Roots of Modern Technology

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An Elegant Survey of the Basic Mathematical and Scientific Concepts

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Chapter 1
Explaining Modern Technology

A couple of years ago, on a subway ride in Berlin, I overheard a conversation between two students, and I remember one of them saying, “Last semester I took Dr. Anderson’s course on computer electronics. But I should have stayed in bed instead of getting up early each Tuesday morning in order to attend his eight o’clock lecture, since nobody learned anything from this course. This semester, I am attending the lectures of Dr. Heymann, and she is the best professor I ever had. Her explanations are so illustrative that even the most complex subjects have become absolutely clear to me.” This made me think back to my own years in high school and college, and I remembered my math teacher as the one whom I still would grade an A plus, while at the same time, I remembered some other teachers who did a rather poor job. I am convinced that almost all of my readers have experienced extremely good teachers and others who weren’t worth the salary they were paid.

During my over thirty years of teaching engineering courses, I certainly tried to be a good teacher. But it was not until I was approaching retirement that I explicitly considered the question about how to become a good teacher. Although I don’t have the complete answer to this question, at least I am convinced that there is a specific condition that must be satisfied under all circumstances: whenever a teacher enters the classroom, he must consciously consider the fact that his way of viewing the world may differ substantially from the views of the students. This makes him aware that he must try to view the world through the eyes of his students.

As a professor of engineering sciences, I am, of course, mainly interested in the different ways a person may view the world with respect to its technological aspects. Thus, it happened that I not only considered the differences between my view and the views of my students, but I also asked myself what the differences were between my view and the views of my wife and my three year old grandson. This made me aware of the fact that, for a little child, all technical devices belong to the natural world. A child doesn’t wonder about the buttons which can be pushed to turn the light on or off, or that there is a cell phone which can be used to talk to Grandma. The child doesn’t ask where the cell phone comes from or why a car can move so much faster than a running horse. For the child, cell phones and cars are as natural as apples and horses. We might say that for a little child, either everything is a miracle or nothing is a miracle.

When my philosophical considerations had reached this point, it only required a short further step for me to come up with the question of how someone who had died hundreds or thousands of years before our time, would view our world. Obviously, my thinking had been influenced by a novel I had read some years ago, and a movie I had seen at about the same time. The German novel “Pachmayr” [SPO] and the French movie “Hibernatus” were created around 1970, and both works involved complications which occur when a person who died long ago suddenly comes to life in the present time, and still remembers everything about his former life. In the movie, the time gap was one hundred years, while in the novel it was almost five hundred years. Thus, it seemed quite natural to me to ask myself how a person from the middle ages or from classical antiquity would react if he would be confronted with a world where there are electric lights, pills for headaches, food freezers, microwave ovens, airplanes, television sets, automobiles, mobile phones and computers. Instead of considering this question with respect to an abstract person, I wanted to think of a real person who had lived in the past. Finally, I chose the Greek philosopher Socrates (469-399 BC) because he is the perfect example of someone who continued asking critical questions until he either reached the fundamental essence of the subject, or realized that the limits of knowledge had been reached for the person being asked. Thus, the following chapters are my answer to the question, “How could I help Socrates understand our present world?”

What Socrates Would Ask Me

In part, Socrates would be in the same situation as a newborn child who must learn about the world. The difference between the situations for the newborn and Socrates is that Socrates would be able to recall the rich experiences of his former life. The only things Socrates must learn are how the present world differs from that of his former life.

An important guideline for my teaching efforts always has been the saying, “A picture is worth a thousand words.” Therefore, as you will see, a major part of the text of this book is just a sequence of comments about diagrams. For example, consider Fig. 1.1 where rectangles represent six areas of knowledge which I placed in levels, one on top of the other like the floors of a building.

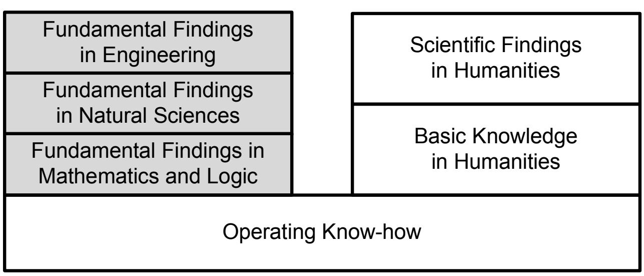

Fig. 1.1 Layers of knowledge

The different types of knowledge shown must be acquired from bottom to top. This means that a newborn child must first acquire the knowledge which I called "Operating Know-how." This is the kind of knowledge needed to operate complex technical equipment. Actually, for our Socrates as for any newborn child, the unknown world is like a new machine which they must learn to use. This learning does not require any understanding since it consists only of remembering causal relationships. Thus a child not only learns what happens when certain buttons are pressed – the light goes on, the door bell rings or the garage door opens – but it will also soon know in which drawer grandma keeps the candy. Learning one's native language also means acquiring operating know-how, since this is nothing but remembering the causal relationships between certain acoustic patterns and types of human behavior. One characteristic of all kinds of operational know-how is that it completely loses its value when its owner is placed in a new environment. There was a time when I often had to travel for professional reasons, and in those days it frequently happened that I could not immediately drive away from the airport in a rental car because the buttons and switches for the wipers and the headlights were not where I expected them to be. Each time in these situations I had to acquire new operating know-how which was valuable only as long as I drove that particular make and model of car. Ask yourself how much of what you actually know is valuable only when you are in certain environments. Even knowledge about the location of the nearest toilet is of that type.

