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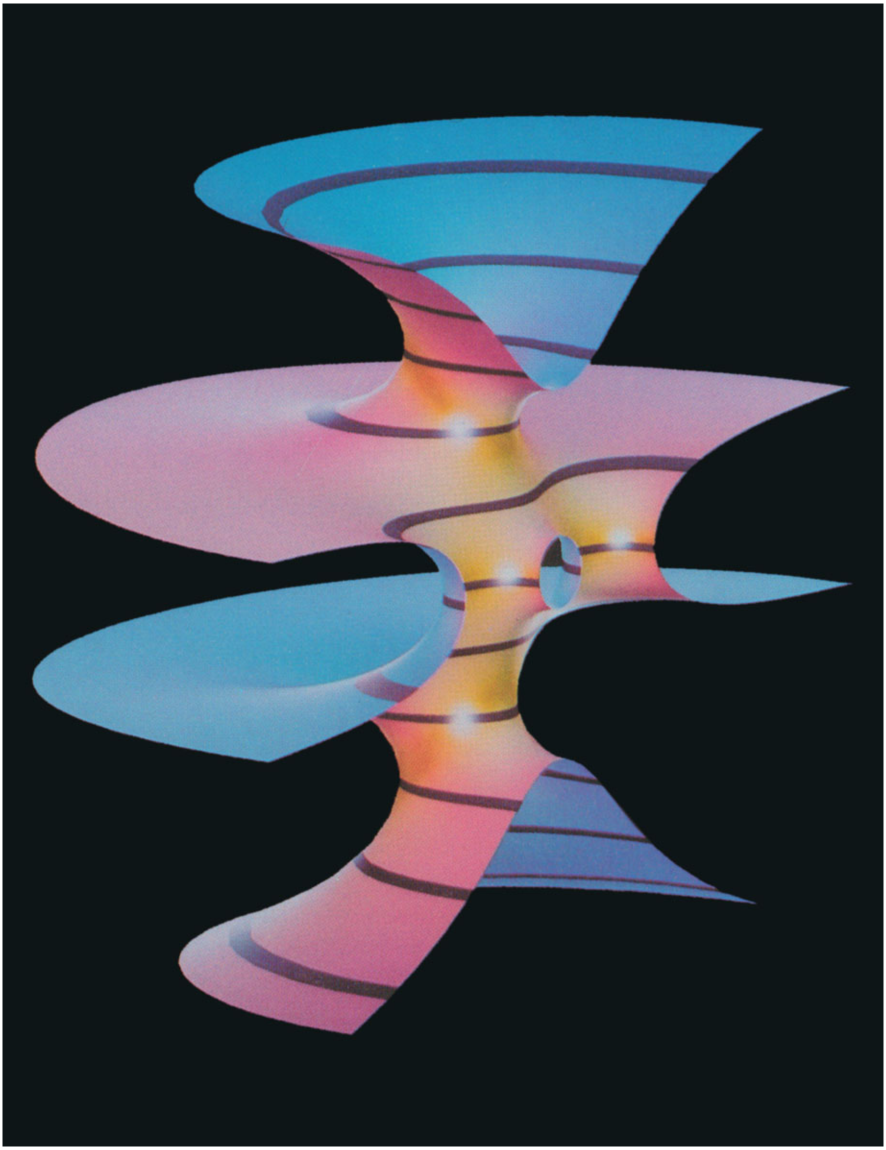

Half of an analog to Costa's surface which is stationary in a configuration consisting of four nearly semicircular arcs and a plane. Courtesy of K. Polthier

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Regularity of Minimal Surfaces

With assistance and contributions by A. Küster

Revised and enlarged 2nd edition

With 62 Figures and 4 Color Plates

Springer

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Preface

This book is the second volume of a treatise on minimal surfaces consisting of altogether three volumes which can be read and studied independently of each other. The central theme is *boundary value problems for minimal surfaces* such as Plateau's problem. The present treatise forms a greatly extended version of the monograph *Minimal Surfaces I, II* by U. Dierkes, S. Hildebrandt, A. Küster, and O. Wohlrab, published in 1992, which is often cited in the literature as [DHKW]. New coauthors are Friedrich Sauvigny for the first volume and Anthony J. Tromba for the second and third volume.

The four main topics of this second volume are *free boundary value problems*, *regularity of minimal surfaces* and their *geometric properties*, and finally a new method is introduced to show that minimizers of area are immersed. Since minimal surfaces in $\mathbb{R}^3$ are understood as harmonic, conformally parametrized mappings $X : \Omega \to \mathbb{R}^3$ of an open domain Ω in $\mathbb{R}^2$, they are real analytic in Ω, and so the *problem of smoothness* for X is the question how smooth X is at the boundary $\partial\Omega$ if X is subject to certain boundary conditions. However, even if X is "analytically regular", it might not be "geometrically regular" since it could have branch points. We investigate how X behaves in the neighbourhood of branch points, and secondly whether such points actually exist. In addition we describe geometric properties of minimal surfaces in $\mathbb{R}^3$ or, more generally, of H-surfaces in an n-dimensional Riemannian manifold. This book can be read independently from the preceding volume of this treatise although we use some terminology and results from the previous material.

We thank E. Kuwert, F. Müller, D. Schwab, H. von der Mosel, D. Wienholtz, and S. Winklmann for pointing out errors and misprints in [DHKW] which are corrected here. Particularly we are indebted to Frank Müller for some penetrating contributions to Chapter 3, and to Albrecht Küster who supplied a considerable part of Chapter 1 (which was taken from Vol. 1 of [DHKW]). Special thanks also to Ruben Jakob who carefully read and corrected Chapters 4 and 6, and to Klaus Steffen and Friedrich Tomi for their valuable comments to the Scholia of Chapter 6. Furthermore we thank Klaus

Bach, Frei Otto, and Eric Pitts for providing us with photographs of various soap film experiments. Thanks also to M. Bourgart, D. Hoffman, J.T. Hoffman, and K. Polthier for permitting us to reproduce some of their computer generated figures.

The continued support of our work by the Sonderforschungsbereich 611 at Bonn University as well as by the Hausdorff Institute for Mathematics in Bonn and its director Matthias Kreck was invaluable. We also thank the Centro di Ricerca Matematica Ennio De Giorgi in Pisa and its director Mariano Giaquinta for generous support of our work.

We are especially grateful to Anke Thiedemann and Birgit Dunkel who professionally and with untiring patience typed many versions of the new text.

Last but not least we should like to thank our publisher and in particular our very patient editors, Catriona Byrne, Marina Reizakis, and Angela Schulze-Thomin, for their encouragement and support.

Duisburg	*Ulrich Dierkes*
Bonn	*Stefan Hildebrandt*
Santa Cruz	*Anthony Tromba*

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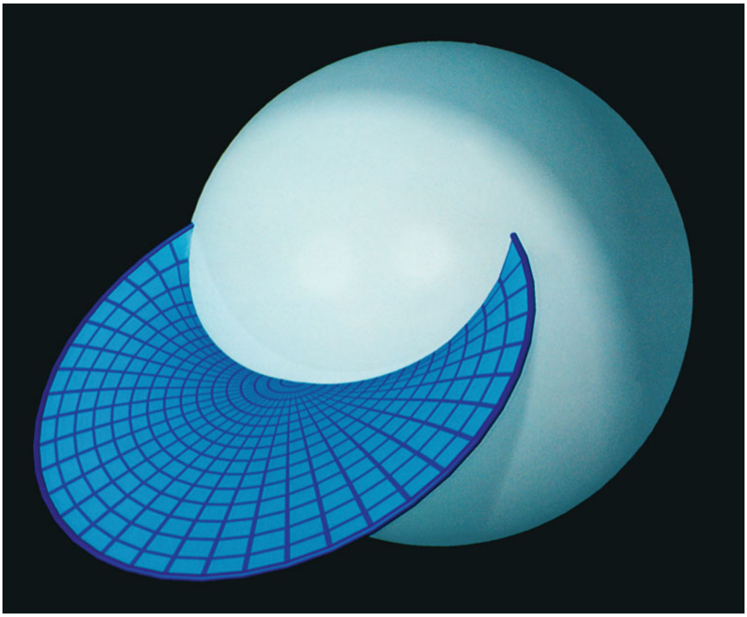

