressed from the most basic ideas in number and geometry to the magnificent edifice that we know today as mathematics.

This ICMI Study is predicated on the premise that we can duplicate some of this sense of entitlement and mastery among the public, both within and without school, by the use of mathematical challenges.

0.3 Debilitating and enabling challenges

Already, mathematics often challenges both school children and the general population, but for reasons that often discourage and alienate.

Difficulties may result from poor curricular and pedagogical design that shroud in mystery what should be clear and overwhelms with tedious detail what should be surveyable.

Schools may be afflicted with teachers who themselves are uncertain of their mathematics, or whose mathematical training ill equips them to anticipate possible stumbling blocks to learning even basic mathematics.

Sometimes, students get locked into a conceptualization that is not at all productive, so that they cannot move beyond their frustration.

Is it possible for students to learn mathematics and the general public to understand it if they must accept someone else's formulation of it? Or should we bring in lay people as participants, providing a direct opportunity to grapple with the ideas and devise their own patterns of thought?

If the latter, is it through the posing of challenges, problems and investigations, that the learner is brought into a collaboration with the teacher, each alive to his own particular responsibilities?

0.4 What is a challenge?

For the purpose of the Study, we will regard a challenge as a question posed deliberately to entice its recipient to attempt a resolution, while at the same time stretching their understanding and knowledge of some topic. Whether the question *is* a challenge depends on the background of the recipient; what may be a genuine puzzle for one person may be a mundane exercise or a matter of recall for another with more experience.

Furthermore, a challenge may or may not be appropriate. An inappropriate challenge is one for which the background of the recipient is so weak that she may not understand what is at stake or does not possess or is unable to create the tools needed to engage with it. A good challenge is one for which the person possesses the necessary mathematical apparatus or logical skill, but needs to use them in a nonstandard or innovative way.

A good challenge will often involve explanation, questioning and conjecturing, multiple approaches, evaluation of solutions for effectiveness and elegance, and construction and evaluation of examples.

Following Danesi (2002), we can expect a latent willingness for people to accept challenges provided a suitable stimulus can be devised. This is an optimistic message which, as we shall see in the following chapters, has already been validated in many individual situations across the globe. Indeed, we have seen enough instances that we can analyze what is likely to be effective and what is likely to be counterproductive.

We see in *mathematical challenge* an idea that will revitalize discourse about the role of mathematics in the educational culture. In schools, it may help to equip students to face future challenges in life by fostering desirable attributes such as patience, persistence and flexibility, to learn content more richly and exploit connections, to identify and develop their mathematical capabilities, to become self-actualized and confident, to experience the pleasure of engagement and the joy of success and to participate in a community of learning.

We can see through challenge a kind of relationship between a learner and a learning opportunity, mediated by the engagement of the individual. In the diagram below, we list the ingredients in the sphere of the learner alongside activities that could be in the learning environment that affect them.

LEARNER		LEARNING OPPORTUNITIES
development		exploring
cognition	E	inquiring
knowledge	N	problem solving
metacognition	G	developing, valuing, comparing, evaluating multiple approaches
beliefs	A	explaining
motivation/beliefs	G	generating ideas and questions
values	E	conjecturing
desire to learn and succeed	M	constructing forms
	E	judging effectiveness and elegance
	N	
	T	

The opening chapter provides many examples of challenges. We look at where they come from, what they are about and what makes them work. It is hoped through this chapter to indicate to the educator or teacher in mathematics who may not have an advanced background in the subject the scope of challenges, and to encourage them to think through some of these themselves as a way of honing their ideas about their possible use with students and the public.

We emphasize that good challenges are like good musical compositions or poems; they infiltrate into the broader culture and are passed down from one generation to another.

Some will have broad appeal, while others will be treasured only by the cognoscenti.

The second chapter studies challenges beyond the classroom. Over the last century, particularly in the last fifty years, many different occasions for being challenged mathematically have been introduced, from newspapers and magazines, and an ever-increasing variety of competitions, to the work of artisans in the creation of topological and other puzzles (the Rubik's cube being a notorious example), as well as to mathematics “museums”, clubs and web sites. The scope of these is examined and several examples are studied in detail.

