Mathematical Lives

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Editors

Mathematical Lives

Protagonists of the Twentieth Century
From Hilbert to Wiles

Translated by Kim Williams

Springer

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Preface

Mathematical knowledge is increasing at a dizzy rate. In the course of the last 50 years more theorems have been proven that in the preceding thousand years of human history: to give an idea of the order of magnitude, every year in specialized journals alone tens of thousands of research articles are published, and just as many more are made available on the Internet. Even given that the great part of these results are understandable and interesting only to specialists, others represent fundamental intellectual conquests, solving irksome problems or famous conjectures, establishing unexpected connections between various theories or discovering new horizons for research. Furthermore, in many cases these steps forward in mathematics, even those of seemingly limited importance, reverberate in other scientific disciplines, giving rise to innovative conceptual developments or finding surprising technological applications.

Only weak echoes of this fervid intellectual activity reach the general public. The newspapers might carry the news of Andrew Wiles's proof of Fermat's last theorem, or the contorted events surrounding the solution of Poincarè's conjecture by Grisha Perelman, but aside from the sporadic cases mathematics remains more or less ignored. Thus, ironically, in precisely the period of its most florid growth mathematics appears at once extremely fragile, almost a victim of its own excesses of specialisation, relegated to a secondary role in the science of our culture, indeed – in the opinion of the most pessimistic – at risk of extinction as a science in its own right. A few years ago, Gian Carlo Rota commented, "at the end of the second millennium, mathematics seriously risks dying. Among the many threats to its survival, those that loom the largest seem to me to be the crass ignorance of its results, and the widespread hostility towards its practitioners. Both of these are facilitated by the reluctance of mathematicians to push themselves beyond the restricted confines of their own discipline and by their reluctance to translate the esoteric contents into exoteric slogans, which is instead imperative in the age of means of mass communication and public relations".

Whether or not one agrees with these gloomy prophecies, the fact remains that it is not at all easy to coin "exoteric slogans" in order to render the hard-to-digest abstractions of mathematics appetizing to the largest possible number of palates. Physics, biology, and even chemistry can take advantage of concepts that are certain to be attractive – the "secrets of the universe", the "wonders of life", the "mysteries of the molecule" – which, no matter how many times they are served up, still have a grip on the collective imagination (if we can use that expression) and can be used as a point of departure even for works of serious and rigorous popular science. But what are the secrets, the wonders, the mysteries unveiled by mathematics, if not those that appear as such, in all their fascination, only to the eyes of those trained in this discipline?

In an attempt to illustrate the richness of the mathematics of the twentieth century without resorting to slogans or propaganda, the present volume has a new approach: to bring to the forefront some of the protagonists of this extraordinary intellectual adventure, who have put at our disposal new and powerful instruments for investigating the reality around us. There are at least two distinct reasons for making this choice. Above all, the desire to give credit where credit is due. Little has been written on the people – men and women – whose ideas have made possible such deep scientific changes, and they run the risk of remaining in the shadows along with their results. Although many have heard of Russell, Gödel, von Neumann or Nash, how many know about Emmy Nöther, Schwartz, Grothendieck or Atiyah? Secondly, the desire to demonstrate the falsity of a widespread and deeply-rooted belief. It is often held that mathematicians are in every way similar to the extravagant personalities that populate the flying island of Laputa in the Swift's *Gulliver's Travels*. You'll recall that the inhabitants of this land are so lost in mathematical and musical thoughts and concoctions that they can neither talk nor follow anyone else's discussion, and constantly risk falling off some cliff or banging their heads against some obstacle. For this reason they are always accompanied by servants to rescue them, who capture their Masters' attention by touching them on the lips, ears or their eyes with a kind of rattle tied to the end of a stick. Nothing could be further from the truth: mathematicians, bizarre as their behaviour might sometimes appear, have no need at all of solicitous servants to bring them back to reality, because in general their curiosity is vigilant and open to the multiplicities of the world. Many of the portraits contained in this volume present people with strongly charisma, with wide ranging cultural interests, impassioned about defending the importance of their own research, sensitive to beauty, attentive to the social and political problems of their times.