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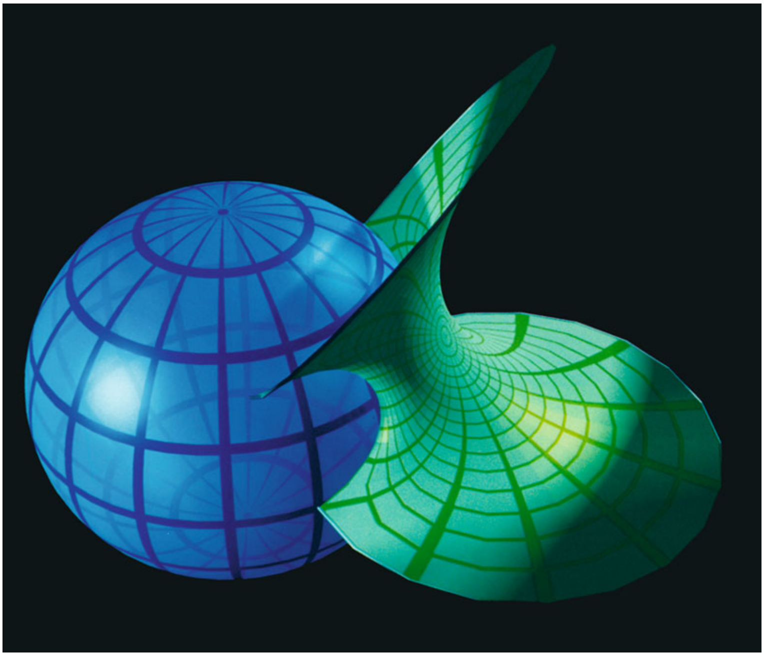

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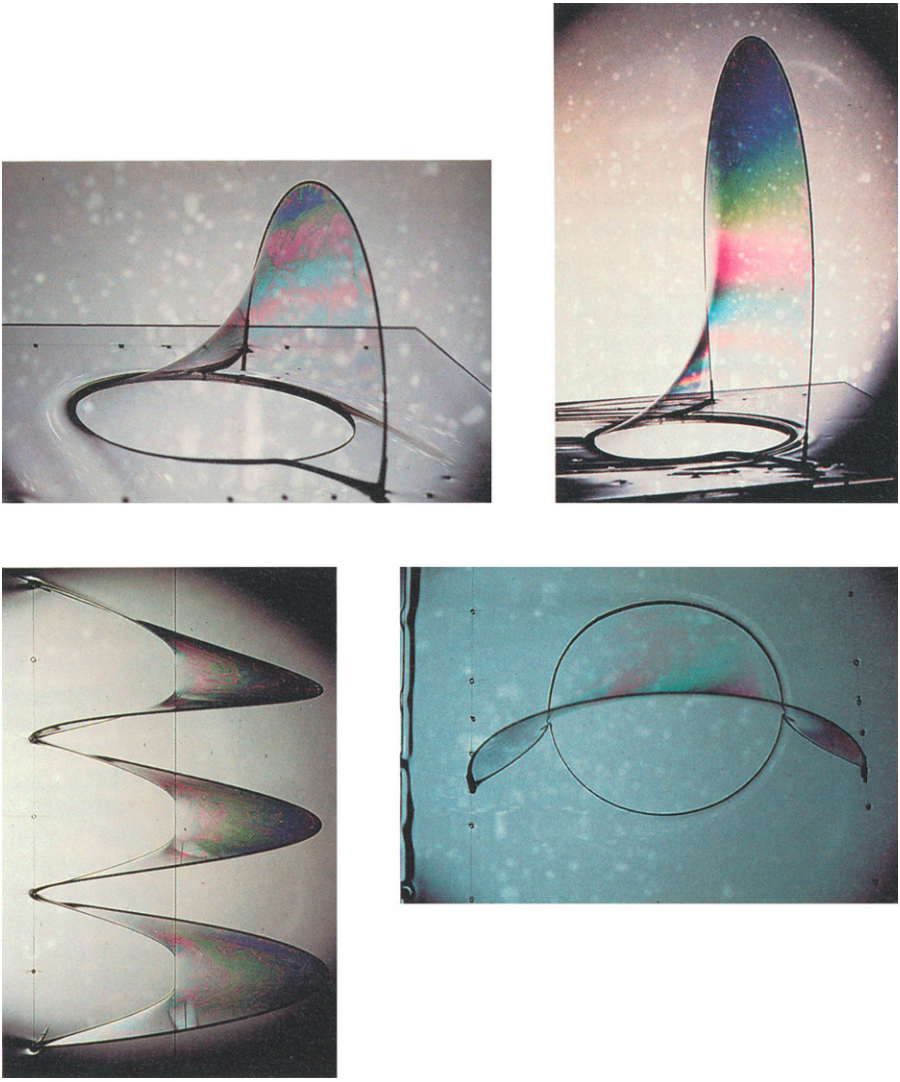

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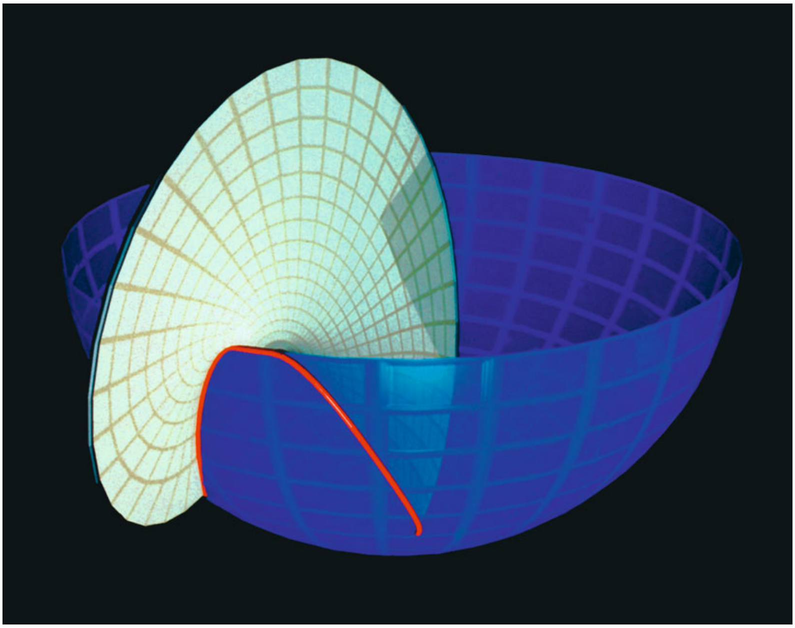

