Heinz Klaus Strick

Mathematics is Beautiful

Suggestions for People Between 9 and 99 years

to Look at and Explore

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Heinz Klaus Strick
Leverkusen, Germany

The translation was done with the help of artificial intelligence (machine translation by the service DeepL.com). A subsequent human revision was done primarily in terms of content.

ISBN 978-3-662-62688-7 ISBN 978-3-662-62689-4 (eBook)
https://doi.org/10.1007/978-3-662-62689-4

Responsible Editor: Iris Ruhmann
This Springer imprint is published by the registered company Springer-Verlag GmbH, DE part of Springer Nature.
The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

Preface

Not everyone thinks of mathematics as something to enjoy when talking about it. But mathematics has many exciting and aesthetically pleasing aspects to offer. In this book, I have tried to show some of these beautiful things in mathematics.

During my work as a mathematics teacher, I have always endeavored to loosen up my lessons to a certain extent. Unfortunately, even in the most exciting mathematics lessons tedious and dry phases cannot be avoided.

For such relaxation and enrichment, there are questions that could be classified as *mathematical games,* or even *brain teasers* whose solutions lead to amazing insights.

Thus, for example, after the treatment of the inscribed angle theorem in elementary geometry, regular star figures can be examined (Chap. 1) or regular polygons can be laid out using diamonds (Chap. 10). Searching the greatest common divisor of two numbers is more entertaining if one interprets this as the dissection of a rectangle (Chap. 3). Mental arithmetic is not to everyone's taste, but surprisingly, you can discover interesting structures in the world of numbers with just a few arithmetical tricks (Chap. 7). Solving quadratic equations and linear systems of equations is usually not very exciting – unless you use these methods to explore wonderful figures with touching circles *(Kissing circles,* Chap. 15) or to deal with the question of the tessellation of rectangles by squares of different sizes (*Squaring the square*, Chap. 14). In addition to the *Kissing circles* problems from Japanese temple geometry (*Sangaku*) are examined.

Several of the topics addressed in the book are aimed at younger students. Experience has shown that thread pictures (*Curve stitching*, Chap. 6) are extremely fascinating – even if the theoretical background can only be conveyed at the end of secondary school or even afterwards. Playing with pentominoes (Chap. 5) encourages a strategic and logical approach. And smart 10-year-olds can understand that weighing with a fixed, very limited set of balance weights (Chap. 9) conceals arithmetic in the ternary numeral system.

In the first years of school, children already learn to determine the areas of simple geometric figures; it is all the more astonishing, that a completely different way of measuring can be chosen: the area inside a polygon can be calculated when the vertices are points of a square-ruled paper: you only have to count the lattice points of the boundary and those lying inside the figure (Chap. 11). As an introduction to the subject this chapter also includes studies of rectangles and other simple figures on grid paper.

However, studying beautiful mathematics can also mean looking at colored patterns or designing one's own patterns. Patterns made of colored stones (Chap. 2) were already studied 2500 years ago. When coloring circular rings (Chap. 4) and equally large subareas of regular polygons (*Area divisions*, Chap. 8) you can develop your own imagination and perhaps even discover new patterns.

At the end of the book, there are two more extensive chapters on the derivation of power sum formulas (Chap. 16) and on the Pythagorean theorem (Chap. 17). They make clear how new ideas on a topic have been developed over the centuries.

Unfortunately, there was no room in this book for other topics. I am aware that a selection could have been different. (For example, if you miss the "Golden Ratio": at least some aspects can be found in Chaps. 3 and 13, but here will be a lot more in the second volume of "Mathematics is beautiful").

The chapters can be read independently of each other. At least when starting with the individual topics, the simplest possible approach was chosen; for this none or only a little background knowledge from school lessons is required.

It is an important concern of the book that – by reading this book – many young people find their way to mathematics and at the same time those readers, whose school days are some time ago, remember again and discover something new. The numerous references to further sources for information on the Internet as well as to further literature should help here. The "solutions" to the problems described in the individual sections *Suggestions for reflection and for investigations* are published on the author's website: https://www.mathematik-ist-schoen.de/mathematics-is-beautiful/.

This book was written for everyone who enjoys mathematics or wants to understand why the book bears this title. It is also aimed at teachers who want to give their students additional or new motivation to learn.

Even though each chapter contains – graphically emphasized – theorems, rules, and formulas, that is, the typical elements of a mathematics book, this is not a textbook of mathematics. Proofs of theorems are only based on examples – it was always more important to me to convey the underlying ideas than pointing out the formal conclusions.

The abundance of graphics in this book should encourage you to develop your own ideas about the objects presented:

Viewing, thinking, trying out, varying, researching, wondering.

The fact that most of the graphics were created using the LOGO programming language may be criticized, as the graphic resolution that can be achieved with this software is certainly not optimal. Besides the licensing issue, the decisive factor for my decision was my own positive teaching experiences with the concept of the programming language, which the inventor Seymour Papert (*Mindstorms*) himself considered suitable for primary school.