In spite of the inevitable omissions (which we openly knowledge, but as Marcel Schwob observed in the preface to his *Imaginary Lives*, "the art of biography consists precisely in choice"), what we have sought to document is mathematics' central position in the culture – and not only scientific – of our day, in a continuous play of exchanges and references, and correspondences and suggestions. For this reason, in the pages that follow we have made space for not only biographical portraits of the great mathematicians but also for literary texts, which allow us to glimpse this subterranean contiguity. We have even included two intruders (or so they appear, at least at first glance) – Robert Musil and Raymond Queneau –, authors for whom mathematical concepts represented a valuable auxiliary for investigating the modalities of the “new relationship between the phantasmatic lightness of ideas and the weight of the world” (to quote Calvino), to resolve the disagreement between “soul and precision”.

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Editors' Note

The present volume is based on number 50–51 (December 2003–March 2004) of the journal *Lettera matematica PRISTEM*, with modifications, amplifications and significant additions.

We are grateful to Maria Poggi and Geraldine d'Alessandris for their nice and expressive drawings.

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Umberto Bottazzini

In an isosceles triangle, if the ratio between the base angle and the vertex angle is an algebraic but irrational number, is the ratio between the base and the side always transcendent?

The simplicity of this question is deceiving. This is not an exercise in Euclidean geometry that can be solved by a bright student, but is instead a translation into geometric terms of the fact that the exponential function exp(iπz) must always be a transcendental number for irrational algebraic values of z. David Hilbert thought that this was "highly probable", although providing a proof of it seemed to be an "extremely difficult" undertaking. Thus he added it to the list of problems for "future generations" that he presented in Paris on 8 August 1900 during the second International Congress of Mathematicians.

"Who of us would not be glad to lift the veil behind which the future lies hidden; to cast a glance at the next advances of our science and at the secrets of its development during future centuries?", exclaimed Hilbert at the beginning of his talk. "What particular goals will there be toward which the leading mathematical spirits of coming generations will strive? What new methods and new facts in the wide and rich field of mathematical thought will the new centuries disclose?"

It was a unique moment. The Congress, on the cusp of two centuries, offered the mathematician from Göttingen a chance to "look over the problems which the science of today sets", and invite the mathematicians of "future generations" to put themselves to the test. His talk defined an epoch. However, for those who imagine Hilbert reading his talk, soon to be become legend, to a hall filled with the most authoritative mathematicians of the times, reports about the congress contain some surprises. According to Gino Fano, the audience was not actually very large. Many of the participants didn't attend. There were about ten Italians: Peano and his followers (Amodeo, Padoa, Vailati), a couple of high school teachers, and then Levi-Civita and Volterra, who gave the opening address. Of the Germans, neither Klein nor Nöther were present, nor were any of the mathematicians from Berlin. Even among the French, leading mathematicians such as Hermite, Picard, Jordan, Goursat, Humbert and Appell failed to attend the Congress sessions. Hilbert's talk was one of those in the section entitled *Bibliographie et Histoire. Enseignement et*

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methodes with historian Moritz Cantor presiding. Hilbert limited himself to presenting about 10 of the 23 problems that appeared in the text prepared for publication. Thus, the volume of the acts of the Congress contains a note saying that “an enlarged version of the talk of Mr Hilbert, because of its great importance, has been included among the lectures”.

David Hilbert

Hilbert’s observations regarding methodology in the introduction to the problems shed light on his conception of mathematics and its development. “The deep significance of certain problems for the advance of mathematical science in general and the important role which they play in the work of the individual investigator are not to be denied”, he said, and continued, “An old French mathematician said: ‘A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street’. This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem . . . [it] should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts”. The failure to solve a problem often depends on “our failure to recognize the more general standpoint from which the problem before us appears only as a single link in a chain of related problems”. Once the right level of generality is found, not only does the problem show itself to be more accessible, but often the right methods to solve problems related to it also appear. An unlimited faith in the capacity of human reason led Hilbert to formulate a kind of “general law” for our thinking, to establish a kind of axiom that it was possible to find a solution for any mathematical problem whatsoever. "In mathematics there is no *ignorabimus*", he stated optimistically – perhaps *too* optimistically – in defiance the notorious statements by Emil Du Bois-Reymond.