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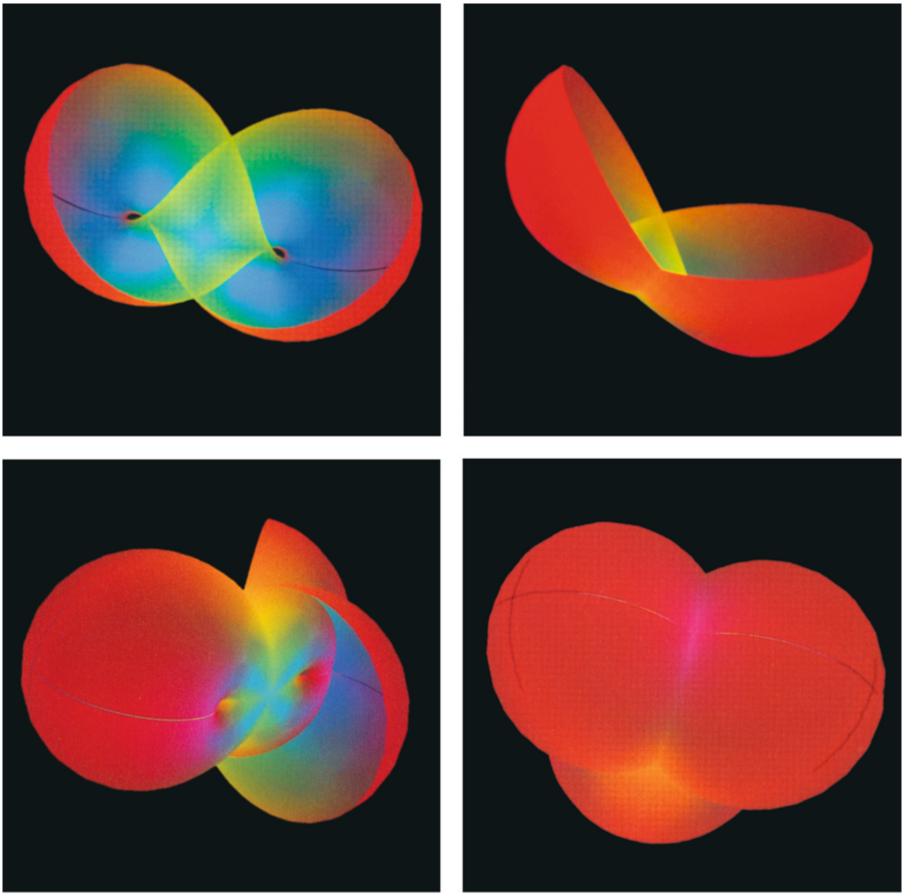

Plate IV. Construction of Wente's compact H-surface from the three building blocks. Courtesy of D. Hoffman and J.T. Hoffman

Introduction

We begin this volume with a survey on minimal surfaces with entirely free boundaries. In the Sections 1.1–1.3 Courant's existence result is described. It follows the derivation of the *transversality condition* at the free boundary in Section 1.4 and of the *condition of balancedness* that are to be satisfied by stationary surfaces. Further results and examples in Sections 1.6–1.10 conclude this chapter.

In Chapter 2 we investigate the boundary behaviour of minimal surfaces subject to Plateau boundary conditions or to free boundary conditions. Roughly speaking we show that a minimal surface is as smooth at the boundary as the data of the boundary conditions to which it is subject. There is a basic difference between grappling with the regularity problem for area minimizing surfaces or for merely stationary solutions of boundary value problems. For disk-type minimizers it is always possible to derive a priori estimates, while examples show that it is generally impossible to establish a priori estimates for stationary solutions of free boundary problems. Thus it becomes necessary to apply indirect methods if one wants to prove boundary regularity of minimal surfaces subject to free boundary conditions.

For a more complete understanding of the boundary behaviour of minimal surfaces one has not only to investigate their class of smoothness at the boundary, but it is also necessary to find out whether singular points occur at the boundary and, if so, how a minimal surface behaves in the neighbourhood of such points. This question is tackled in Chapter 3. If a minimal surface $X(w), w = u + iv$, is given in conformal parameters u, v, then its singular (= nonregular) points are exactly its branch points w_0, which are characterized by the relation $X_w(w_0) = 0$. In Chapter 3 we derive asymptotic expansions of minimal surfaces at boundary branch points which can be seen as a generalization of Taylor's formula to the nonanalytic case. Moreover, we also derive expansions of minimal surfaces with nonsmooth boundaries (e.g. polygons) at boundary points which are mapped onto vertices of the nonsmooth boundary frame.

Asymptotic expansions and boundary smoothness are very useful if one wants to treat subtle geometric and analytic problems. Furthermore they are indispensable for the derivation of *index theorems* and for the investigation of the *Euler characteristic* of minimal surfaces. Topics of this kind will be discussed in Volume 3.

The long Chapter 4 could have been labeled as *geometric properties of minimal surfaces*. First we derive *inclusion theorems* for minimal surfaces in dependence of their boundary data. Such results, obtained in Sections 4.1 and 4.2, are more or less sophisticated versions of the maximum principle. They lead to interesting nonexistence results for connected minimal surfaces and H-surfaces whose boundaries consist of several disjoint components, as it is seen in Sections 4.3–4.5. Here we even discuss the situation for higher dimensional surfaces and for solutions of variational inequalities, obtained from *obstacle* problems. Inclusion principles for such solutions are the fundament for results ensuring the existence of minimal surfaces and H-surfaces solving Plateau's problem in Euclidean space or in a Riemannian manifold respectively, see Sections 4.7 and 4.8. Of particular interest are the *Jacobi field estimates* obtained in Section 4.8.

Isoperimetric inequalities for minimal surfaces solving either Plateau's problem or a free boundary value problem are derived in Sections 4.5 and 4.6. The simplest kind of such an inequality was already stated in Section 4.14 of Vol. 1; for the sake of completeness we repeat here the derivation. Furthermore, in Section 6.4 of Vol. 1 an isoperimetric inequality for harmonic mappings $X : \Omega \to \mathbb{R}^3$, due to Morse & Tompkins, was derived, which plays an essential role in Courant's theory of unstable minimal surfaces.

In Chapter 5 we investigate an extension of the isoperimetric problem, the so-called *thread problem*, and prove the existence and regularity of minimal surfaces with *movable boundary parts of fixed lengths*, which in soap film experiments are formed by very thin threads.

The last chapter contains a new approach to the celebrated result that a minimizer of area in a given contour has no interior branch points. The novelty consists particularly in the fact that, in certain cases, relative minimizers of Dirichlet's integral are shown to be free of nonexceptional branch points, and this is achieved by a purely analytical reasoning.

The Scholia serve as sources of additional information. In particular we try to give credit to the authorship of the results presented in the main text, and we sketch some of the main lines of the historical development. References to the literature and brief surveys of relevant topics not treated in our text complete the picture.

Our *notation* is essentially the same as in the treatises of Morrey [8] and of Gilbarg and Trudinger [1]. Sobolev spaces are denoted by H^k_p instead of $W^{k,p}$; the definition of the classes C^0, C^k, C^∞ and $C^{k,\alpha}$ is the same as in Gilbarg and Trudinger [1]; C^ω denotes the class of real analytic functions; $C^\infty_c(\Omega)$ stands for the set of C^∞-functions with compact support in Ω. For greater precision we write $C^k(\Omega, \mathbb{R}^3)$ for the class of C^k-mappings $X : \Omega \to \mathbb{R}^3$, whereas the corresponding class of scalar functions is denoted by $C^k(\Omega)$, and similarly for the other classes of differentiability. Another standard symbol is $B_r(w_0)$ for the disk $\{w = u + iv \in \mathbb{C} : |w - w_0| < r\}$ in the complex plane. On some occasions it is convenient to switch several times from this meaning of B to another one. Moreover, some definitions based on one meaning of B have to be transformed mutatis mutandis to the other one. This may sometimes require slight changes but we have refrained from pedantic adjustments which the reader can easily supply himself.

Part I

Boundary Behaviour of Minimal Surfaces

Chapter 1

Minimal Surfaces with Free Boundaries

This chapter is centered on the proof of existence theorems for minimal surfaces with completely free boundaries. We approach the problem by applying the direct methods of the calculus of variations, thus establishing the existence of minimizers with a boundary on a given supporting surface S. However, this method does not yield the existence of stationary minimal surfaces which are not area minimizing. As certain kinds of supporting surfaces are not able to hold nontrivial minimizers, our method is restricted by serious topological limitations. For example, it does not furnish existence of nontrivial stationary minimal surfaces within a closed convex surface. It seems that the techniques of geometrical measure theory are best suited to handle this problem. Unfortunately they are beyond the scope of our lecture notes, but we shall at least present a survey of the pertinent results in Section 1.8 as well as an existence result for the particular case of S being a tetrahedron. There the reader will also find references to the literature.