Despite the tremendous amount of operating know-how Socrates would need in order to survive in our present world, this is not the kind of knowledge he would ask me to provide. Nor would his questions be related to the two areas of knowledge which are on the right hand side in Fig. 1.1, since these subjects are just what Socrates had experienced in his former lifetime, namely politics, sociology, economics and arts. He was involved in wars in his time, and so he would not be surprised to learn that since then many more wars have been fought, people have had to leave their countries and move to other regions, and empires have grown and vanished. He had known sculptors, painters and poets and would consider it normal that in the meantime many new artists have created new works, and that aesthetic criteria have undergone evolutionary changes. Certainly it would be very interesting for him to learn that Athens is no longer the center of the universe, that the earth is a globe, that Columbus had discovered America, that Marco Polo had traveled to China and that today there are precise maps of the entire earth. But probably all of these changes would not cause him to say, "I don't understand it." The same can be said about the fact that letterpress printing was invented and that our children learn to read and write in elementary school. Although historians say that Socrates never produced any written text, Socrates certainly would appreciate the fact that today it is quite easy to obtain a fundamental education in humanities just by reading a few books. Today there are even books on the market with titles such as “Everything an Educated Person Should Know.” But even after Socrates had read these books and had acquired all that knowledge, he would still be dissatisfied and see the need to ask for my help. He would come and say, ”All this knowledge does not help me in the least to understand the strange phenomena I regularly encounter almost all the time, and wherever I go. You press a button and a huge stadium gets bright; then you press the same button and the stadium gets dark again. You press other buttons and the church bells in the tower start to ring or heavy doors swing open. Objects looking like extremely long houses on wheels with hundreds of people sitting inside move at high speeds on rails, although no apparent reason for the movement can be discerned. And, of course, the devices and systems which you call television and mobile telephones remain absolute mysteries to me. On the other hand, I have been with you long enough to be sure that mankind is still completely human and has not turned into a crowd of demigods. Therefore, I am looking for someone who is willing and able to clearly explain to me the fundamental findings which enabled humans to create such mysterious devices and systems.”

In my present situation as a retired professor of engineering, nothing could be more gratifying to me than to be asked by Socrates to give him these explanations. Erwin Chargaff whom, in many respects, I consider a model of a very wise man, once wrote [CH 1]: “A real explanation requires two persons, one who explains and another one who understands.” From my long experience as a university professor, I must add another requirement: “Understanding an explanation requires a person who wants to understand it.” In the case of Socrates I can be sure that he wants to understand and that he would not refuse to make the necessary effort. The effort I must impose for the reader is not more than the careful study of the 440 pages of this book. I certainly would have preferred to reach my goal with fewer pages, and I really tried hard to do this. But I had to guide Mr. Socrates and my other readers from bottom to top over the three stacked grey areas of knowledge on the left hand side of Fig. 1.1. This stack tells us that the knowledge about fundamental engineering findings can be acquired only after one has gained a certain insight into the fundamental findings in natural sciences, and that these require knowledge about fundamental findings in mathematics and logic. Therefore, this book is structured as a sequence of three groups of chapters with each group corresponding exactly to one of the grey areas of knowledge in Fig. 1.1.

You will not find any information about the internal details of technical products in this book. The purpose of this entire book is to provide the knowledge which leads the reader to believe that these products can be conceived and built. No one would ever have invested a minute in the development of a technical product if he or she had not been convinced about its possible success. This conviction about possible success has always been based on the level of knowledge a person had reached. From a particular level of knowledge, they could look down and see all the findings or results which would help them solve all the problems which might turn up in the course of the development process. Someone who begins the development of a technical product has a vague idea of the overall solution that can be implemented if certain types of problems can be solved. Solving any of these problems does not require any findings outside of the plateaus of knowledge which have already been reached and can be seen from the place where the developer is standing. In most cases, solving all these problems still requires hard work. Sometimes good luck is needed to come up with a reasonable solution. Never does any magic come into play. Thus, it is my goal to lead my readers to that small set of plateaus of knowledge, the plateaus on which the technicians walk when they are looking for solutions of problems with which they are confronted when developing new products. While walking around on these plateaus is part of the job of professionals, I shall not take my readers on such excursions. Instead, I shall help them climb the rock walls which lead to these plateaus. In the course of history, each of these rock walls has been conquered for the first time. But since then, the walls have been climbed often, and hooks which make it easier to go up have been hammered into the rock. There are no cable cars leading up to the plateaus of knowledge, but with the help of an experienced mountain guide, even inexperienced mountaineers can reach the plateaus with reasonable effort.

Omitting Irrelevant Subjects Is an Art

When the physicist Richard Feynman (1918-1988) was given the Nobel Prize, a journalist asked him to summarize in three sentences what he got the prize for. This request certainly was for an explanation far below an acceptable limit of brevity, and Feynman answered, “If I could explain this in three sentences, it certainly would not be worth the Nobel Prize. But I could explain the essence of my work to your educated readers on two pages.”

Although I did not win the Nobel Prize, I am somehow in a similar situation since I am expected to present my subject in a minimum number of pages. I have omitted everything that could be omitted without reducing the comprehensibility and the intellectual depth of the presentation. Since this book is restricted to findings which are fundamental for our present technical systems, I could omit all findings from mathematics and natural sciences which have not yet been applied to any technical products. Therefore this book does not contain the theories of cosmology which explain the origin of the world. If you are interested in the so-called “big bang” theory, you must look in other books. By the way, did you know that Albert Einstein commented on the idea of the big bang with the words, “This theory is as absurd as having an encyclopedia originate from the explosion of a printing shop?” Among the subjects I omitted in this book is the theory of strings, which is an attempt to combine cosmology with quantum theory, and Darwin's theory of evolution which is an explanation of the origin of living species.

But there are subjects which I omitted even though they are technologically relevant – for instance acoustics and geometric optics. My decisions not to discuss them are based on my judgment that, in these cases, the corresponding knowledge plateaus can be reached without a mountain guide. I also omitted most of the historical details of the processes which finally resulted in the knowledge we have today. The way in which I will guide you up the rock walls will always lead directly to the plateau we want to reach. Certainly it would have been nice to show you all the dead end roads which have been tried in the past. For example, consider the definition of electricity which a student of the German philosopher Georg Wilhelm Friedrich Hegel (1770-1831) once wrote [HEG]:

> "Electricity is the pure purpose of the shape which liberates itself from it, which begins to cancel its indifference, since electricity is the immediate emergence, or the existence which does not yet emerge from or be conditional upon the shape, or is not yet the dissolution of the shape itself, but is the superficial process wherein the differences leave their shape but still are dependent on it and are not yet independent with it."

Today, we can only laugh about this, and it remains absolutely mysterious how someone could ever dream up such nonsense. Perhaps Professor Hegel said something reasonable, but a student taking notes confused all of it. Certainly we can take this definition as a hint about how difficult it was in those days to gain a fundamental insight about the basis for today's electrical engineering.

When I discussed my intention to write this book with friends and relatives, some of them spontaneously expressed their doubts about whether I could succeed at all during a time when "the total knowledge of mankind doubles every few years." Why am I not afraid of this doubling? It is simply because it is not a doubling of our knowledge, but only of the available information, which can be acquired by searching and reading. Admittedly, people very often use the term "knowledge of mankind." But this is not a correct concept, because knowledge is only what one person knows. In order to be able to get access to and use the enormous wealth of information stored in libraries and on the internet, a person must possess some knowledge, because someone who knows nothing cannot ask questions. We cannot expand the range of our understanding unless we have already understood some fundamentals. Do you know why, for example, the sky is blue on a sunny day? Actually, I don't know this myself, but I do know exactly where I could look or whom I could ask to get a comprehensive answer if the need arose.