Among the classic problems, Hilbert noted Johann Bernoulli's brachistochrone problem, which had led to the birth of the calculus of variations, and Fermat's last theorem, which gave rise to Kummer's theory of ideal numbers and their generalisations to all algebraic fields through the work of Dedekind and Kronecker. The three body problem, which in recent times had led Poincaré to the discovery of "fruitful methods and far-reaching principles", was of an entirely different nature. For Hilbert, as mathematical problems Fermat's theorem and the three body problem were situated at "opposite poles": "the former a free invention of pure reason, belonging to the region of abstract number theory, the latter forced upon us by astronomy and necessary to an understanding of the simplest fundamental phenomena of nature". Like the three body problem, Hilbert observed: "Surely the first and oldest problems in every branch of mathematics spring from experience and are suggested by the world of external phenomena". This was the case for the operations of counting or the classic problems of geometry, the duplication of the cube or the quadrature of the circle. However, "in the further development of a branch of mathematics, the human mind, encouraged by the success of its solutions, becomes conscious of its independence. It evolves from itself alone, often without appreciable influence from without, by means of logical combination, generalization, specialization, by separating and collecting ideas in fortunate ways, new and fruitful problems, and appears then itself as the real questioner". Thus Hilbert explained the origins of the problem of the distribution of prime numbers, Galois's theory of algebraic invariants, and the theories of Abelian and automorphic functions. In short, "almost all the nicer questions of modern arithmetic and function theory".

He goes on, "In the meantime, while the creative power of pure reason is at work, the outer world again comes into play, forces upon us new questions from actual experience, opens up new branches of mathematics, and while we seek to conquer these new fields of knowledge for the realm of pure thought, we often find the answers to old unsolved problems and thus at the same time advance most successfully the old theories. And it seems to me that the numerous and surprising analogies and that apparently prearranged harmony which the mathematician so often perceives in the questions, methods and ideas of the various branches of his science, have their origin in this ever-recurring interplay between thought and experience".

Thus, in the continual interaction between unfettered creations of the mind and knowledge of the phenomena of the external world, Hilbert finds the fundamental dynamic of mathematical development, along with the driving force behind the process of the mathematisation of the other sciences. The rigour of the proofs, a particular characteristic of mathematics – considered by Hilbert to be "a universal philosophical necessity of our understanding" – was also required in the treatment of the most delicate problems of analysis and those questions that originate in the external world, in the world of empirical experience.

The text of Hilbert's Paris talk belies the caricature often used to portray the Hilbertian concept of mathematics, reducing it to a purely formal game with meaningless symbols. To be sure, "to new concepts correspond, necessarily, new signs", observed Hilbert. But these signs are chosen so as to recall the phenomena that generated them. Thus, for example, "the arithmetical symbols are written diagrams and the geometrical figures are graphic formulas; and no mathematician could spare these graphic formulas". On the other hand, he continued, "we do not habitually follow the chain of reasoning back to the axioms in arithmetical, any more than in geometrical discussions". When addressing a new problem, "we apply, especially in first attacking a problem, a rapid, unconscious, not absolutely sure combination, trusting to a certain arithmetical feeling for the behaviour of the arithmetical symbols, which we could dispense with as little in arithmetic as with the geometrical imagination in geometry". He had put this vision to the test in his own research on the foundations of geometry, the subject of a course he taught and of the volume entitled *Grundlagen der Geometrie*, which appeared in 1899 as a *Festschrift* on the occasion of the inauguration of the monument to Gauss and Weber in Göttingen.

In the introductory explanation to the *Grundlagen*, Hilbert declared that he considered "three different systems of objects", called respectively points, lines and planes. He added that "the exact and complete description" of the relations between the three were entrusted to the axioms. The logician Gottlob Frege objected that in so doing, the axioms were given the task usually assigned to the definitions. Frege was convinced that the axioms of geometry were true statements, the knowledge of which "grows out of a cognitive source that is of an extra-logical nature, which we might call spatial intuition".

For Hilbert, in contrast, the axioms were not statements that were true in themselves. The criteria for establishing the truth and existence of mathematical objects was entrusted to the proof of the non-contradictoriness of the axioms (and of their consequences). He retorted to Frege's criticism, saying, "All theories are only a frame, a layout of concepts that are together with their necessary mutual relationships", which can be applied to "infinite systems of fundamental entities". These fundamental entities can be thought of arbitrarily. In order to obtain all of the propositions of the theory it was sufficient that the relationships between the fundamental entities be established by the axioms. The axiomatic method shed light on the deductive weave, the way in which axioms and theories depend on each other. In Hilbert's eyes, this was its essential value. Of course, if we want to apply a theory to the world of phenomena, then "a certain amount of good intention and certain sense of measure" was necessary. Instead, applying an axiomatic theory to phenomena other than those for which the theory was ideated required "an enormous amount of bad intention".