In the following we shall describe Courant's method for proving the existence of a nontrivial and minimizing minimal surface whose boundary lies on a given closed supporting surface. This problem is more difficult than the Plateau problem or the semifree problem treated in Chapter 4 of Vol. 1 because an arbitrary minimal sequence will shrink to a single point. In order to exclude this phenomenon, we have to impose suitable topological conditions on the boundary values of admissible surfaces. For instance, one could assume that the boundary values are continuous curves on S which are contained in a prescribed homotopy class. This approach would, however, lead to a rather difficult problem. One would first have to prove that a suitable minimizing sequence tends to a limit with continuous boundary values, and then one would have to show that these boundary values lie in a prescribed homotopy class. Therefore we abandon this idea.

Instead we show in Section 1.1 how a kind of homotopy class can be set up for surfaces X which are of class $H_2^1(B, \mathbb{R}^3)$ and have their boundary values on S. We shall also prove by way of example that the problem of prescribed

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homotopy class need not have a solution. In Section 1.2 we set up the classes of admissible functions for which we can solve the minimum problem and in which we are able to find nondegenerate solutions.

The free boundary problem will be solved in Section 1.3; the supporting set S may look as bizarre as the one in Fig. 1 or as simple as the catenoid. The gist of our reasoning consists in an indirect argument showing that the limit of a suitable minimizing sequence satisfies the prescribed topological condition, and therefore it will be a nondegenerate solution of the minimum problem.

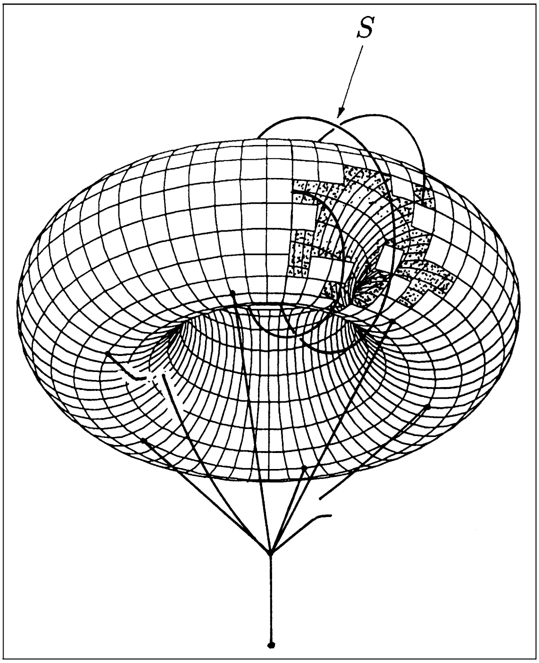

Fig. 1. A bizarre supporting set

The remaining part of the chapter will deal with additional properties of minimal surfaces with free boundaries.

In Section 1.4 we give a precise definition of a *stationary minimal surface* X whose free boundary lies on a given support surface S. Here we do not require X to be a minimizer. It will be investigated how the condition of being stationary is linked with the condition that X intersects S perpendicularly at its free trace Σ, provided Σ does not touch the boundary of S. This discussion is used in Section 1.5 to set up necessary conditions for the existence of stationary minimal surfaces with boundary on S. This will lead us to a class of non-existence results which explain, for example, why soap films in a funnel always run to its narrow end.

In Section 1.6 we prove the existence of three embedded stationary surfaces with their boundaries on a tetrahedron, following the discussion of B. Smyth. This is a case where the minimizing approach cannot be used.

Section 1.7 is concerned with stationary surfaces whose boundaries lie on a sphere. We shall prove Nitsche's result that flat disks are the only solutions to this problem that are of the type of the disk.

After a report on the existence of stationary minimal surfaces with boundaries on a convex surface (Section 1.8), in Section 1.9 we shall present some results concerning uniqueness and nonuniqueness of minimal surfaces with a free boundary on a given support surface. In particular, we construct a family of minimizing minimal surfaces with boundaries on a regular, real analytic surface of the topological type of a torus which are nonisometric to each other. Moreover, we discuss some finiteness results of Alt & Tomi for minimizers with boundaries on a real analytic supporting surface.

1.1 Surfaces of Class H_2^1 and Homotopy Classes of Their Boundary Curves. Nonsolvability of the Free Boundary Problem with Fixed Homotopy Type of the Boundary Traces

Let us fix some closed set S in $\mathbb{R}^3$. Then we want to define the class $\mathcal{C}(S)$ of surfaces $X \in H_2^1(B, \mathbb{R}^3)$ with boundary values $X|_{\partial B}$ on S. The parameter domain B will be chosen as the unit disk:

$$B = \{w = u + iv\colon |w| < 1\}.$$

In the following we shall usually pick an ACM-representative[1] for a given Sobolev mapping X. If we work with polar coordinates r, θ about the origin, i.e., $w = re^{i\theta}$, this means that we choose a representative $X(r, \theta)$ such that $X(r, \cdot)$ is absolutely continuous for almost all $r \in (0, 1)$, and that $X(\cdot, \theta)$ is absolutely continuous for almost all $\theta \in (0, 2\pi)$. Thus X is in particular a continuous function on almost all circles $C_r = \{w \in \mathbb{C}\colon |w| = r\}$.

Any function $X \in H_2^1(B, \mathbb{R}^3)$ possesses a *trace* (or *boundary values*) ξ on ∂B which is of class $L_2(C, \mathbb{R}^3), C := \partial B$, and we have both

$$\lim_{r \to 1-0} X(r, \varphi) = \xi(\varphi) \quad \text{for almost all } \varphi \in [0, 2\pi] \tag{1}$$

and

$$\lim_{r \to 1-0} \int_0^{2\pi} |X(r, \varphi) - \xi(\varphi)|^2 \, d\varphi = 0. \tag{2}$$

---

[1] ACM stands for absolutely continuous in the sense of Morrey; cf. Morrey [8], Lemma 3.1.1.

However, the trace $\Sigma = \{\xi(\varphi)\colon \varphi \in [0, 2\pi]\}$ of an arbitrary Sobolev function $X \in H_2^1(B, \mathbb{R}^3)$ will in general not be a continuous curve, whereas the curves

$$\Sigma_r := \{X(r, \varphi)\colon 0 \le \varphi \le 2\pi\}$$

are absolutely continuous for a.a. $r \in (0, 1)$. As we cannot formulate topological conditions for a possibly noncontinuous curve Σ, we shall use the continuous curves Σ_r as a substitute. In view of (1) and (2) we can expect that conditions on curves Σ_r close to Σ express conditions on Σ in an appropriate sense.

We begin, however, by defining the class $\mathcal{C}(S)$ of surfaces with boundary values on a supporting set S. *We assume once and for all that supporting sets S are closed, proper, and nonempty subsets of $\mathbb{R}^3$. However, if a boundary configuration contains other parts besides S, we allow S to be empty.*

Definition 1. *Let S be a supporting set in $\mathbb{R}^3$. Then we denote by $\mathcal{C}(S)$ the class of functions $X \in H_2^1(B, \mathbb{R}^3)$ whose L_2-trace $\xi := X|_C$ sends almost every $w \in C = \partial B$ into S.*

For any closed set S in $\mathbb{R}^3$, $S \ne \emptyset$, and for any number $\mu > 0$, we define the *tubular μ-neighbourhood* $T_\mu = T_\mu(S)$ of S by

$$T_\mu(S) := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, S) < \mu\}. \tag{3}$$

Then we can formulate our first result on surfaces of class $\mathcal{C}(S)$ which will shed some light on their boundary behaviour.

Theorem 1. *Let S be a supporting set in $\mathbb{R}^3$, and suppose that X belongs to $\mathcal{C}(S)$. Then, for every $\mu > 0$ and every $\varepsilon > 0$, there is a subset $\mathcal{I} \subset (1 - \varepsilon, 1)$ of positive measure such that, for all $r \in \mathcal{I}$, the curve $\Sigma_r = \{X(r, \varphi)\colon 0 \le \varphi \le 2\pi\}$ is a closed continuous curve which is contained in the tubular neighbourhood $T_\mu(S)$ of S.*

Note that other curves Σ_r, $r \in (1 - \varepsilon, 1) \setminus \mathcal{I}$, may stay arbitrarily far from $T_\mu(S)$ as can be shown by simple examples; cf. Fig. 1.