I recently read the following statement of the Austrian philosopher Konrad Paul Liessmann in the German journal "Research and Teaching" [LI]:

> "What a person can or should know today is no longer determined by some standard theories of education, but mainly by the market which constantly changes. This knowledge can be produced and acquired rapidly, but just as rapidly be forgotten."

Nothing of what he said in this statement applies to the fundamental findings I present to you in this book. These findings and their presentation could not be produced rapidly; they required a laborious process of hard work. They cannot be acquired rapidly, since in spite of all the support of an experienced mountain guide, a lot of effort is still needed to climb up the steep walls of understanding. However, those who invest this effort and reach the plateaus of knowledge shall never forget the essential findings, and shall be glad to be able to refer to them for the rest of their lives. That which is discussed in this book will not be affected by a constantly changing market.

No One Should Be Afraid of Formulas

In 1905, Albert Einstein climbed Mount Sinai. When he arrived at the top, he was shrouded by a cloud and heard a voice saying to him, "Albert, take this slab of stone, carry it down the mountain, and then read what I wrote on it and explain it to your people." When Albert Einstein came down from the mountain, he read "$E=mc^2$," and then he explained this formula to the people. Did it happen this way? We all know that this never happened. Why then does almost every author of a science book for non-professionals introduce this formula as if it had come from heaven? The reason for this can be found in the preface to the book "A Brief History of Time" by the English physicist Stephen Hawking [HA 1]. There he wrote:

> "I was told that each formula in the book would halve the number of its readers. Therefore I decided to refrain from including any formulas. But finally, I made one exception, $E=mc^2$."

Publishers of books on science for non-professionals apparently believe that such books do not sell well to a broad public if they contain mathematical formulas. Why do publishers have this belief? Unfortunately, mathematics has acquired the image that it can be understood only by an extremely small minority of people who are gifted in a very exotic way, and that therefore it is quite natural for mathematics to remain closed for the rest of mankind. The German philosopher Arthur Schopenhauer (1788-1860) contributed to establishing this image of mathematics by writing [SCH]: "The talent for mathematics is a very specific and unique one which is not at all parallel to the other faculties of a human head, and indeed has nothing in common with them." And in the book "Lies in Educational Politics," the German author Werner Fuld [FU] wrote: "Does not Schopenhauer's statement call to mind our years in school, and especially those schoolmates from whom we could gratefully copy each math paper, but whom, on the other hand, we considered rather dumb if not even mentally deficient?" Obviously, the author Fuld must have had some problems with his lessons in math, and he sought to get applause from those of his readers with similar experiences. By his statements, he reinforces a prejudice which can be easily disproved. Of course, I myself do know some extremely unworldly mathematicians whom Mr. Fuld presumably would call "mentally deficient." I would be more specific about such mathematicians and say, "They cannot communicate adequately." But it is a severe logical sin to generalize from a small number of pathological cases. Think of the many engineers who move with great competence through many fields of mathematics; Mr. Fuld would have great difficulty in finding such pathological cases among them.

How about the belief that a rare and exotic talent was required for understanding mathematics and having fun dealing with it? Here the situation is the same as in all kinds of arts: a rare and special talent is required for creating a valuable work of art, but almost everybody is gifted enough to understand the work and enjoy it. Only a very few people can be good composers or writers, but many can enjoy listening to the music and reading the novels. If you think mathematics is extremely difficult and requires a rare talent which you don't have, this might be true concerning the creation of mathematical works. But if you don't understand a mathematical work presented to you, this might not be a consequence of not having a particular talent. Very often it will be the consequence of the fact that the "performing artist" was not gifted enough to communicate the work to others.
The Spanish philosopher Ortega y Gasset (1883-1955) once wrote [OYG 1]:

> "Mathematicians exaggerate the difficulty of their subject a bit. If it appears so incomprehensible today, it is because the necessary energy has not been applied to simplifying its teaching."

Although I think that Ortega's statement is correct, I also think that there is something special about dealing with mathematics. It is not primarily a question of a good memory, but it requires the motivation to struggle for understanding. Throughout this entire book, I shall refer to the rock walls and plateaus of knowledge, where acquiring the understanding of mathematical findings corresponds to conquering a rock wall leading to a plateau. In contrast to this, acquiring knowledge about history, literature or art corresponds to walking around in a plane or on a plateau where the hikers need only a good memory to help them remember all the places they have been. But these hikers will never experience the moment of joy which causes the mountaineers to shout "Hooray!" when they reach the top of the mountain.

Attempts to teach findings in natural science and engineering without using mathematics are similar to attempts to teach the evolution of styles in painting to a blind person. Formulas are just statements in a special type of language which help to state facts about abstract structures as briefly and precisely as possible. Any attempt to state these facts by exclusively using natural language would result in bulky and confusing phrases which nobody would understand except for their authors. Formulas are needed especially for defining concepts about higher levels of abstractions which otherwise could not be defined at all. For instance, there would not be a relativity theory, or any type of electrical engineering, because they depend heavily on abstract concepts like imaginary numbers and four-dimensional spaces which can be defined only by using formulas. These concepts are completely formal and even Einstein and all the other geniuses in physics and mathematics depend on formulas to deal with these concepts.

In the course of decades of academic teaching, I became strongly convinced that I finally know why so many people are scared of formulas: they believe that mathematicians, in contrast to normal people, have a sixth sense for formulas. They don't believe that what the teacher tells them about a certain formula is all that could be said about it. They believe that there is something "behind the formula" to which they don't have any access. Throughout the entire book, I keep emphasizing that nobody needs a sixth sense to understand a formula, because there is never anything hidden behind it.

In about 1925, the Spanish philosopher Ortega y Gasset, whom I mentioned above, wrote an essay entitled "Mission of the University". In this essay, I found the following paragraph which hopefully motivates you to go through all the remaining chapters of this book [OYG 2]:

> "Consider the gentleman who professes to be a doctor, a magistrate, a general, a philologist, or a bishop, i.e., a person who belongs to the leadership class of society. If he is ignorant about what the physical universe is today, he is a perfect barbarian, no matter how well he may know his laws, or his medicine or his Holy Fathers. It is certain that all the other things he does in life, including parts of his own profession which transcend its proper academic boundaries, will turn out unfortunately. His political ideas and actions will be inept. He will bring an atmosphere of unreality and cramped narrowness to his family life, and this will warp the upbringing of his children. With his friends, he will emit thoughts that are monstrosities and opinions that are a torrent of drivel and bluff."