The problems proposed by Hilbert touched on a variety of questions: in the first place, the foundations of analysis (problems 1 and 2), geometry (problems 3, 4 and 5), and the axiomatisation of physical theories (problem 6). The first problem concerned the nature of continuum: "every infinite system of real numbers, that is, every infinite set of numbers (or points) is either equivalent to the set of all natural numbers 1, 2, 3, ... or equivalent to the set of all real numbers, and as a consequence, of the continuum". From the proof would have followed the proof of Cantor's *continuum hypothesis*, according to which 'the number of real numbers is the level of infinity immediately above countable infinity'. According to Hilbert, the "key to the proof" might perhaps lie in Cantor's statement that every infinite set could be well ordered. The set of real numbers, in natural order, was certainly not a well-ordered set. However, Hilbert asked, was it possible to find for that set a different order such that each of its subsets had a prime element? In other words, was it possible to find a well-ordered sequence for the continuum?

Before any mathematician could respond to that question, Bertrand Russell pointed out an antinomy that posed a serious threat to the foundations of the entire construction of Cantor's set theory. The question posed by Hilbert thus came to be entwined with the more general question of the basic principles of Cantor's set theory and gave rise to an enormous mass of studies both logical and foundational, in which many of Hilbert's students and collaborators were involved, beginning with Ernst Zermelo who, in 1904, provided a first axiomatisation of set theory and shed light on the role of the so-called "axiom of choice". With particular regard to Cantor's continuum hypothesis, a first significant result was obtained by Kurt Gödel who, in 1938, proved that the (generalised) continuum hypothesis could not be disproved from the axiom of choice and other axioms of set theory. However, it was not until 1963 that Paul Cohen demonstrated that it couldn't be proven by those axioms either.

The second problem proposed by Hilbert was intimately related to the first. In the *Grundlagen* he had shown that the non-contradictoriness of the axioms of Euclidean geometry was related to the axioms of the arithmetic of real numbers, in the sense that, as he explained, "every contradiction in the deductions of the axioms of geometry must be traced back to arithmetic" of real numbers. Thus, he continued, "this makes a direct method for the proof of non-contradictoriness of the axioms of arithmetic necessary", essentially the axioms for the usual rules of calculation with the addition of an axiom of continuity (that is, the *axiom of Archimedes* and a new *completeness axiom* stated by Hilbert in a then recent work which established the impossibility of an Archimedean extension of the line of real numbers and modified an essential point of the system of axioms established in the first edition of the *Grundlagen*).

Hilbert attributed a decisive role to the proof of non-contradictoriness as a criterion for the existence of mathematical objects. A few months earlier, in reply to Frege's criticism of the axiomatic formulation of the *Grundlagen*, he had written, "If arbitrarily established axioms are not contradictory in any of their consequences, then they are true, and then defined entities exist by means of those axioms. I consider this to be the criterion for truth and existence". He now declared publicly, "If contradictory attributes are assigned to a concept, I say that mathematically that concept does not exist". Hilbert had amazed the world with proofs of an existential nature some 10 years earlier (the 1888 *basis theorem* and the 1890 *theorem of zeroes*). The hoped-for proof of the non-contradictoriness of the axioms of arithmetic would have proven the existence of both the real numbers and the continuum. Because the consistency of geometry and of analysis could be traced back to that of arithmetic, the direct proof of the non-contradictoriness of the axioms of arithmetic would have guaranteed the consistency of the whole of mathematics. The second problem was in fact a statement of this ambitious program, which Hilbert and his students would pursue through the 1920s, before Gödel's incompleteness theorem of 1931 proved that the task was an impossible one in terms of how it had been formulated by Hilbert, which led to its being drastically revised.

The next three problems were inspired by Hilbert's own research on the foundations of geometry. In the *Grundlagen* Hilbert had shown that in plane geometry the axioms of congruence (without resorting to the axiom of continuity) were sufficient to prove the congruence of straight line figures. Gauss had already noted that, instead, the proof of theorems of solid geometry such as that of Euclid – prisms of equal height and triangular bases are proportional to their bases – depends on the method of exhaustion, that is, in the final analysis, to an axiom of continuity. In problem three, Hilbert asked to be shown "two tetrahedra of equal bases and equal heights that cannot be subdivided into congruent tetrahedra". The proof was produced 2 years later by Hilbert's student Max Dehn (1878–1952).