We shall prove Theorem 1 in several steps.

Lemma 1. *For any closed set S in $\mathbb{R}^3$, the function $d_s := \operatorname{dist}(\cdot, S)$ is Lipschitz continuous on $\mathbb{R}^3$ with a Lipschitz constant less than or equal to one.*

Proof. For arbitrary points $P_1, P_2 \in \mathbb{R}^3$ there exist points $Q_1, Q_2 \in S$ such that

$$\begin{aligned} d_s(P_1) &= |P_1 - Q_1| = \inf_{Q \in S} |P_1 - Q|, \\ d_s(P_2) &= |P_2 - Q_2| = \inf_{Q \in S} |P_2 - Q|. \end{aligned}$$

Therefore we obtain

$$d_S(P_2) \le |P_2 - Q_1|$$

and

$$d_S(P_2) - d_S(P_1) \le |P_2 - Q_1| - |P_1 - Q_1| \le |P_2 - P_1|,$$

and analogously

$$d_S(P_1) - d_S(P_2) \le |P_1 - P_2|.$$

Therefore we have

$$|d_S(P_1) - d_S(P_2)| \le |P_1 - P_2| \quad \text{for all } P_1, P_2 \in \mathbb{R}^3. \qquad \square$$

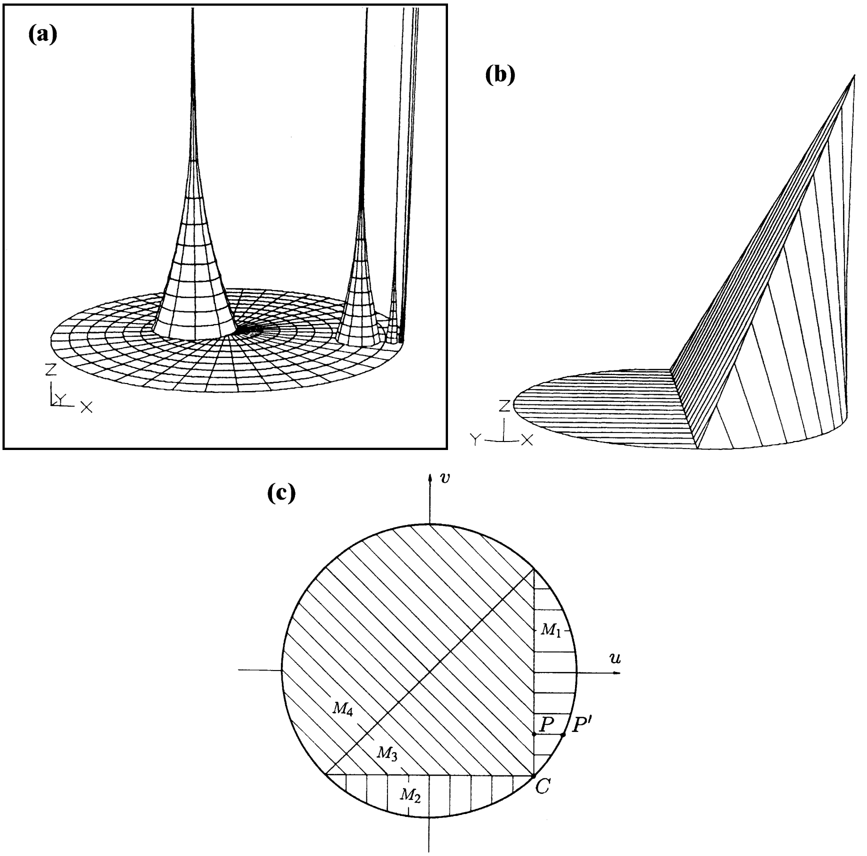

Fig. 1. **(a)** The graph of a bizarre function $f \in \overset{\circ}{H}{}_2^1(B)$ which has infinitely many peaks congruent to a part of the graph of $\log|\log|w||$. These peaks converge to the point $w = 1$ on ∂B. Given $\varepsilon > 0$ and $\delta > 0$, there is a set of values $r \in (1-\delta, 1)$ of positive measure such that the absolute values of f on C_r remain less that ε; see Lemma 4. This is a borderline case of the boundary behaviour of functions of class $\overset{\circ}{H}{}_p^1(B)$. For $p > 2$ they are continuous up to ∂B, and therefore their values on *all* circles sufficiently close to ∂B remain close to zero. For $p < 2$ there may be *no* such circle, as is shown by the function depicted in **(b)** and **(c)** which belongs to $\overset{\circ}{H}{}_p^1(B)$ for all $p \in (1, 2)$ and has a discontinuity at $C \in \partial B$

Lemma 2. *A function* $X \in H_2^1(B, \mathbb{R}^3)$ *belongs to* $\mathcal{C}(S)$ *if and only if the scalar function* $d_S \circ X$ *is an element of the space* $\overset{\circ}{H}{}_2^1(B)$ *of functions* $f \in H_2^1(B)$ *with generalized boundary values zero.*

Proof. Note that $X \in H_2^1(B, \mathbb{R}^3)$ implies $d_S \circ X \in H_2^1(B)$. Then the assertion follows from well-known properties of functions of class $\overset{\circ}{H}{}_2^1(B)$ (see Gilbarg and Trudinger [1]). ☐

Lemma 3. *Let* X *belong to* $H_2^1(B, \mathbb{R}^N), N \geq 1$. *Then, for any two numbers* $\mu > 0$ *and* $\delta > 0$, *there is an* $\varepsilon > 0$ *with the following property:*

If $\mathcal{I}' = [\theta_1, \theta_2]$ *is an angular interval with* $\theta_2 - \theta_1 = \delta$, *then there exists a subset* $\sigma \subset \mathcal{I}'$ *of positive measure such that*

$$|X(1, \theta) - X(r, \theta)| \leq \mu$$

holds for all $\theta \in \sigma$ *and for all* $r \in (1 - \varepsilon, 1)$. *In fact, we can choose* ε *as*

$$\varepsilon = \min \left\{ \frac{1}{2}, \frac{1}{4} \frac{\mu^2 \delta}{D(X)} \right\}. \tag{4}$$

Proof. From

$$r \int_{\theta_1}^{\theta_2} \int_r^1 |X_\rho(\rho, \theta)|^2 \, d\rho \, d\theta \leq 2D(X)$$

we conclude that there is a subset $\sigma \subset [\theta_1, \theta_2]$ of positive measure such that

$$\int_r^1 |X_\rho(\rho, \theta)|^2 \, d\rho \leq \frac{2}{r\delta} D(X)$$

holds for all $\theta \in \sigma$ and for $\delta = \theta_2 - \theta_1$. Moreover, we have

$$\begin{aligned} |X(1, \theta) - X(r, \theta)| &\leq \int_r^1 |X_\rho(\rho, \theta)| \, d\rho \\ &\leq \sqrt{1 - r} \left(\int_r^1 |X_\rho(\rho, \theta)|^2 \, d\rho \right)^{1/2} \end{aligned}$$

for $\theta \in \sigma$ and $0 < r < 1$, whence

$$|X(1, \theta) - X(r, \theta)| \leq \{2r^{-1}(1 - r)\delta^{-1} D(X)\}^{1/2} \quad \text{for } \theta \in \sigma.$$

Choosing ε as in (4), the assertion follows at once. ☐

Lemma 4. *Let* f *belong to* $\overset{\circ}{H}{}_2^1(B)$. *Then, for any* $\mu > 0$ *and any* $\varepsilon > 0$, *the set* $\mathcal{I} := \{r \colon 1 - \varepsilon < r < 1, |f|_{0, C_r} < \mu\}$ *has positive measure.*

Proof. Suppose that the assertion were false. Then we would have $D(f) > 0$, and there were numbers $\varepsilon > 0$ and $\mu > 0$ such that

$$|f|_{0, C_r} \geq \mu \tag{5}$$

for almost all $r \in (1-\varepsilon, 1)$. Without loss of generality we can assume that

$$0 < \mu < \sqrt{D(f)}$$

holds true.