Of course, no examination of this topic would begin to cover it without acknowledging the intensive intervention of technology. Technology not only serves to augment the effectiveness of traditional resources, such as books and journals, lectures and schools, but it also provides a handy and extensive library of information and problems; it provides electronic tools to aid learning, experimentation and understanding on a scale impossible to conceive of until now.

Its novelty and power has made possible brand new interactive programs and the capacity for in-depth investigations in number theory, combinatorics, probability and geometry by students. With the modern computer, new areas of mathematics have been opened out, such as dynamical systems and stochastic processes, and some of this is accessible to students. Moreover, the Internet has made it convenient for mathematicians and students to collaborate easily.

Chapter Three takes stock of these developments, sorting out what is available, assessing how they relate to existing methods of pedagogy and dissemination, analyzing the issues at stake, and describing ways in which they can support the use of challenge.

In Chapter Four, the promotion of mathematical learning through the use of challenges is examined through several case studies. The role of the teacher is critical, for she must seek out appropriate material, calibrate and orchestrate the challenges given, ensure that students are properly prepared to meet them, analyze what makes them successful and see that the classroom environment is conducive to an effective experience.

Chapter Four touches briefly on an area that this author feels has not been given due attention—the use of mathematical fallacies. Two decades ago, he persuaded the editors of the *College Mathematics Journal* to initiate a department, “Fallacies, Flaws and Flimflam”, devoted to the collection of flawed mathematical proofs and solutions in the hope that this might be of use to teachers. His own experience suggests that it can be a serious problem for students to troubleshoot a flawed argument, that the attempt to do so may lead to an appreciation of some rather subtle mathematical points, and that students come to appreciate the need for care. Indeed, the reader may have been in the position of marking a student solution, “smelling a rat”, but being hard pressed to winkle out the error, for good students can make interesting mistakes. Even “howlers”—manifestly incorrect or inappropriate techniques that lead to a correct answer—can lead to a fruitful investigation into the situations for which they may actually “work” and the reasons for this. One source of such material is Movshovitz-Hadar and Webb (1998).

In Chapter Five, the emphasis is on the cooperative facing of a challenge by some community of students and their teachers or mentors, whether it be in a mathematics laboratory, special schools, a school assembly, a classroom or a jamboree where teams compete in the consideration of an experiment or research problem.

The success of any educational regime in the schools depends on a well-prepared corps of teachers. If the use of challenges in schools is to be successful, then the formation and professional development of teachers needs to be suitably reformed. Teachers need to be convinced that what they are required to do authentically represents mathematics and is of lasting benefit to the student. They must be prepared to reassess the way in which they interact with the students.

Most importantly, they must come to see themselves as practitioners of mathematics, sharing their own experience and joy of mathematics. Thus, in their training, teachers need to grapple with mathematical challenges themselves, so that they know how to support their students and can model good mathematical behavior. Such considerations are the burden of Chapter Six, which opens with a discussion of the nature of challenges in school and why they are important. Some examples of challenges, their design, and the responses they elicit are given.

Psychological considerations are important. What is it that might prevent teachers from using challenging problems? What effect does their knowledge and beliefs have on their willingness and ability to handle challenging problems? What can be said about the development of the brain? The chapter next deals with the factors that lead to effective pedagogy and the role of professional development.

Finally the chapter provides a description of some pre-service and in-service programs. In China, the new curriculum that promotes the use of challenging problems has resulted in a Shanghai study, “Teacher Action Education”, to promote the effectiveness of teachers in adopting the spirit of the reforms and achieving student learning goals.

In Münster, Germany, cadet teachers directly experience mathematical challenges and have to reflect on their own mental processes; their formation involves teamwork, critical discussion of videotaped lessons and research, resulting in the production of educational materials. Similarly in New Zealand, a numeracy development project brought about extensive professional development.