Another of Hilbert's students, Georg Hamel (1877–1954) had successfully taken on the fourth problem. Hilbert had drawn attention to the geometry developed by Minkowski in the 1896 *Geometrie der Zahlen*, in which all of the axioms of ordinary geometry were valid (including the axiom of parallels) with the exception of the axiom of congruence of triangles, which was replaced by the axiom of triangular inequality. Hilbert himself, in 1895, had studied a geometry in which all of the axioms of Minkowski's geometry were valid, except the axiom of parallels. Convinced of their importance for number theory, theory of surfaces and the calculus of variations, Hilbert now called for a systematic study of the geometries in which all of the axioms of Euclidean geometry were valid except for the axiom of triangular congruence (axiom III, 5 of *Grundlagen*), which was substituted by triangular inequality, taken as a particular axiom. Hamel proved that the only possible geometries were elliptic (in the case of an integer plane) or hyperbolic such as the kind studied by Minkowski and Hilbert. The problem was in any case formulated by Hilbert in terms that were quite vague, and in the decades that followed this gave rise to numerous studies on particular classes of geometries.

In his work on continuous transformation groups, Lie had established a system of axioms for geometry and resolved the problem of how to determine all the n-dimensional manifolds that admit a group of rigid motions, in other words, the problem posed by Riemann and Helmholtz as to the characterisation of the rigid motions of bodies. Lie had assumed that the transformations of his groups would be differentiable functions. In 1898 Klein had expressed doubts as to whether this hypothesis was necessary, and now, in problem five, Hilbert took up the question once more, asking himself if, as far as the axioms of geometry were concerned, the hypothesis of differentiability was inevitable or if instead this was a consequence of other geometric axioms.

More than an actual problem, Hilbert's sixth problem provided a guideline for research. Using as a model the studies on the principles of arithmetic and geometry, Hilbert invited mathematicians "to treat in the same manner, by means of axioms, those physical sciences in which mathematics plays an important part". What he had in mind was the kind of probabilistic concepts introduced by Clausius and Boltzmann in the kinetic theory of gas, and Mach's and Boltzmann's research on the foundations of mechanics. Starting at the beginning of the twentieth century, first with Minkowski and then, after his friend's premature death in 1909, with his assistants, Hilbert studied problems of theoretical physics with growing interest for a couple of decades. He taught courses and gave lectures on particular topics; he published important works, such as his 1915 paper, which appeared just a few weeks after that of Einstein, in which he obtained the equations for general relativity and exhorted his students and collaborators to engage in this kind of research. Also situated in that field was Emmy Nöther's 1918 theorem – regarding the calculus of variations – of fundamental importance in modern mathematical physics, which put the number of parameters of a subgroup of invariants for lagrangian systems in relation to number of laws of conservation that could be derived for those systems. With Richard Courant he wrote the treatise *Matematische Methoden der Physik* (1924), which became a classic. Books by his students, such as *Gruppentheorie und Quantenmechanik* (1928) by Hermann Weyl, and *Mathematische Gundlagen der Quantenmechanik* (1932) by John von Neumann are considered to be among the most significant results produced in the spirit of Hilbert's sixth problem. However, as Weyl himself admitted, in spite of the great results achieved, "Hilbert's plans in physics never matured".

As far as probability theory is concerned, the axiomatisation hoped for by Hilbert took shape in the Russian school of Bernstein and Kolmogorov, in the context of modern measure theory.

After the problems concerning foundations, Hilbert passed to a consideration of specific problems, beginning with number theory, the discipline around which his research in recent decades had focussed, culminating in the publication of the *Zahlbericht* (1897). This is reflected above all in problem seven, which we mentioned earlier and other problems correlated to it (which were solved in the 1930s by Gelfond), and in problem eight regarding the distribution of the prime numbers and the *Riemann hypothesis*, which is perhaps the most important conjecture still open in mathematics today.