Because of (6) we infer that, for almost all $r \in (1-\varepsilon, 1)$, there is an angle $\theta(r)$ such that

$$|f(re^{i\theta(r)})| \geq \mu.$$

Furthermore we choose some $\delta \in (0,1)$ such that

$$\varepsilon' := \min\left\{\frac{1}{2}, \frac{\mu^2\delta}{16D(f)}\right\} \tag{6}$$

satisfies $0 < \varepsilon' < \varepsilon$. By Lemma 3, every angular interval $\mathcal{J}'$ of width δ contains an angle θ' such that $f(\cdot, \theta')$ is absolutely continuous and that

$$|f(re^{i\theta'})| < \tfrac{1}{2}\mu \quad \text{for all } r \in (1-\varepsilon', 1).$$

Conclusion: For almost all $r \in (1-\varepsilon', 1)$, there exist angles $\theta(r)$ and $\theta'(r)$ with $|\theta(r) - \theta'(r)| < \delta$ and

$$|f(re^{i\theta(r)})| \geq \mu, \quad |f(re^{i\theta'(r)})| \leq \frac{\mu}{2}.$$

Thus

$$\frac{\mu}{2} \leq \left| \int_{\theta(r)}^{\theta'(r)} |f_\theta(re^{i\theta})| \, d\theta \right|$$

and consequently

$$\frac{\mu^2}{4\delta} \leq \int_0^{2\pi} f_\theta^2(re^{i\theta}) \, d\theta.$$

Thus

$$\begin{aligned} \int_{\{1-\varepsilon' < |w| < 1\}} |\nabla f|^2 \, du \, dv &\geq \int_{1-\varepsilon'}^1 \int_0^{2\pi} \frac{1}{r^2} f_\theta^2(re^{i\theta}) r \, d\theta \, dr \\ &\geq \int_{1-\varepsilon'}^1 \left(\int_0^{2\pi} f_\theta^2(re^{i\theta}) \, d\theta \right) dr \geq \frac{\varepsilon'\mu^2}{4\delta}. \end{aligned}$$

Because of (6), we have

$$\int_{\{1-\varepsilon' < |w| < 1\}} |\nabla f|^2 \, du \, dv \geq \frac{\mu^4}{64D(f)}$$

for $0 < \delta \ll 1$, and $\varepsilon' \to 0$ as $\delta \to +0$. This is impossible for an H_2^1-function. □

Proof of Theorem 1. The assertion of Theorem 1 is now an immediate consequence of the Lemmata 1–4. □

Remark 1. The assertion of Lemma 4 holds for trivial reasons if $f \in \overset{\circ}{H}{}^1_p(B)$ and $p > 2$, because Sobolev's embedding theorem yields that $f \in C^0(\overline{B})$ and $f = 0$ on ∂B. The assertion turns out to be false if $p < 2$, as one can find examples of functions $f \in \overset{\circ}{H}{}^1_p(B), p < 2$, such that near ∂B the function $|f(w)|$ is bounded away from zero by an arbitrary constant (cf. Fig. 1).

Now we want to give a reasonable definition for a homotopy class of a boundary mapping $\xi(\theta) = X(1, \theta)$ of a surface X of class $\mathcal{C}(S)$ which is not necessarily continuous on B. To this end we consider the curves $\Sigma_r = \{X(r, \theta)\colon 0 \le \theta \le 2\pi\}$ for r close to one which are absolutely continuous and lie in a tubular neighbourhood T_μ of S. By Theorem 1, there exist sufficiently many of them: In fact, for any number $\varepsilon \in (0, 1)$ there is a set $\mathcal{I} \subset (1 - \varepsilon, 1)$ of positive measure such that, for every $r \in \mathcal{I}$, the mapping $X(r, \cdot)$ is absolutely continuous and $\Sigma_r \subset T_\mu$.

Now we can state the following result:

Theorem 2. *Let T_μ be the μ-neighbourhood of some closed set S in $\mathbb{R}^3$, and suppose that $X \in \mathcal{C}(S)$. Then for $\delta := \frac{1}{4}\pi\mu^2 > 0$, the following holds true:*

If $r_1, r_2 \in (0, 1)$ are two radii such that

(i) *the Dirichlet integral of X over the annulus*

$$\Omega(r_1, r_2) := \{w \in \mathbb{C}\colon r_1 < |w| < r_2\}$$

is at most δ;

(ii) *the curves $X|_{C_1}$ and $X|_{C_2}$ with $C_k := C_{r_k} = \{w\colon |w| = r_k\}$ are absolutely continuous, and their traces $\Sigma_k := X(C_k)$ are contained in $T_{\mu/2}$;*

(iii) *there is an angle θ such that the curve $X(r, \theta), r_1 \le r \le r_2$, connecting Σ_1 and Σ_2 is absolutely continuous and that its trace lies in $T_{\mu/2}$; then the curves $X|_{C_1}$ and $X|_{C_2}$ are homotopic in T_μ.*

Recall that two closed continuous curves $\gamma_1\colon C \to T_{\mu/2}$ and $\gamma_2\colon C \to T_{\mu/2}$ are *homotopic in T_μ* if there is a continuous map $H\colon C \times [0, 1] \to T_\mu$ such that $H(\cdot, 0) = \gamma_1$ and $H(\cdot, 1) = \gamma_2$. The mapping H is called a *homotopy.*

Furthermore, a closed curve $\gamma\colon C \to T_{\mu/2}$ is *contractible in T_μ* if it is homotopic in T_μ to a constant map or, equivalently, if it extends to a continuous map $\overline{B} \to T_\mu$.

Remark 2. Close to $C = \partial B$, the angle θ appearing in condition (iii) can be found by virtue of Lemma 3.

The *proof of Theorem 2* can be reduced to proving the following

Lemma 5. *Let T_μ be the μ-neighbourhood of some closed set S in $\mathbb{R}^3$, and set $\delta := \frac{1}{4}\pi\mu^2$. Suppose, moreover, that X is a mapping of class $H^1_2(B, \mathbb{R}^3) \cap C^0(\partial B, \mathbb{R}^3)$ whose boundary curve $X|_{\partial B}$ is contained in $T_{\mu/2}$ and which satisfies $D(X) < \delta(\mu)$. Then the curve $X|_{\partial B}$ is contractible in T_μ.*

In fact, let r_1 and r_2 be two radii as in Theorem 2, and let $\theta \in [0, 2\pi)$ be an angle as in (iii) of the theorem. Then we consider a conformal map τ of B onto the slit annulus

$$\{w = re^{i\varphi}\colon r_1 < r < r_2, \varphi \in [0, 2\pi), \varphi \neq \theta\}$$

and apply Lemma 5 to the surface $Z := X \circ \tau$, thus obtaining that $Z|_{\partial B}$ is contractible in T_μ. A straightforward reasoning now implies that the curves $X|_{C_1}$ and $X|_{C_2}$ are homotopic in T_μ.

Proof of Lemma 5. We begin by choosing a mapping $Y\colon \overline{B} \to \mathbb{R}^3$ which is harmonic in B, continuous on $\overline{B}$, of class $H_2^1(B, \mathbb{R}^3)$ and satisfies $Y - X \in \mathring{H}_2^1(B, \mathbb{R}^3)$ and $Y = X$ on ∂B. We know that $D(Y) \leq D(X)$. Since $X(\partial B) = Y(\partial B)$ is contained in $T_{\mu/2}$, there exists a strip $U = \{w\colon 1 - \varepsilon < |w| \leq 1\}$ about the boundary $C = \partial B$ such that $Y(U) \subset T_{\mu/2}$. Then we can find a regular, real analytic curve $Y|_{C_r}, r \in (1 - \varepsilon, 1)$, which is homotopic to $X|_C = Y|_C$ in $T_{\mu/2}$. Thereafter we can find a sequence $\{\Gamma_k\}$ of smooth closed Jordan curves Γ_k given by smooth topological mappings $\Phi_k\colon C \to \Gamma_k$ such that $|\Phi_k - \Phi|_{2,C} \to 0$ as $k \to \infty$ holds for the mapping $\Phi\colon C \to \mathbb{R}^3$ defined by $\Phi(e^{i\theta}) := Y(re^{i\theta})$.