Likewise, at Northern Kentucky University in the USA, pre-service teachers take a course that embraces work on challenging problems alone and in groups, with sharing of solutions and preparation of examples to be used in schools.

Chapter Seven follows this up with a look inside the classroom. The issue is one of priorities, as teachers have to make sure that they cover the syllabus and prepare students for various tests. Nevertheless, it is argued that challenges should and can be part of the classroom experience. However, in many situations this goes against the grain of the existing culture. Teachers need to examine their own attitudes and be prepared to interact with the pupils in a less authoritarian way.

Those responsible for curriculum design and assessment need to question whether their policies inhibit or promote an authentic and productive mathematical experience in the classroom. This is the theme of Chapter Eight in which assessment issues in the framework of challenge are considered in four countries, and assessment is evaluated as to its role in promoting learning and its implications for curriculum and assessment. The chapter concludes with some research questions needing further investigation.

While challenges have always been part of mathematical exposition in some small way, they have now come to the forefront in our conception of classroom practice and public exposition. This Study has intervened at a time when there has been a lot of activity and experience that can be assessed. The time has come for a gathering of the available materials and the formulation of research and field trials involving the use of challenges that will allow us to move forward in a sound and measured way.

I am particularly indebted to Jean-Pierre Kahane, Roza Leikin, Ralph Mason and Peter Taylor for contributing perspectives that informed this chapter.

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Chapter 1
Challenging Problems: Mathematical Contents and Sources

Vladimir Protasov, Mark Applebaum, Alexander Karp, Romualdas Kašuba, Alexey Sossinsky, Ed Barbeau, and Peter Taylor

This chapter gives many examples of challenging problems, categorized according to the settings in which they occur, both in and beyond the classroom. Some challenges arise as extensions of normal classroom practice; other challenges, some of very recent origin, become popular among the general population, while still others are created especially for contests for different populations of students. In some cases, a detailed discussion of their origins and uses is provided. It should become clear to the reader the extent to which they betoken the vitality and the creativity of the mathematical enterprise and show how much of a "human endeavor" mathematics is. The final section provides a summary of the place of context and content in the creation and use of challenges. This chapter focuses on challenges given in the form of individual problems; challenges that occur in extended investigations are mentioned briefly in the final section and will be considered in later chapters.

1.1 Introduction

Challenges in mathematics are not new. Once people began to observe numerical and geometric patterns and sought to account for them, difficult problems emerged naturally to challenge their wits and to force them to organize their knowledge and explain the underlying concepts more precisely. During the Renaissance, mathematicians who had discovered a new technique might show off their knowledge by posing a challenge problem that others, not privy to their strategy, could not solve. In the Enlightenment, prizes were offered for those who could make some progress on some pressing mathematical problem of the day.

In modern times, the first challenges aimed specifically at students were posed in magazines and in competitions, with Hungary being an early leader. These

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challenges were created especially for this purpose, so that it became a battle of wits between the mathematicians who set them to confound the students, and the students who might arrive at a solution, often in an unanticipated way.

However, as the popularity of games like bridge, topological puzzles, and Sudoku and other puzzles in the popular press indicate, there is a taste for challenges of a mathematical type among non-mathematicians as well. Indeed, in almost every culture there is an ancient tradition of posing mathematical problems.

It is therefore plausible that both tradition and culture support the introduction of challenges both into the classroom and into public events to enhance the appreciation of and facility in mathematics among both students and the general population.

However, the mathematician or teacher who introduces challenges into an environment must be aware of the particular circumstances. In the classroom, it is important to be as inclusive as possible, while for extracurricular activities, where participation is voluntary, the educator cannot force anyone to take part and must select material carefully to ensure success.

As we shall see in the final section of this chapter, the research literature has devoted insufficient attention to the issue of the appropriate selection of challenges. The aim of the next four sections is to encourage educators to engage and contribute to the discussion of challenges and their use. We provide a set of problems used in a variety of situations and earnestly encourage the reader first to try them, then reflect on the thought processes that they evoke, what they convey about the nature of mathematics and how different types of recipients might respond.