If you now follow me through the coming chapters and finally climb all the rock walls to the plateaus of knowledge, you certainly will not belong to the kind of people Ortega was talking about.

Now it's time to tackle the first wall. Let's go!

Part I
Fundamentals of Mathematics and Logic

Chapter 2
Mathematicians Are Humans Like You and Me – They Count and Arrange

"Math – no, thanks!" said my daughter and the majority of all people I've met during the course of my life. How about you – do you also belong to those who haven't liked mathematics since they encountered it in high school? Let's assume we are not talking about mathematics but about a lamb roast, and you say, "I have never liked a lamb roast." Do you think a gourmet cook would accept this just as it was said? No, he would suspect that your aversion to a lamb roast was the consequence of the fact that you never have been served an optimally prepared lamb roast. Now let me persuade you to taste the meal of mathematics which I have prepared especially for you.

In this chapter, I shall present a view of mathematics "from high above" as if mathematics were a continent over which we could fly in a satellite. Think of Europe which actually can be seen from a satellite. It is evident that from such a far distance we cannot see any details, but only the rough topography of the continent. Sometimes we want to look a little bit closer which in our analogy means that we then use binoculars.

The textbooks for the mathematics courses from my years in high school have long disappeared, but I still remember that on their covers were printed certain strange words which at the time nobody explained to me, words like Arithmetic, Algebra and Analysis. Only the word Geometry referred to the world I knew. In our analogy, these words on the covers of math books correspond to the names of the countries printed on the maps of Europe. When we study a map of Europe, we are interested in the structure of the continent, that is, how the outlines are shaped, where the borders between the countries are located, which way the rivers flow and where the mountains and the big cities are located. In this situation, we do not ask what Europe is good for and how it could be applied. In the same way, we first have to get to know the structure of the "math continent." We should not yet ask what it could be used for. Of course, we already know that we had no reason to study this continent except that it had become fundamental for physics and technology.

What a Number "Sees" When It Looks into a Mirror

The mathematician Leopold Kronecker (1823-1891) said, "The natural numbers have been created by the dear God and everything else is the work of humans." In the following paragraphs, we shall reconstruct this human act of creation. For each day in the creation of the world, the bible includes the closing statement: "And God saw that it was good." I hope that we can write at the end of our creation story: "And we saw how wonderfully and logically one result implied the next."

The *natural numbers* are so natural that even small children quite intuitively grasp the corresponding concepts. The grandchild goes up the stairs holding grandma's hand and counts: "One, two, three, …." The natural numbers serve to count, and that means that a position in a sequence is assigned to each element of a set, and that the size of the set is found at the same time. Of course, the steps of a staircase have their positions in a sequence before we count them. By counting, we only assign names to these positions, for after there is a sixth step there is a seventh, etc. The name which we assign to the last position is the name of the number of steps in the staircase. When we count, we pass through all positions of a sequence, and therefore the most natural way of counting consists of sequentially saying the names of the numbers. We can remember the sequence of these names as if they were words in a rhythmic song: "One, two, three, four, five, six, seven – all the angels look down from heaven." It is true, though, that we also count things which don't have a natural sequential order: think of apples in a basket. In such a case, we produce an artificial order which is destroyed when we are done counting. For example, we pour the apples from the basket onto the table and then put them, one after the other, back into the basket "singing" our numbers song.

Although the natural numbers and their application in counting are the nucleus of mathematics, the true concept of mathematics was not born until addition, subtraction, multiplication and division were performed as abstract operations. They are called the four basic arithmetic operations where the word "arithmetic" refers to the Greek word "arithmos" for number. People always used to illustrate these operations by thinking of containers which hold countable objects, for example apples in baskets. *Addition* means pouring the contents from one container into a second container on top of its contents. The inverse of this operation is called *subtraction*, and this means that a part or all of the contents of one container is removed and put into another container which originally was empty. Of course, we cannot take out more apples than had been in the basket originally. Only magicians in a circus can demonstrate that they can take 8 balls out of a bag which originally seemed to contain no more than 5 balls. Subtraction is the first operation which motivates us to create a new number, "*Zero*," which is the number of apples remaining in a basket after we have removed all of the apples which were there originally. Zero can occur only as a result of computing and not when we count. Therefore, it does not belong to the natural numbers, but to those numbers which Kronecker said were not created “by the dear God.”

In addition and subtraction, we deal only with objects of the same type, for example, apples or steps. In *multiplication* and *division*, however, objects of two different types are counted, for example baskets and apples. Multiplication is illustrated by the idea that we originally have ‘b’, a certain number of baskets, each containing an equal number ‘a’ of apples. As an example, think of 5 baskets each containing 8 apples. By pouring the contents of all of these baskets into one basket, we obtain the so-called product, namely 40 apples. Another way to illustrate a product is with a rectangular floor which is covered by many smaller rectangularly shaped tiles, and where we wish to find the total number of tiles. Two numbers must be known, namely how many tiles fit into one row and how many tiles fit into one column. From these two numbers, we can get the total number of tiles by multiplication.

Division is based on the idea that the apples in one basket must be equally distributed into a given number of originally empty baskets. If we are not allowed to cut apples into parts it will quite often occur that an equal distribution is impossible. This can be done in the case of 3 baskets if 15 apples are to be distributed, but not if we start with 17 apples.

While addition and multiplication can be performed on any pair of natural numbers, this is not true for subtraction and division. This is a consequence of the restriction that the result of the operation must be a natural number or zero. Now we lift this restriction and allow the objects to be distributed to be cut into parts if necessary. At this point we no longer think of apples but of sticks of butter, since these can be cut easily and precisely. When such a stick is cut into three parts of equal length, it is said that each part has one third the length of the original stick. “One third” is formally written 1/3. Such a form consisting of two numbers separated by a slash is called a *fraction*. The number in front of the slash is called the numerator; it stands for the number of objects to be distributed. The number behind the slash is called the denominator; it stands for the number of containers or receivers among which the objects are to be distributed. In our example, we wanted to distribute 17 sticks equally among 3 receivers. First, we perform the easy part of the task and distribute 15 sticks with each receiver getting 5 of these. Then we cut each of the two remaining sticks into three parts of equal length and distribute the resulting six thirds among the three baskets with each basket receiving two of the thirds. The operation of distributing 17 sticks among 3 receivers can be written formally as $17/3 = 5 + 2/3$.