These two problems are among those that Hilbert presented during his talk in Paris. The others he mentioned were the first, second, sixth, thirteenth, sixteenth, nineteenth and twenty-second. With the complex problems comprised between the ninth and the eighteenth, he passed from number theory to problems of algebra or algebraic geometry. The tenth problem, for example, asked for a procedure that would be able to determine by means of a finite number of operations whether or not a given Diophantine equation with n unknowns has integer solutions. Instead, the 13th asked for a proof showing that the generalised seventh-degree equation can be solved with functions of only two parameters. Establishing rigorous foundations for Schubert's enumerative calculus in geometry was the aim of the 15th problem, while problem 16 regarded the topology of algebraic curves and surfaces. The next two problems were also geometric in nature. Extending a theorem established in the final chapter of the *Grundlagen*, which was dedicated to the possibility of constructions with compass and straightedge, problem 17 asked if for any definite form (that is, a rational integer function of n variables that takes only non-negative values over the reals) could be expressed as a quotient of the sum of squares. Problem 18 asked for an extension of Poincaré's (and Klein's) results regarding Lobachevsky's plane (and space) groups of motions to n-dimensional Euclidean space. To this was correlated a question that was "important to number theory and perhaps sometimes useful to physics and chemistry: How can one arrange most densely in space an infinite number of equal solids of given form, e.g., spheres with given radii or regular tetrahedra with given edges...?"

In the final group of problems, Hilbert took topics in analysis into consideration. In the 19th problem, he questioned "whether all solutions of regular variation problems must necessarily be analytic functions", while the 20th problem regarded whether or not there exist solutions to partial differential equations with certain boundary conditions.

In the 21st problem, inspired by Riemann's and Fuchs's results, Hilbert asked for proof of the existence of a linear differential equation with given singular points and monodromic group. The next to last problem regards an extension of Poincaré's uniformisation in the theory of automorphic functions. Last but not least, with the 23rd problem Hilbert calls for "further development of the calculus of variations".

Looking at the 23 problems as a whole, it is possible to see that the original studies outline the scheme of development for some of the most important branches of twentieth-century mathematics. While some of the problems were stated clearly and precisely, in other cases Hilbert instead urged young mathematicians to create new theories or research programs. From this point of view, the more than 60 doctoral theses written under his direction between 1898 and 1915 are revealing. Eleven of his students wrote a thesis on questions of number theory, and three of their topics were related to the 12th problem – Hilbert was inspired by Kronecker's *Jugendtraum*, or youthful dream – which regarded the development of the parallels between fields of algebraic numbers and fields of algebraic functions. Ten of the theses dealt with the foundations of geometry and problems of algebraic geometry in strict correlation to the16th problem. However, almost half of his doctoral students deal with the topics in analysis that Hilbert was predominantly interested in up to the First World War, above all the calculus of variations (in particular with *Dirichlet's principle*) and the theory of integral equations. In the 1920s, five of the nine theses overseen by Hilbert dealt with the foundations of mathematics and proof theory. These concerned the development of ideas outlined in the second problem, to which Hilbert dedicated the final phase of his work, tying his name to the so-called formalist program of the foundations of mathematics.

Hilbert's 23 Problems

At the second International Congress of Mathematicians, which took place in Paris in 1900, David Hilbert presented 23 problems that were unsolved at the time in various areas of mathematics. In his opinion, these were the problems to which the attention of the researchers of the new century would be drawn.

1. *Cantor's problem of the cardinal number of the continuum (the continuum hypothesis)*: Is there set whose size is strictly between that of the integers and that of the continuum? In 1938 Gödel proved that the continuum hypothesis is consistent with Zermelo–Fraenkel set theory; in 1963 Cohen proved that its negation is as well.
2. *The compatibility of the arithmetical axioms*: Gödel showed in 1931 that no proof of its consistency can be carried out within a system as rich as arithmetic.
3. *The equality of two volumes of two tetrahedra of equal bases and equal altitudes*: Max Dehn found a counterexample in 1902.
4. *Problem of the straight line as the shortest distance between two points*: Construct all the metric geometries in which the lines are geodesics. Solved in 1901 by Georg Hamel.
5. *Lie's concept of a continuous group of transformations without the assumption of the differentiability of the functions defining the group*: Is it possible to avoid the hypothesis that the transformations are differentiable to introduce the concept of continuous transformation groups according to Lie? Solved for particular transformation groups by John von Neumann in 1933 and, in the general case, by Andrew Gleason and independently by Deane Montgomery and Leo Zippin in 1952.
6. *Mathematical treatment of the axioms of physics*: In particular, the axiomatisation of those areas, such as mechanics and probability theory, in which mathematics is essential. Results were produced by Caratheodory (1909) in thermodynamics; von Mises (1919) and Kolmogorov (1933) in probability theory; John von Neumann (1930) in quantum theory; Georg Hamel (1927) in mechanics.
7. *Irrationality and transcendence of certain numbers*: In particular, if a^b is transcendent when base a is algebraic and exponent b is irrational. An affirmative answer was given by Gelfond in 1934 and (independently) by Schneider in 1935.
8. *Problems of prime numbers*: In particular Riemann's hypothesis on the zeroes of Riemann's "zeta function" relative to the distribution of primes.
9. *Proof of the most general law of reciprocity in any number field*: Resolved for a special case by Teiji Takagi in 1920, and more generally by Emil Artin in 1927.
10. *Determination of the solvability of a Diophantine equation*: Is there a univeral algorithm for their solution? A negative answer was provided by Yuri Matiyasevich in 1970.
11. *Quadratic forms with any algebraic numerical coefficients*: Solved by Helmut Hasse in 1923.