A good challenge can be thought of as a work of art, similar to a poem, a musical composition or a painting. All of these are of no account without an appreciative audience. Accordingly, the creativity and elegance of a challenge should be matched by the discernment to calibrate it so that the challenge entices but does not overwhelm those with whom the challenger wishes to make communion.

1.2 Challenges within the regular classroom regime

Quite a bit of time in the normal classroom is spent in teaching standard results and techniques, and applying them to stock situations. Understandably, pupils may not appreciate the significance of the work nor retain and use it effectively later.

Accordingly, it may be useful to adapt a mundane question to make pupils think more deeply about structure of this mathematics and shine a light on its salient features. Here are some examples:

Challenge 1.2.1 (Ages 6 to 8): Six-year-old Danny is discussing even and odd integers with his father, a research mathematician and a talented teacher. Danny has learnt what even and odd integers are: “It’s an even number of people when they can split up into pairs and no one will be left out, and an odd number if one person doesn't have a partner to hold hands with."

Danny is asked to prove that the sum of two odd numbers is an even number. After some hesitation and mumbling, his face suddenly lights up and he cries out: "Of course, of course, the two people who didn't have a partner, they find each other, they start holding hands, so the number is even!"

Discussion: If it can be said that one purpose of education in general is to encourage the child to be aware of and discriminating towards the world around her, this is particularly true in mathematics. Necessarily, the mathematics syllabus covers a lot of concepts and procedures; it is a hazard that pupils can fall into a mechanical mode, reciting definitions and performing operations while unaware of the significance and utility of what they are doing.

We need to insinuate into the situation something to increase awareness and fluency. The posing of questions designed to make a pupil pause and take stock is one way to do this.

The reader is invited to formulate her own response as to why the sum of two odds is even, and then think about situations in which this can be asked and how one might expect pupils of various ages to respond.

Two aspects stand out. First, in order to answer the question, the responder needs a workable definition of "even" and "odd". By workable, we mean something that can be appealed to. What would be the understanding of a young child? Perhaps it might be that odd and even numbers alternate in counting, putting 1 in the odd pile, 2 in the even, and so on. How would a child with this viewpoint tackle the question? Perhaps the criterion is the remainder upon division by 2. Would an eight-year-old work with this definition? A secondary student with some knowledge of arithmetic modulo 2 might use this approach. Or would children be likely to follow Danny in pairing off?

The second aspect is that the pupil is being asked to prove a general proposition with infinitely many instances. Normally, one thinks of proofs as occurring much later in the school career, perhaps not until high school geometry. Is a child of Danny's age likely to be "up to" handling such a proof?

That the sum of two odds is even might be accepted through empirical observation—every example shows that it is true. In the same way, every robin has a red breast. Some children might not progress much further than this. But this particular challenge puts on the table a new and different aspect of mathematics. Accompanied by similar future challenges, a child's perception of mathematical truth and the power of reasoning to illuminate what can never be checked directly will be deepened.

Thus, we see how a challenge can originate from the desire of a teacher to induce her charges to probe more deeply into the mathematics.

It is this feeling of elation (Danny's response) that one should try to induce in setting challenges before classes.

Challenge 1.2.2 (Ages 9 to 11): Take the digits 1, 2, 3, 4, 5 and 6. Using each exactly once, form two numbers for which the difference is as small as possible.

Discussion: Generally, children master skills by working examples provided by the teacher. But one might better create a sense of ownership by asking children to create examples of their own. To begin with, a teacher might simply ask pupils to form two numbers from the six digits and find their difference. However, by introducing the optimization question, she provides a goal that induces the pupils to look more carefully into the ingredients of the situation and pay attention to what factors govern the size of the difference.

There are a number of realizations that students will reach, probably implicitly. The first is that to make the difference small, the numbers chosen should both have three digits. The teacher may wish to draw out explicitly why this is so.