The above statement which says that the division of 17 by 3 cannot be performed is now revised by introducing a new type of number. If it is possible to cut objects into parts of equal size, it is certainly possible to distribute any natural number of such objects among any natural number of receivers. By definition, the number of objects one receiver gets is said to be a number though it may not be natural. Thus, 1/3 and 17/3 are numbers of a new type, called "fractions," created by us. The set of numbers which we discussed up to now is the union of the natural numbers, the zero and those fractions which do not result in natural numbers.

We eliminated the impossibility of division for certain pairs of numbers by allowing objects to be cut into parts of equal size. Now we eliminate the impossibility of subtraction for certain pairs of numbers by a formal trick. At this point, we cannot see a way to subtract 8 from 5 since this would mean that we can take 8 objects out of a container which originally contained only 5 objects. As long as we are restricted to operations in the real world, we have no way of overcoming this impossibility. But in mathematics it has sometimes been very common to leave the real world and enter the so-called *formal* world. In this formal world, objects and operations are defined without any reference to reality. The formal *objects* are "created" just by giving them names, and the formal *operations* are defined just by writing down which formal objects should be the result of the formal operation on other formal objects. The first formal objects which I shall now introduce are the so-called *negative numbers*. Imagine that we could place the numbers which we already have in front of a mirror. All these numbers are ordered according to their "weight," i.e., for each pair of different numbers we can always say which of the two numbers is greater. Therefore, we can imagine a straight line onto which we place the numbers ordered according to their weight. This line begins with the position of the zero, and from there on the natural numbers follow at equal distances. The fractions will be placed between the positions of these numbers. Now we imagine we could put this straight line in front of a mirror in such a way that the zero position touches the mirror, and that the line stands at a right angle (90 degrees) to the mirror surface. Now we see this straight line twice, both in front of and behind the mirror surface. Thus, each number in front of the mirror has its partner in the mirror. Since the zero sits on the mirror surface, it is the only number which does not have a separate partner. Obviously, we must now introduce a way to make clear whether we are talking about a number in front of the mirror or about its partner behind the mirror. For example, we could identify the partner of five by calling it "mirrored five." Or, at least in the case of writing, we could use black ink for the numbers in front of the mirror and red ink for their partners in the mirror. This use of colors is actually used when people denote the financial condition of a company by writing monetary numbers in black or red. This indicates that a company is making either a profit or a loss. Mathematicians use the *minus sign* to identify the numbers in the mirror (-1, -2, -3, etc.). In this way they indicate that such a number can be obtained only as the result of a subtraction where more objects are taken out of a container than have previously been in it. Such numbers are called *negative numbers*.

I have very often heard the statement, “Negative numbers don’t mean anything to me since I cannot relate them to any picture from the real world.” People say this with the same kind of regret as if they were talking about a personal weakness: “I cannot make attractive drawings.” They believe that mathematicians relate much more meaning to the concept of negative numbers than a “normal person.” But this belief is not justified. The description above prescribed what picture you should use to relate to the concept of negative numbers, and this is exactly the picture all mathematicians have. A negative number is a mirrored positive number – and nothing more. Once you can accept this idea of a number sitting in front of a mirror and seeing its reflection, you are well-prepared to accept another type of reflection which will be introduced later.

The reasons for “creating” fractions and negative numbers came up when we found that a particular arithmetic operation could not be performed on all pairs of known (natural) numbers unless we allowed the results to be numbers which were of a type different from those we already knew. Consequently, we must now ask whether arithmetic operations with numbers of the new types (fractions and negative numbers) again force us to create new types of numbers. There is no space or reason here for presenting all the cases which have to be considered in order to answer this question. The answer is that no new types need to be introduced. Figure 2.1 represents the results obtained in our process of creating numbers. Besides the natural numbers originally given, there are four new types of numbers, namely the zero, the positive fractions, the negative whole numbers and the negative fractions. All these numbers together are called *rational numbers* which comes from the Latin word ‘ratio’. In the figure, the four arithmetic operations are symbolized by rectangles, each of which has two arrows entering and one arrow leaving. The two arriving arrows symbolize the two numbers which are the inputs of the operation, and the leaving arrow symbolizes the result. In each case, the result is a rational number. Division is the only operation which does not allow the input to be any pair of rational numbers; there is the restriction that zero not be allowed to be the denominator. It does not make any sense to request that a certain number of objects be distributed without specifying the number of receivers for the distribution.

Although the results shown in Fig. 2.1 do not imply any need for finding new types of numbers, the rational numbers actually do not yet represent the end of the process of creating numbers. The old Greek mathematicians found that there are certain relations between the lengths of lines in geometric shapes which are not based on rational numbers. The two best-known examples are the relation between the diagonal and the edge of a square, and the relation between the circumference

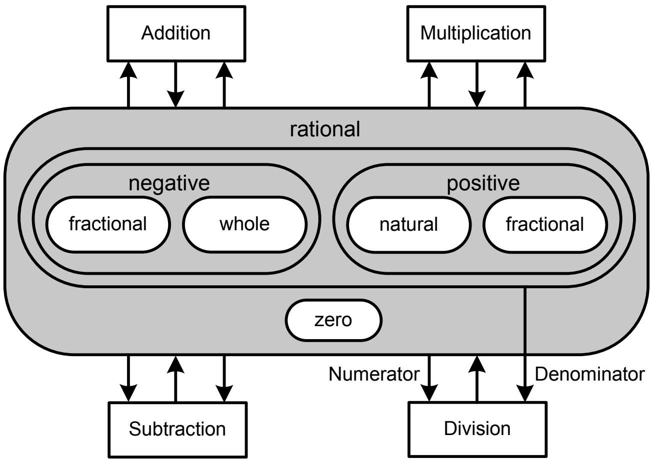

Fig. 2.1 The rational numbers and the four arithmetic operations

and the diameter of a circle. It is impossible to express these relations exactly as fractions, although they can be approximated arbitrarily closely by fractions. Numbers of this type are called *irrational* numbers. Rather good approximations are 707/500 for the relation of the square and 3927/1250 for the relation of the circle. All over the world, the relation between the circumference and the diameter of a circle is symbolized by the Greek letter π (“pi”). The set of numbers which we get by gathering the rational numbers and the irrational numbers together is called the set of *real numbers*.