Now let $Z(w) := Y(rw)$ and $Z_k(w)$ be the harmonic extensions to B of the boundary values Φ and Φ_k respectively, and let X_k be a solution of the variational problem $\mathcal{P}(\Gamma_k)$. Then the maximum principle implies $|Z_k - Z|_{0,\overline{B}} \to 0$ as $k \to \infty$ and, applying the estimate of Lemma 7 in Section 2.1 together with the Arzelà–Ascoli theorem, we also obtain $|Z_k - Z|_{1,\overline{B}} \to 0$ as $k \to \infty$. This implies

$$\lim_{k \to \infty} D(Z_k) = D(Z).$$

Consequently we have

$$A(X_k) = D(X_k) \leq D(Z_k) \to D(Z) = D_{B_r}(Y) \leq D(Y) \leq D(X).$$

By assumption, we have also

$$D(X) < \delta(\mu) = \tfrac{1}{4}\pi\mu^2,$$

whence

$$A(X_k) = D(X_k) < \pi(\mu/2)^2 \tag{7}$$

is satisfied for k sufficiently large.

If for one of these k the minimal surface X_k were not contained in T_μ, then there would exist some $w \in B$ such that $X_k(w) \notin T_\mu$. We choose a conformal selfmapping of B satisfying $\tau(0) = w$ and note that all the boundary values of $X_k \circ \tau$ lie outside the ball of radius $\mu/2$ centered at $X_k(w) = X_k(\tau(0))$. Then we infer from Vol. 1, Section 3.2, Proposition 2 that

$$A(X_k) \geq \pi(\mu/2)^2$$

which contradicts (7). Thus we have shown that

$$X_k(\overline{B}) \subset T_\mu \quad \text{for all } k \gg 1. \tag{8}$$

Moreover, every minimal surface X_k furnishes a topological mapping of C onto $\varGamma_k$ (see Vol. 1, Section 4.5, Theorem 3). Thus $X_k|_C$ furnishes a parameter representation of $\varGamma_k$ equivalent to $\varPhi_k$, and we infer from (9) that $\varPhi_k$ is contractible in T_μ for $k \gg 1$. Since $X|_C$ is homotopic in T_μ to all of the $\varGamma_k$ with $k \gg 1$, we infer that $X|_C$ is contractible in T_μ. ☐

Recall now that $\mathcal{C}(S)$ has been defined as the class of all surfaces $X \in H_2^1(B, \mathbb{R}^3)$ having their boundary values $X|_C$ on a closed subset S in $\mathbb{R}^3$ (see Definition 1).

We now denote by $\tilde{\varPi}_1(S)$ the *set of all homotopy classes of closed paths* in S. (For details, we refer for instance to Schubert [1], or to Greenberg [1].)

Assumption (A). *Suppose that there is a number $\mu > 0$ such that the inclusion map $S \to T_\mu$ of the closed set S into its μ-neighbourhood T_μ induces a bijection from $\tilde{\varPi}_1(S)$ to $\tilde{\varPi}_1(T_\mu)$.*

For example, this assumption is fulfilled for sufficiently small $\mu > 0$ if S is a smooth compact submanifold of $\mathbb{R}^3$.

Let $\mu > 0$ be a number as in Assumption (A), and recall that the curves $X|_{C_r}$ are absolutely continuous for almost all $r \in (0, 1)$.

If $X \in \mathcal{C}(S)$, then there is a number $\varepsilon > 0$ such that any two curves $X|_{C_r}$ and $X|_{C_{r'}}$ contained in $T_{\mu/2}$ and with $r, r' \in (1-\varepsilon, 1)$ define the same homotopy class in $\tilde{\varPi}_1(T_\mu)$; this homotopy class will be viewed as *homotopy class of the boundary values* $X|_C$. It is denoted by $[X|_C]$ and will be called *the boundary class of a surface* $X \in \mathcal{C}(S)$. Because we have a bijection

$$\tilde{\varPi}_1(T_\mu) \leftrightarrow \tilde{\varPi}_1(S),$$

we can view the class $[X|_C]$ as an element of $\tilde{\varPi}_1(S)$. If the mapping $X\colon C \to \mathbb{R}^3$ is continuous, then $[X|_C]$ coincides with the usual homotopy class of $X|_C$.

Note that the definition of the homotopy class $[X|_{\partial B}]$ does not depend on the particular *ACM*-representative of X that we have chosen since any two of them coincide on almost all circles C_r.

Moreover, the definition $[X|_C]$ is even independent of μ in the following sense: Suppose that the inclusion maps $S \to T_\mu$ and $S \to T_{\mu'}$ induce two bijections $\tilde{\varPi}_1(S) \leftrightarrow \tilde{\varPi}_1(T_\mu)$ and $\tilde{\varPi}_1(S) \leftrightarrow \tilde{\varPi}_1(T_{\mu'})$. Then both constructions with respect to μ and μ' lead to the same class $[X|_C]$ in $\tilde{\varPi}_1(S)$.

Indeed, according to the definition we first have to choose an $\varepsilon > 0$ such that any two of the curves $X|_{C_r}, r \in (1-\varepsilon, 1)$, lying completely in $T_{\mu/2}$ (or in $T_{\mu'/2}$) are homotopic in T_μ (or in $T_{\mu'}$). This ε may be the same for μ and μ' because decreasing ε does not change the class $[X|_{\partial B}]$. If, say, $\mu' \le \mu$, then we find in $(1-\varepsilon, 1)$ a subset $\mathfrak{J}'$ of positive measure or radii r such that the curves $X|_{c_r}, r \in \mathfrak{J}'$, are completely contained in $T_{\mu'/2}$ and that any two of them are homotopic in $T_{\mu'}$. Therefore all these curves $X|_{C_r}$ define a homotopy class α' in $\tilde{\Pi}_1(T_{\mu'})$ which corresponds to the boundary class $[X|_{\partial B}]' \in \tilde{\Pi}_1(S)$ which is constructed by means of $T_{\mu'}$.

On the other hand, all curves $X|_{C_r}, r \in \mathfrak{J}'$, are contained in $T_{\mu/2} \supset T_{\mu'/2}$, and any two of them are homotopic in $T_\mu \supset T_{\mu'}$. Therefore all these curves $X|_{C_r}, r \in \mathfrak{J}'$, define a homotopy class $\alpha \in \tilde{\Pi}_1(T_\mu)$ which by the definition of ε corresponds to the boundary class $[X|_{\partial B}] \in \tilde{\Pi}_1(S)$ defined by means of T_μ. Since the inclusion $T_{\mu'} \to T_\mu$ induces a bijection $\tilde{\Pi}_1(T_{\mu'}) \to \tilde{\Pi}_1(T_\mu)$ which maps α' to α, the boundary classes $[X|_{\partial B}]$ and $[X|_{\partial B}]'$ and identical. □

Collecting our results and inspecting Chapter 4 of Vol. 1, we obtain the following

Theorem 3 (Natural boundary classes). *Let S be a subset of $\mathbb{R}^3$ such that for some $\mu > 0$ the inclusion $S \to T_\mu$ induces a bijection $\tilde{\Pi}_1(S) \to \tilde{\Pi}_1(T_\mu)$ between the corresponding sets $\tilde{\Pi}_1$ of homotopy classes of closed paths in S and T_μ respectively.*

(i) *Then for every surface $X \in \mathcal{C}(S)$ a boundary homotopy class $[X|_{\partial B}] \in \tilde{\Pi}_1(S)$ is defined in a natural way.*

(ii) *If σ is a closed curve in S which is not contractible in S and if $[\sigma] \in \tilde{\Pi}_1(S)$ denotes its homotopy class, then every minimizer of the Dirichlet integral $D(X) = \frac{1}{2}\int_B |\nabla X|^2 \, du \, dv$ in the class*

$$\mathcal{C}(\sigma, S) := \{X \in \mathcal{C}(S) \colon [X|_{\partial B}] = [\sigma]\} \tag{9}$$

is a minimal surface.