The second is that the leading digit of the larger number (minuend) exceeds the leading digit of the smaller (subtrahend) by 1. The third is that the digits after the leading one of the subtrahend should form as large a number as possible while the digits after the leading one of the minuend should form as small a number as possible. The answer is given by $412 - 365 = 47$.

This is an example of what a teacher can do with a simple computation. The set of digits could be varied and both the largest and smallest sums, differences, products, quotients and exponentials formed by two numbers using those digits can be found. Another nice problem that lends itself to group work is to take the ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 and use them to construct three numbers, the largest of which is the sum of the other two. What are the smallest and largest such sums?

1.2.1 Challenge from observation

Sometimes a challenge can be built on an interesting observation. There are likely many pupils who have noted with interest that the sum and product of two 2s are the same.

The teacher can exploit this by asking pupils to find other examples of pairs of numbers that have the same sum and product. The numbers involved can be just positive integers, or rationals.

Here is another way to generalize the property:

Challenge 1.2.3 (Ages 10 to 15): Determine all possible ways of finding two pairs of positive integers such that the sum of each of the pairs is equal to the product of the other.

Discussion: Take a few minutes to think about this "two pairs" problem. Do we need to know something about the relative sizes of the sum and the product of two positive whole numbers? How easy is it to generate an example? How would we know that we have a complete set of examples?

In tackling this challenge, one makes the key observation that for one of the pairs, the sum must be at least as great as the product. It may be that some children have never considered this possibility, having dealt only with situations for which the opposite is true. The number 1 might not have been used as a multiplier. We can see the possibility of counteracting a bias that multiplication always makes things larger, a prejudice that can later confound a child's handling of products of numbers other than integers.

Pre-algebra students may solve this problem informally and intuitively using trial and error. With algebra, a more systematic attack is possible. The condition that $a + b \geq ab$ can be converted to $(a - 1)(b - 1) \leq 1$, which for unequal positive integers a and b implies that one of the variables is equal to 1. Or one can draw from the conditions $a + b = cd$ and $ab = c + d$, the equation $(a - 1)(b - 1) + (c - 1)(d - 1) = 2$. The only possibilities are $\{(2, 2), (2, 2)\}$ and $\{(2, 3), (1, 5)\}$.

The challenge can be extended. What happens if we allow negative integers? Or we could ask that the product of each pair is twice (or some other multiple of) the sum of the other.

1.2.2 Challenge from a textbook problem

Sometimes a challenge can be made from a standard textbook problem that will encourage students to look beyond a merely algorithmic approach to a more holistic stance towards a problem.

Challenge 1.2.4 (Ages 11 to 14): (This problem appeared in a first algebra text.) A man is standing in a theatre line. 5/6 of the line is in front of him and 1/7 of the line is behind him. How many people are in the line altogether? Without setting up an equation, argue that the answer must be 42.

More generally, we can pose this situation. A man is standing in a theatre line with the fraction x of the line in front of him and the fraction y behind, where x and y are fractions written in lowest terms. If x and y are such that the problem makes sense, then that answer must be the least common multiple of the denominators of x and y.

Discussion: This problem was originally posed in a graduate course on problem solving for practicing secondary teachers, as part of a discussion on the creative use of textbook exercises. Is it clear that the answer must be 42? Would the reader have realized this without being prompted? Certainly, one can set up an equation and solve it; the exercise came from a text chapter on this very topic. But does the algebraic formalism reveal aspects of the situation that are worth noting?

The key observation is that, in the original problem, the number of people in the line must be divisible by 6 and 7, so that the numbers before and behind are integers. So the total number of people is a multiple of 42. Why must it be 42 itself, rather than a larger multiple? Expressing the reason clearly and completely is an expository challenge for even the brightest students.

1.2.3 Increasing fluency with fractions

While we are on the topic of fractions, there are some challenges that can be used to help students become more fluent with them and gain some understanding of inequality relations among them.

A good area for this Study is that of Egyptian fractions, those whose numerators are 1.