There is a fundamental difference between the square’s diagonal relation and the circle’s number π. In the case of the diagonal relation d, a so-called arithmetic equation exists which exactly defines this number. Although the concept of equations will be presented in a later section of this book, you will easily understand the following equation: $d*d=2$. This equation says that the multiplication of d with itself has the result 2. In the case of the number π an arithmetic equation defining the number does not exist. The types of numbers which cannot be defined by an arithmetic equation are called *transcendental* numbers. The prefix “trans” indicates that something beyond certain limits is considered. The transcendental numbers lie beyond any considerations which start with natural numbers and lead to a definition of the number we are interested in. In later sections we shall encounter this type of number again.

Before describing the last part of the process of creating numbers, we stay for a while on the playground of the real numbers. In this sentence, I used the word “playground” deliberately in order to make it clear to you that many mathematical findings have been found by playing around and not by searching for solutions to serious problems. Naturally, children beginning to play do not question what their play might be good for, and this attitude is also often very true for mathematicians. We could even think that the following biblical quotation was meant for mathematicians: "Unless you become like little children, you shall not enter into the kingdom of heaven." (Matthew 18, 3). Therefore, let's begin to play.

Name of the operation	Reduction to known operations	Formal representation
Multiplication	Addition of multiple equal summands: Product = 2 + 2 + 2 + 2 + 2	Product = 10 = 5 $*$ 2
Division (Inversion of Multiplication)	Search for the Factor Q in the Multiplication: 10 = 5 $*$ Q	Quotient = 2 = 10:5 = $\frac{10}{5}$
Power	Multiplication of multiple equal factors: Power = 2 $*$ 2 $*$ 2 $*$ 2 $*$ 2	Power = 32 = 2^5
Root (First Inversion of Power)	Search for the Base R in the Power: 32 = R^5	Fifth root of 32 = 2 = $\sqrt[5]{32}$
Logarithm (Second Inversion of Power)	Search for the Exponent L in the Power: 32 = 2^L	Logarithm of 32 to the Base 2 = 5 = $\log_2 32$

Fig. 2.2 Powers, roots and logarithms

The table in Fig. 2.2 summarizes the results of our playing around with numbers and operations, and leads us to new concepts and operations. We see that the division operation is the inverse of the multiplication operation, and that multiplication gives us the idea to introduce the concept of *power*. By formally replacing the plus sign in the definition of the multiplication by the multiplication symbol, we obtain the definition of the power relation. In the example, we write the power operation as 2^5 where the 2 is called the *base* and the 5 is called the *exponent*. These words refer to facts from our every day lives: the position of the basement is underneath the rest of the house, and the position of something which is exposed is above in order to make sure that it can be seen by everybody. Multiplication has only one type of inversion because the order of the two factors can be reversed without having an effect on the result. In contrast to this, the power relation has two types of inversions because reversing base and exponent may change the result: $2^5=32$ and $5^2=25$. The first type of inversion of the power relation provides the concept of *roots*. The symbol used in formulas for the operation "root" is an abstraction from the letter r which makes us think of the word root. The second type of inversion of the power relation provides the concept of the "*logarithm*". The name of this operation comes from the two Greek words, "logos" and "arithmos".

The new operations introduced in Fig. 2.2, namely the power, the root and the logarithm help us to dream up the next steps in our play activities. In Fig. 2.2 we restricted ourselves to using only natural numbers. Now we ask whether it makes sense to ask for the results of these operations if their operands are no longer natural numbers, but negative numbers or fractions. If we define a power b^n as a chain of n numbers which all have the value b and are connected by multiplication symbols, the expressions b^0, b^{-3} or $b^{4.5}$ cannot be interpreted as meaningful powers. But Fig. 2.3 shows that it really makes sense to define powers with exponents which are negative numbers or fractions.

$$\frac{2^7}{2^4} = \frac{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot 2\cdot 2\cdot 2}{\cancel{2}\cdot\cancel{2}\cdot\cancel{2}\cdot\cancel{2}} = 2^3$$

Conclusions: $\boxed{\frac{b^m}{b^n} = b^{m-n}}$ $\boxed{b^0 = 1}$ $\boxed{b^{-m} = \frac{1}{b^m}}$

$$(3\cdot 3)\cdot(3\cdot 3)\cdot(3\cdot 3)\cdot(3\cdot 3)\cdot(3\cdot 3) = (3^2)^5 = 3^{10}$$
$$(3\cdot 3\cdot 3\cdot 3\cdot 3)\cdot(3\cdot 3\cdot 3\cdot 3\cdot 3) = (3^5)^2 = 3^{10}$$

Conclusions: $\boxed{(b^m)^n = (b^n)^m = b^{m\cdot n}}$ $\boxed{\left(b^{1/m}\right)^m = b^1 = b}$

$\boxed{b^{1/m} = \sqrt[m]{b}}$

Fig. 2.3 Powers whose exponents are not natural numbers

From Fig. 2.3 follows $b^0=1$, $b^{-3}=1/b^3$ and $b^{4.5}=b^{9/2}=(b^{1/2})^9=(\sqrt{b})^9$. The values of these powers are real numbers, i.e., they are elements of the set of numbers which we have already found. But we are forced to create a new type of number when we ask for the square root of a negative number, i.e., when we ask for a number x which satisfies the equation $x^2 = -1$. This number x cannot be a real number since the square of a real number is always a positive number, even if the original number is negative: $(-1)*(-1) = +1$. In a textbook written in 1768 by the brilliant mathematician Leonhard Euler (1707-1783), I found the amusing statement [EU]: "When we have to compute the square root of a negative number, we certainly are in an embarrassing situation." We now remember the situation when we used a mirror to create the negative numbers. A method which was successful once could be successful again. Therefore, we will take all of the real numbers and place them in front of a mirror, but we cannot proceed in the same way as we did when we created the negative numbers. Otherwise, we would not be able to distinguish between the negative numbers and the new type of numbers. When creating the negative numbers, the straight line with the numbers was placed at a right angle to the surface of the mirror. Now the line with the numbers to be mirrored must be placed at a different angle. A reasonable choice turns out to be 45 degrees.

As in the previous case, the zero is the number which touches the surface of the mirror. In this way, we get a partner in the mirror for every real number in front of it except for the 0. Since the numbers in front of the mirror are called real numbers, the numbers in the mirror could have been called unreal numbers, but Leonhard Euler suggested the name "*imaginary number*s" and introduced the letter i to identify the number whose square is -1. That which I pointed out when we created the negative numbers by using a mirror can be repeated here: there is nothing strange or miraculous about imaginary numbers, they are just real numbers in a mirror, and the only property we assign to them is that their square is a negative real number: i∗i=-1, 2i∗2i=-4, 3i∗3i=-9, etc. You should not think that mathematicians know more about imaginary numbers than what I just told you.