12. *Extension of Kroneker's theorem on abelian fields to any algebraic realm of rationality*: Solved by Shimura and Taniyama in 1959.
13. *Impossibility of the solution of the general equation of the seventh degree by means of functions of only two arguments*: Generalises the impossibility of solving a fifth-degree equation by roots. Answered in the negative by Kolmogorov and Arnol'd in 1961: a solution is possible.
14. *Proof of the finiteness of certain complete systems of functions*: A first counter-example was provided by Nagata in 1958.
15. *Rigorous foundation of Schubert's enumerative calculus*: precisely determine the limits of the validity of the numbers that Hermann Schubert had determined on the basis of the principle of special position, by means of his enumerative calculus. Solved.
16. *Problem of the topology of algebraic curves and surfaces*: In particular, developing Harnack's methods and Poincaré's theory of limited cycles.
17. *Expression of definite forms by squares*: In 1927 Emil Artin proved that a positive definite rational function is the sum of squares.
18. *Building up of space from congruent polyhedra*: Solved (but Penrose found non-periodic solutions).
19. *Are the solutions of regular problems in the calculus of variations always necessarily analytic?* Partially solved in 1902 by G. Lötkeyeyer and more generally in 1904 by S. Bernstein. General solution by De Giorgi in 1955 and by J.F. Nash Jr. independently some months later.
20. *The general problem of boundary values*: do variational problems with particular boundary conditions have solutions? Resolved.
21. *Proof of the existence of linear differential equations having a prescribed monodromic group*: Partially resolved by Hilbert in 1905, and by Deligne for other special cases in 1970. A negative solution was found by Andrej Bolibruch in 1989.
22. *Uniformization of analytic relations by means of automorphic functions*: Solved in 1907 by Paul Koebe.
23. *Further development of the methods of the calculus of variations.*

The Way We Were

The Protagonists of the "Italian Spring" in the First Decades of the Twentieth Century

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After the trial run in Zürich (1897), the *International congresses of mathematicians* officially began with Paris (1900) and Heidelberg (1904). The third was held in Rome (1908). This order was not random, nor was it dictated only by contingencies. The fact is, at the beginning of the twentieth century Italian mathematics was considered the third world "power", immediately after the great and traditional French and German schools. The same classification holds, almost completely unchanged, at the beginning of the 1920s. American mathematician G. D. Birkhoff, particularly attentive to the situations of European research centres (and interested in consolidating collaborations with them for the definitive launch of the mathematics of the United States) does not hesitate to place Rome immediately after Paris, even before Göttingen.

But who was in Rome in those years? Who were the mathematicians who made it possible for Italian mathematics to compete with the more famous schools of Europe (and therefore, for the moment, of the world)?

After the Italian Unification and the successive transfer of the capital to Rome, the political leaders had made it part of their policy to bring the most vivacious aspects of culture to the city. This also included the scientific culture. The first to arrive among the mathematicians referred to by Birkhoff was Guido Castelnuovo, who transferred to Rome in 1891. In truth a little more than 30 years would have to pass before the Italian school of algebraic geometry would regroup in the capital, but in the end – though with a bit more effort and ill will that foreseen – it made it. In 1923, Federigo Enriques and Francesco Severi came to Rome as well. Vito Volterra arrived to the capital from Turin in 1900 and was immediately charged with giving the inaugural address at the beginning of the academic year. The choice of argument was not a given: Volterra chose to speak "*On the attempts to apply mathematics to the biological and social sciences*". Tullio Levi-Civita would arrive in Rome shortly after Volterra, in 1909, but for the time being he didn't want to leave the tranquillity of Padua. He transferred only after World War I (1918), after he had married, and after a first period spent in Rome, following the defeat at Caporetto.

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