Let us denote the minimum problem

$$D(X) \to \min \quad \text{in } \mathcal{C}(\sigma, S) \tag{10}$$

by $\mathcal{P}(\sigma, S)$.

In general one encounters serious difficulties if one tries to solve the problem $\mathcal{P}(\sigma, S)$. For instance, the classes $\mathcal{C}(\sigma, S)$ are not necessarily closed with respect to weak convergence in H_2^1; yet this fact was crucial for the existence proof carried out in Section 4.6 of Vol. 1.

All basic difficulties of this problem can already be seen in the comparatively simple case that we shall consider next. The reader who is not interested in the details of the following discussion may very well skip it since it is not anymore needed in the later sections.

Let us choose a *torus* T in $\mathbb{R}^3$ as the prescribed supporting surface, and consider the corresponding variational problem

$$\mathcal{P}(\sigma, T): \qquad D(X) \to \min \text{ in } \mathcal{C}(\sigma, T).$$

To be precise, let T be the torus in $\mathbb{R}^3$ which is obtained by revolving the circle

$$\{(x,y,z) : y=0, (x-R)^2+z^2=r^2\}, \quad 0<r<R,$$

about the z-axis (see Fig. 2). Denote by $\sigma_1, \sigma_2 \colon [0,2\pi] \to T$ the two circles

$$\sigma_1(t) = (R - r\cos t, 0, -r\sin t)$$

and

$$\sigma_2(t) = ((R-r)\cos t, (R-r)\sin t, 0).$$

Finally let $P = \sigma_1(0) = \sigma_2(0) = (R-r, 0, 0)$ be the base point of T.

Note that in this case the assumption made in the construction of the boundary classes $[X|_{\partial B}]$ of a surface $X \in \mathcal{C}(T)$, namely that the inclusion map $T \to T_\mu$ induces a bijection $\tilde{\Pi}_1(T) \leftrightarrow \tilde{\Pi}_1(T_\mu)$, is satisfied for all sufficiently small μ since for these μ the above inclusion $T \to T_\mu$ is a homotopy equivalence.

In general the set $\tilde{\Pi}_1(M)$ of all equivalence classes of (freely) homotopic closed curves in a topological space M is different from its fundamental group $\Pi_1(M,*)$; but if Π_1 is Abelian and if M is connected, then the canonical map $\Pi_1(M,*) \to \tilde{\Pi}_1(M), [\sigma] \to [\sigma]$, is indeed a bijection (cf. Schubert [1]).

The fundamental group of the torus T is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, and it is freely generated by $[\sigma_1]$ and $[\sigma_2]$. Therefore, in the case of the torus, the class $\mathcal{C}(T)$ of all H_2^1-surfaces with boundary values on T is the disjoint union of the classes $\mathcal{C}^{k,l}, k, l \in \mathbb{Z}$, of surfaces $X \in \mathcal{C}(T)$ whose boundary class $[X|_{\partial B}]$ can be represented by the closed path $\sigma_1^k \cdot \sigma_2^l$. (First k-times along σ_1, then l times along σ_2, negative powers denote reversal of orientation.)

Now we can state our *nonexistence result.*

Theorem 4. *Let T be the torus defined before.*

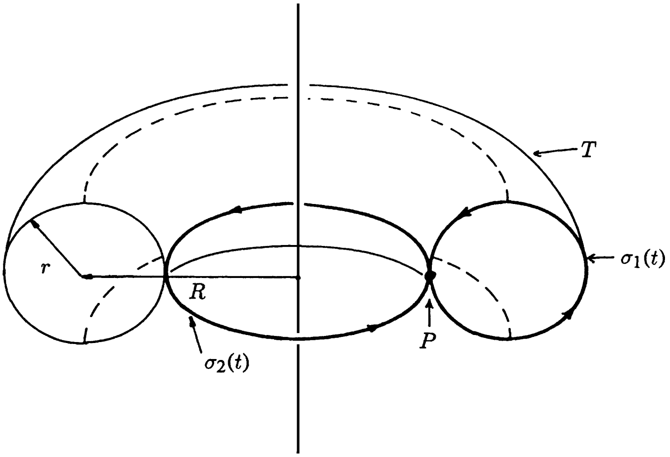

Fig. 2. The points and curves on a torus T used in the study of minimizing sequences for the Dirichlet integral of surfaces with free boundaries on T whose boundary curves have a prescribed homotopy class

(i) *For all* $k, l \in \mathbb{Z}$ *the numbers* $d_{k,l} := \inf\{D(X) \colon X \in \mathcal{C}^{k,l}\}$ *are given by*

$$d_{k,l} = \pi\{|k|r^2 + |l|(R-r)^2\}.$$

(ii) *The variational problem*

$$D(X) \to \ \min \quad \text{in } \mathcal{C}^{\,k,l}$$

has a solution if and only if $k = 0$ *or* $l = 0$.

For the proof of Theorem 4 we shall need the following

Lemma 6 (A formula for the oriented area). *Assume that the boundary values of a mapping* $X = (X^1, X^2) \in H_2^1(B, \mathbb{R}^2)$ *are contained in* $\mathbb{R}^2 \setminus B_\rho(w_0)$. *Then the boundary class* $[X|_{\partial B}] \in \tilde{\Pi}_1(\mathbb{R}^2 \setminus B_\rho(w_0))$ *is well defined, and it is characterized by the winding number* $U([X|_{\partial B}], w_0)$. *If* $\Omega := \{w \in B \colon X(w) \in B_\rho(w_0)\}$, *then we have for the oriented area*

$$A_\Omega^0(X) := \int_\Omega X_u \wedge X_v \, du\, dv$$

of the mapping X *the formula*

$$A_\Omega^0(X) := \int_\Omega \{X_v^1 X_v^2 - X_v^1 X_u^2\}\, du\, dv = \pi\rho^2 U([X|_{\partial B}], w_0).$$

Proof of Lemma 6. Approximating H_2^1-mappings $Z \in H_2^1(B, \mathbb{R}^2)$ by smooth mappings, we obtain the following two formulas that are well known for smooth maps:

(i) For almost all $R \in (0,1)$, the oriented surface area of Z is given by

$$A_{B_R}^0(Z) = \frac{1}{2}\int_0^{2\pi} \{Z^1 Z_\theta^2 - Z^2 Z_\theta^1\}\, d\theta.$$

(ii) If Z is absolutely continuous on ∂B_R, then

$$U(Z|_{\partial B_R}, 0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{Z^1 Z_\theta^2 - Z^2 Z_\theta^1}{|Z|^2}\, d\theta$$

unless $Z = 0$ somewhere on ∂B_R. Of course, $0 < R < 1$ and $Z = Z(Re^{i\theta})$, etc.

Let us now prove the lemma. We may assume without loss of generality that $w_0 = 0$. Moreover, for $0 < \varepsilon < \rho$, let $\pi_\varepsilon \colon \mathbb{R}^2 \to \overline{B}_{\rho-\varepsilon}(0)$ denote the radial projection

$$Z \mapsto \begin{cases} Z & \text{if } |Z| < \rho - \varepsilon, \\ \frac{Z}{|Z|}(\rho - \varepsilon) & \text{otherwise}, \end{cases}$$

and set $Y^\varepsilon := \pi_\varepsilon \circ X$, which is again of class $H_2^1(B, \mathbb{R}^2)$ since π_ε is Lipschitz continuous. The boundary values of Y^ε are contained in $\mathbb{R}^2 \setminus B_{\rho-\varepsilon}(0)$, and we have