Challenge 1.2.5 (Ages 11 to 15): Solve the following equation for the natural numbers x, y, z:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1.$$

Discussion: There is an obvious solution for this: $(x, y, z) = (3, 3, 3)$. Some pupils may also know about $(x, y, z) = (2, 3, 6)$. The challenge here is to get an exhaustive set of solutions.

The first task in meeting this challenge is to reduce the level of complication by taking advantage of symmetry. Without loss of generality, we can assume that $x \leq y \leq z$. Making this additional assumption is a mathematical gambit that would probably not be taught as a regular part of the syllabus, but one does not require much experience in problem solving to see how such assumptions become standard.

Ordinary syllabus problems also tend to be directive in the sense that the student follows a standard procedure and goes directly after the solution. This particular challenge illustrates that it is often a good idea to get an overview of the situation before beginning the grind of hunting for solutions. We note that the integers required cannot all be very large. In particular, if all the integers exceed 3, the left side must be less than 1. Also it is clear than $x > 1$.

So we have to consider two cases: $x = 3$ and $x = 2$. The first yields only one possibility. The second requires that $y = 3$ or $y = 4$, so we can now list all the solutions, the two mentioned above and $(x, y, z) = (2, 4, 4)$.

If students become interested in this challenge, it can be generalized to having any number, say n, of unit fractions on the left. If there are n fractions on the left, we might ask how large the largest denominator can be.

Challenge 1.2.6 (Ages 11 to 18): Determine all the integers n exceeding 2 for which there exist distinct positive integers x, y, z such that

$$\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}.$$

Discussion: It is a conjecture of Paul Erdös that $4/n$ admits such a representation for all $n \geq 3$. This problem works well with secondary students in particular because most cases can be handled within a short space of time.

Like many challenges, progress depends on making a key observation, in this case that

$$\frac{1}{k} = \frac{1}{k+1} + \frac{1}{k(k+1)}.$$

Many students discover this after playing around a bit. This allows us to dispose immediately of the case that n is an even number: i.e. we can write $n = 2m$. Thus,

$$\frac{4}{n} = \frac{2}{m} = \frac{1}{m} + \frac{1}{m} = \frac{1}{m} + \frac{1}{m+1} + \frac{1}{m(m+1)}.$$

It turns out that generally within an hour a class can arrive at a set of cases that cover every number n that does not leave a remainder 1 when divided by 24. This is a nice example that illustrates how one can make some progress on a natural mathematical question, yet find that some parts of the question remain intractable.

1.2.4 Engaging with algebra

For many secondary students, the study of algebra is quite tedious. The fact that nowadays a large percentage of the population studies high school mathematics and many of these find frustration with algebra probably accounts for its downgrading in the syllabus.

However, the result is that many students do not engage in algebra to a degree that permits fluency and ability to make actual use of it. Without this ability, it is easy for students to see their studies as pointless.

Some students have no desire to study technical mathematics, and, if their ambitions lie in another direction, they should not be penalized for avoiding it. The experience of the remaining students can be enriched if we restore to the standard syllabus more challenging material.

A standard complaint about many students of mathematics is that they tend to be driven by formulae and do not appreciate the structure behind a formula. Factoring polynomials is one way to ameliorate this deficiency.

There is a "trapdoor" aspect to factoring. It is a mundane task to expand the product of two polynomials; to factor this product into its irreducible components requires a range of skills and sensitivity to structure that will help students mature as mathematicians. Here are a few examples:

Challenge 1.2.7 (Ages 13 to 17): Factor

(a) $4(a^2+b^2) + 21b^2 - 20ab - 36$;
(b) $6x^2y - 15y - 5x + 18xy^2$.

Discussion: While there is no grand scheme that allows us to factor any polynomial, nevertheless there are some rules of thumb that students will pick up from experience and allow them to factor with increasing success. Success will also depend on recognizing certain forms and analyzing the structure of the polynomial to be factored. For example, in (a), the student might combine the terms in b^2 and note the difference of two squares, of a linear polynomial and of 6. In (b), one might isolate the terms of like degree and pull out common factors.