However, the process of creating numbers is not finished with the creation of the imaginary numbers, for now, we must check what types of numbers we get as results from arithmetic operations where at least one of the operands is imaginary. The different possible combinations are shown in the table in Fig. 2.4. The thickly-framed field stands out as the one with a result which is neither real nor imaginary. When the pair of operands for an addition or a subtraction contains one real and one imaginary number, the result is a number of a type which does not belong to the types we already know. This means that again we are forced to create a new type of number. These new numbers are called *complex numbers*.

	Addition or Subtraction	Multiplication or Division
Both operands are imaginary.	Imaginary result: 5i + 2i = 7i 5i - 7i = -2i	Real result: 5i ∗ 2i = -10 10i : 5i = 2
One operand is real, the other one is imaginary.	Complex result: 5 + 2i = 5 + 2i 5 - 7i = 5 - 7i	Imaginary result: 5i ∗2 = 10i 10i : 5 = 2i 10 : 5i = -2i

Fig 2.4 Arithmetic operations with imaginary numbers

A complex number is always the sum of its real part and its imaginary part. These two parts stay separate and are not merged in the result in contrast to a normal addition where the two operands can no longer be seen in the result. In normal addition, the two operands +7 and (-4) are merged and don't show up in the result, +3. In contrast to this, in a complex number the real part +7 and the imaginary part 3i keep their identity and stay visible. The real part and the imaginary part are like a husband and wife who do not disappear by being married.

Although it was Leonhard Euler who introduced the symbol i for the square root of -1, he was not the first one who had the idea of "creating" numbers which are square roots of negative numbers. This idea was born about 200 years before Euler began to use such numbers. In 1545, the Italian mathematician Gerolamo Cardano (1501-1576) published a book that introduced the use of complex numbers for solving certain mathematical problems.

Creating one type of number after the other might become boring if you are not a fanatic mathematician. Therefore, you probably will be pleased to hear that now we really have reached the end of this number creation process. The result of an arithmetic operation will always be of one of the number types we created so far if the operands are of these types, too. Addition and subtraction are the simplest cases since here the operation can be performed on the real parts and the imaginary parts independently: (4+3i)+(12+5i)=(16+8i). In the case of multiplication we have to do what is shown in Fig. 2.5.

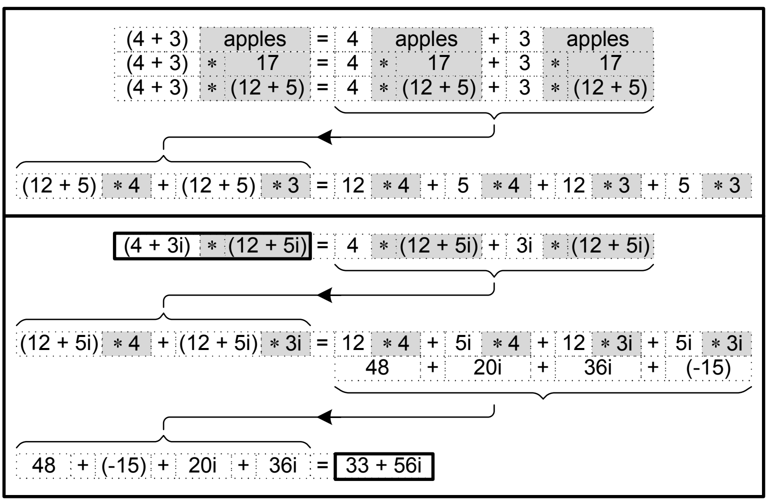

Fig 2.5 Example showing the multiplication of complex numbers

In the upper section of this figure, the method of multiplying two sums is developed. In each of the first three rows, you see the same pattern of white and grey fields. The first row shows that this pattern is not restricted to multiplication. It illustrates the possibility of expanding the expression “four plus three apples” to the longer expression “four apples plus three apples”. The next two rows show that the apples can be replaced by any factor following the multiplication symbol. In the third row, this factor is a sum, and thus we have a product of two sums on the left side of the equation. By reordering the right side of this equation, we obtain the expression on the left side of the fourth row. Here, the method of expanding a white-grey-pattern can be applied twice, and this finally results in four products, the factors of which are no longer sums, but numbers.

In the lower section of Fig. 2.5, the method from the upper section is applied to a product of two sums, each of which has one real and one imaginary summand. You see that the product of these two complex numbers is again a complex number, 33+56i.

Now we have reached the point where I can show you that still another number creator, often unnoticed, was sometimes involved behind the scenes of our number-creation activities. First we look at Fig. 2.6. On the left hand side, all the types of numbers which we know by now are represented in a container view of the kind already used in Fig. 2.1. In this view, you should think of containers into which you are looking from above. There is one big container which contains smaller containers. Certainly, it would be more natural for you to consider if these were not containers for numbers but for flour, sugar and rice. In our case, the containers are for the zero, the positive real numbers, etc. On the right of this container view, the same numbers are represented as points in a plane. The points on the horizontal line represent the real numbers with the negative numbers on the left, the zero in the middle, and the positive numbers on the right. The points on the vertical line below and above the zero represent the imaginary numbers. This

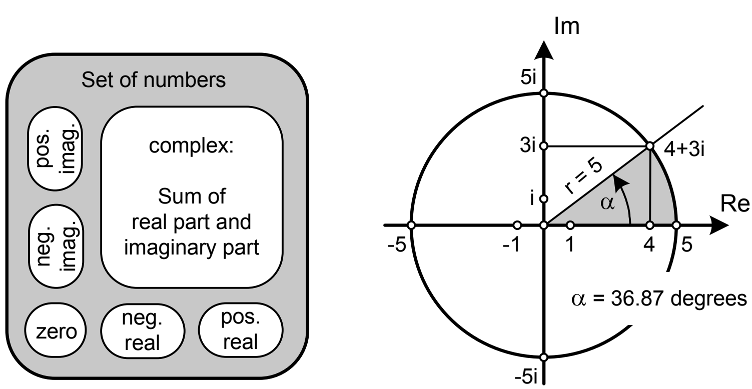

Fig 2.6 Different types of numbers and their representations as points in a plane

vertical line is a copy of the horizontal line turned by 90 degrees. The points in the plane which are neither on the horizontal line nor on the vertical line represent the complex numbers. As an example, I selected and marked the number 4+3i. Instead of identifying this number by providing its real part and its imaginary part, I could have used an alternative identification by providing the radius 5 and the angle 36.87 degrees. The horizontal line of real numbers, the line of the radius and the circle define the grey area which has the shape of a piece of pie. The particular form of identification to choose depends on the actual situation. Please note that a complex number is not a point in a plane, but can be represented as such a point.