In recent years, I have had the pleasure of dealing with a new mathematician every month (https://www.spektrum.de/mathematik/monatskalender/index/). A lot of those “histories”, which, with the help of John O’Connor, are now also available in English and can be downloaded from https://mathshistory.st-andrews.ac.uk/Strick/.

When you deal with the insights and ideas of scholars who have long since passed away, you often cannot help but be amazed. I hope that in this book I have also succeeded in bringing some of these wonderful insights, which have unfortunately often been forgotten, back into consciousness. I have made every effort to provide sufficient suggestions for further study of the topics by the literature references in each chapter and at the end of the book. Fortunately, the quality of the Wikipedia contributions (and the bibliographical references they each contain) has increased significantly in recent years. Sometimes they are even surpassed by the German or French version; therefore, these sources are also mentioned. It is no longer possible for me to state in detail which publications have given me which stimulus. Over the past decades I have worked through a large number of books, whose titles often begin with the words

Recreations, Challenging Problems, Excursions, Adventures …

Most of the time I looked at them from the point of view of whether they contained suggestions for “normal” lessons, for study groups, or as problems for competitions.

At the end of the work on this book, I would like to thank all those who have supported me in the preparation and implementation of the book project:

- To my wife, who patiently put up with the fact that I kept immersing myself in the beautiful world of mathematics,
- To Wilfried Herget (University of Halle), who made numerous suggestions to make the wording of my texts more understandable and revealed gaps in arguments,
- To Manfred Stern †(University of Halle), Peter Gallin (University of Zurich), and Hans Walser (University of Basel) who have given numerous suggestions for this book,
- To John O’Connor (University of St Andrews) who liberally helped so that this book could be published in an understandable translation,
- And not least to Andreas Rüdinger, Iris Ruhmann, Carola Lerch, Snehal Surwade and Jasmeen Kaur from Springer Verlag, who made this book possible.

Leverkusen Germany Heinz Klaus Strick

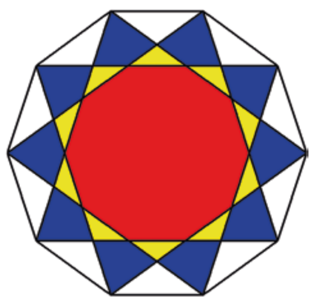
Contents

Regular Polygons and Stars

1

Three things remain with us from paradise:
Stars, flowers and children.

(Dante Alighieri, 1265–1321, Italian poet and philosopher)



1.1 Properties of Regular Stars

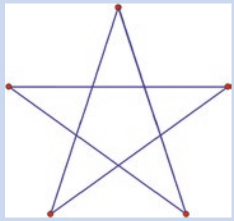
Regular stars are created by connecting vertices of regular polygons according to a certain rule.

Such a rule could be worded as follows:
Connect one vertex of the polygon with the k-next vertex (clockwise).

Example: 5-Pointed Star (Pentagram)
For $n = 5$ and $k = 2$, this means: connect each vertex of a regular 5-sided figure (pentagon) to the second-next vertex (clockwise). Thus a regular 5-pointed star is created.

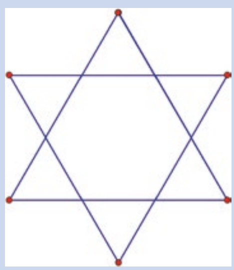
H. K. Strick, *Mathematics is Beautiful*, https://doi.org/10.1007/978-3-662-62689-4_1

No further 5-pointed stars exist, because for $n = 5$ and $k = 3$ you get the same star. Instead of connecting each vertex to the third-next vertex clockwise, you can connect the vertex to the second-next vertex counterclockwise.

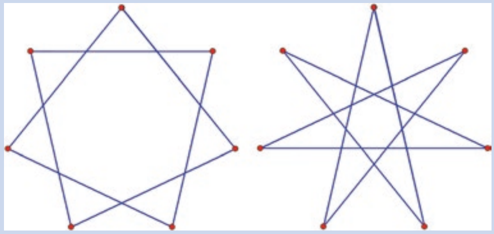


Example: 6-Pointed Star (Hexagram)
Also for $n = 6$ only one type exists. It consists of two 3-sided figures (equilateral triangles), because $2 \cdot 3 = 6$.

If you number the vertices of the n-sided figure clockwise with $P_0, P_1, P_2, P_3, P_4, P_5$, then you get two closed polygonal lines: $P_0 - P_2 - P_4 - P_0$ and $P_1 - P_3 - P_5 - P_1$, with either even or odd indices.

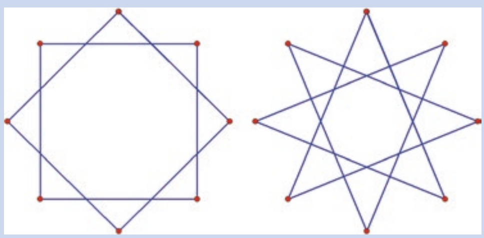


Example: 7-Pointed Stars (Heptagrams)
For $n = 7$ there are two different stars, namely for $k = 2$ and for $k = 3$. If you look closely, you can see that the 7-pointed star for $k = 2$ is also created inside the star for $k = 3$ (also a regular 7-sided figure).