We can observe an analogous situation in elementary calculus, where differentiation of functions can be handled by a set of easily learnt rules, but integration requires more skill and judgment.

Challenge 1.2.8 (Ages 13 to 17): Factor

(a) $a^{10} + a^5 + 1$;
(b) $2(x^5 + y^5 + 1) - 5xy(x^2 + y^2 + 1)$.

Discussion: Part (a) is a nice challenge in which the "breaking down" approach is unlikely to lead to success. A good start is the observation that the polynomial has the form $x^2 + x + 1$, and recognizing that this in turn is a factor of $x^3 - 1$.

So we build up to factor $a^{15} - 1$, which has the given polynomial as a factor. However, this binomial is not only a difference of cubes but also a difference of fifth powers, so that it can be factored according to two different strategies:

$$a^{15} - 1 = (a^5 - 1)(a^{10} + a^5 + 1) = (a^3 - 1)(a^{12} + a^9 + a^6 + a^3 + 1).$$

At this stage, the students need to understand the significance of the result that every polynomial is uniquely given as a product of irreducibles. In particular, $a^2 + a + 1$ is an irreducible factor of $a^3 - 1$, and so must be a factor of $a^{10} + a^5 + 1$.

Indeed, the required factorization is

$$(a^2 + a + 1)(a^8 - a^7 + a^5 - a^4 + a^3 - a + 1).$$

An alternative approach recognizes that all the roots of the given polynomial are those 15th roots of unity that are not 5th roots of unity and that two of its roots are imaginary cube roots of unity.

Part (b) provides a challenge of a different sort. In this case, one can work from the symmetry in x and y. Making the substitution $s = x + y$ and $p = xy$, we find that the polynomial can be rendered as

$$2[s(s^4 - 5s^2p + 5p^2) + 1] - 5p(s^2 - 2p + 1) = 10p^2(s + 1) - 5p(2s^3 + s^2 + 1) + 2(s^5 + 1),$$

which has $s + 1$ as a factor. The desired factorization is

$(x+y+1)(2x^4-2x^3y+2x^2y^2-2xy^3+2y^4-2x^3-x^2y-xy^2-2y^3+2x^2-xy+2y^2-2x-2y+2)$.

Challenge 1.2.9 (Ages 15 to 19): Find the smallest possible value of

$$f(x) = \cos 2x - x\cos x + x^2/8 \text{ for all real } x.$$

Discussion: Here is a challenge designed to help rid students of the habit of blindly following an algorithm without paying attention to any special characteristics of the situation. One can see how the problem was created using a square involving cos x and then made more mysterious by converting $\cos^2 x$ to its equivalent involving cos $2x$.

This is a "wolf in sheep's clothing" sort of challenge. The student tries a standard derivative approach and gets into a mess. How can this be avoided? Noting the middle term, a mixture of x and cos x, one might recall that cos $2x$ can be written in terms of $\cos^2 x$ and see if we can get a perfect square somewhere. Indeed

$$f(x) = 2(\cos x - x/4)^2 - 1.$$

1.2.5 Pedagogies to help development

The foregoing classroom challenges indicate how one can start with straightforward material, and by providing either a twist or asking a natural question, can help authenticate the mathematical experience. In addition, for this to be effective, we need a pedagogy that does not force students to engage with these on their own unsupported, as has so often happened in the past.

Students can often be asked to work in groups so that they can share ideas and can be allowed time to reflect on the problems, knowing that they cannot always be expected to answer questions immediately. Any decision to introduce challenges in the classroom also requires that the whole system of teaching and assessment be reviewed so that different aspects of the classroom experience are not working at cross purposes.

1.2.6 Combinatorics

In addition, there are problems not immediately connected with the syllabus that can be used in the classroom to good effect.

Combinatorics is an area in which little specific background is needed for some situations and children can be expected to be as successful in meeting a challenge as an adult. We discuss a few possibilities.