By applying this representation of complex numbers to the numbers in Fig. 2.5, we get Fig. 2.7. Even if I had not explicitly expressed the relations in the thickly framed fields, you presumably would have soon noticed that the radius of the product is equal to the product of the radiuses of the factors, and that the angle of the product is equal to the sum of the angles of the factors. Isn't this amazing? When complex numbers were created as sums by adding the real parts and the imaginary parts of the numbers, radiuses and angles were not considered at all. Obviously, we were able to create something containing hidden laws of great importance, although we were absolutely unaware of these laws during the creation process. That's what I meant when I said that another number creator was involved behind the scenes in our number creation activities.

	Re + Im	Radius	Angle
1. Factor	4 + 3i	5	36.87 degrees
2. Factor	12 + 5i	13	22.62 degrees
Product	33 + 56i	65 = 5∗13	59.49 degrees = (36.87+22.62) degrees

Fig 2.7 Radiuses and angles for the multiplication in Fig. 2.5

This new insight now makes it easy for us to define the division of two complex numbers as the inverse of their multiplication. Since in the multiplication operation, two radiuses are multiplied and two angles are added, the division operation can be performed by dividing radiuses and subtracting angles. This new insight also indicates how powers and roots of complex numbers can be computed: we have to compute powers or roots of radiuses and we have to multiply or divide angles. But certainly, this is only well-defined for real exponents. It is easy to write 2^i, but it is not at all easy to find out whether this power expression defines a number or whether it does not make any sense at all. The richness of our language makes it possible not only to write down reasonable expressions or statements, but also to construct grammatically-correct sequences of words which are without any reasonable meaning. As an example, consider the expression “the natural number between 17 and 18”. This can be written or said, but it has no meaning since it contradicts the definition of the natural numbers. Thus, the expression 2^i could be nonsense or it could define a number. What I have presented to you so far is totally insufficient for determining which is true. You’ll have to wait until the end of Chapter 3. Then all the results we need for finding the answer will have been presented. However, it might be interesting for you to see the answer even if you cannot understand how it is obtained: 2^i equals the complex number 0.769+0.639i.

I think that now, at the end of the story of the creation of numbers, you will agree that we can say, “We saw how wonderfully and logically one result implied the next.”

Sets Are Everywhere

I still very well remember the time when the so-called set theory was introduced as a subject in elementary school teaching. All of a sudden, parents realized that they no longer could help their children with their homework. Adult education centers reacted by offering evening courses for parents to make them familiar with the modern way to teach mathematics, called the “new math.” One of my classmates from high school became an elementary school teacher, and every now and then we still meet. At one of these meetings, I mentioned set theory and asked her to give me her view of it. I still remembered that mathematics had not been her favorite subject in high school. But I also knew that being an expert in mathematics is not a requirement for understanding the basic concepts of set theory if they are adequately explained. Obviously, she had not been lucky enough to be taught by a good teacher at her college of education. From what she enthusiastically told me, I easily realized that she had not really understood the sense and the purpose of set theory. In the following section, I will tell you what her college teacher should have told my classmate.

The mathematical concept of a *set* corresponds to the idea of a container which contains nothing, little or much, where the set is what’s in the container. In the previous section, I told you the story about the creation of the numbers, and there, in a few cases, I used the term “set” in its mathematical sense. When we look again at figures 2.1 and 2.6, we can imagine looking from above into containers which contain numbers. For example, there is a container for the zero, another one for the positive rational numbers, etc. Of course, the containers for numbers are not real containers like baskets or boxes. Only specific objects can lie in specific containers, like apples in baskets. If the objects are abstract, like numbers, their containers must be abstract, too. Most sets we are interested in are sets of objects of the same type – like sets of apples or sets of numbers. It doesn't make much sense to imagine a container which contains two apples, the fraction 3/5 and the moon. Our interest in sets is restricted to cases where the elements in a set share some common characteristics which are worth talking about. For instance, we talked a lot about the characteristics of complex numbers.

The objects which belong to a set are said to be the *elements* of the set. When we consider a well-defined set and some arbitrary object, it makes sense to ask whether or not the object is an element of the set. For example, we may ask whether the number π is an element of the set of rational numbers, and the answer is no. The symbol $\in$ is used in formulas to state the fact that a given object, x, is an element of a given set, S. The formula for this is written $x \in S$. The shape of the symbol $\in$ resembles the capital letter E to which we can associate the word "enclosed". Accordingly, we can express the fact that x is not contained in S by writing $x \notin S$.

The use of the concept of sets lies in two quite different areas which I call the "language area" and the "infinity area". In the language area, a small number of words with very strict meanings have been selected in order to make communication clearer and unambiguous. It was the language area which motivated the introduction of set theory into elementary school teaching. Mathematicians, however, think the language area is rather trivial. For them, it's the infinity area which makes set theory interesting. We shall now spend some time in the language area and, only at the end, shall we have a short look at the infinity area.

It is appropriate for us to enter the language area like a child in the first grade. At first, we have to determine a *universe* which means only that we have to agree upon the type of objects we are going to talk about. Once we have determined our universe, we will not talk about other types of objects until we explicitly change the universe. Our first universe is shown in Fig. 2.8. It contains 18 geometric shapes which differ in size, shape and color. If we don't want to talk about the universe as a whole, but only about certain selected elements, we may draw container borders within the universe and associate the selected elements with their containers. In Fig. 2.8 you see a border which encloses all white elements, and another border which encloses all triangles. The set of elements within such a border is called a *subset* of the universe. Once we have determined two subsets, we can ask two questions concerning these subsets. For the first question, we ask for the *intersection* of the two subsets which is the set of those elements which are contained in both subsets at the same time. In Fig. 2.8, the intersection shown is the set of all white triangles. Instead of asking for the intersection, we can ask for the *union* of two given subsets which is the set of all elements which are contained in at least one of the two subsets. If their intersection is empty, two subsets are said to be *disjoint*. In this case, there are no elements which are contained in both subsets.