Example: 8-Pointed Stars (Octagrams)
Also for $n = 8$ there are two different stars, that is for $k = 2$ and for $k = 3$.

The 8-pointed star for $k = 2$ also appears inside the star for $k = 3$. It consists of two regular 4-sided figures (squares), because $2 \cdot 4 = 8$.



Example: 9-Pointed Stars (Enneagrams)
For $n = 9$ there are even three different stars.

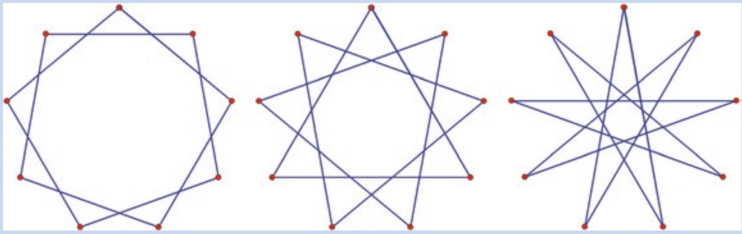
- $n = 9$, $k = 2$: The star can be drawn as a closed polygonal line:

$$P_0 - P_2 - P_4 - P_6 - P_8 - P_1 - P_3 - P_5 - P_7 - P_0$$

- $n = 9$, $k = 3$: The star consists of three regular 3-sided figures (equilateral triangles), because $3 \cdot 3 = 9$.
- $n = 9$, $k = 4$: The star can be drawn as a closed polygonal line:

$$P_0 - P_4 - P_8 - P_3 - P_7 - P_2 - P_6 - P_1 - P_5 - P_0$$

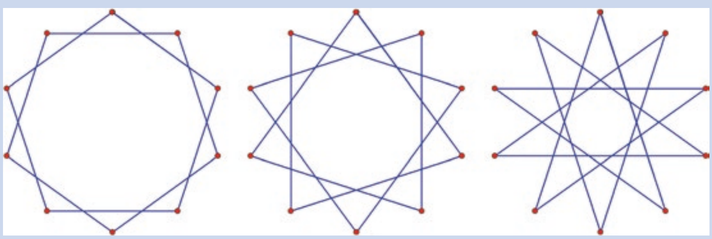
Inside, the stars for both, $k = 2$ and $k = 3$, appear.



Example: 10-Pointed Stars (Decagrams)

There are also three different stars for $n = 10$.

- $n = 10$, $k = 2$: This star consists of two regular 5-sided figures, because $2 \cdot 5 = 10$.
- $n = 10$, $k = 3$: The star can be drawn as a closed polygonal line.
- $n = 10$, $k = 4$: This star consists of two stars of type $n = 5$, $k = 2$. These include the two closed polygonal lines $P_0 - P_4 - P_8 - P_2 - P_6 - P_0$ and $P_1 - P_5 - P_9 - P_3 - P_7 - P_1$.

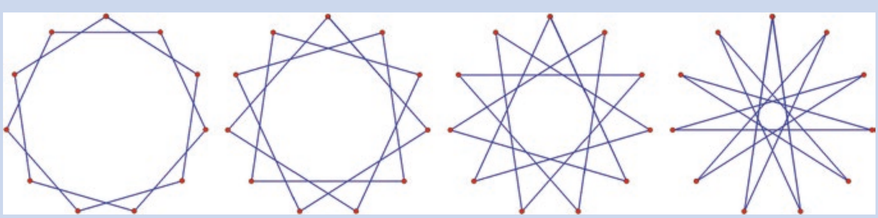


Example: 11-Pointed Stars (Hendecagrams)

For $n = 11$ there are four different stars, namely for $k = 2$, $k = 3$, $k = 4$, and $k = 5$.

All of these stars can be drawn as closed polygonal lines.

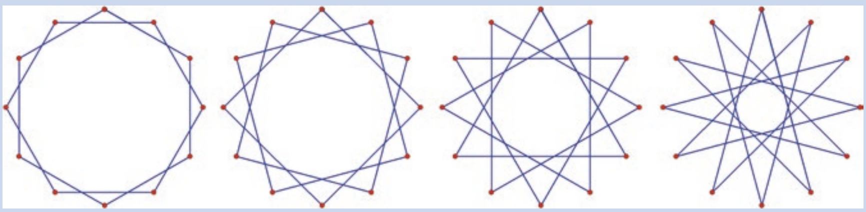
On the inside the stars with smaller k appear respectively.



Example: 12-Pointed Stars (Dodecagrams)
For $n = 12$ there are four different stars:

- $k = 2$: 2 regular 6-sided figures, because $2 \cdot 6 = 12$.
- $k = 3$: 3 regular 4-sided figures (squares), because $3 \cdot 4 = 12$.
- $k = 4$: 4 regular 3-sided figures (equilateral triangles), because $4 \cdot 3 = 12$.

Only the star for $k = 5$ can be drawn as a closed polygonal line.
On the inside the stars with smaller k appear respectively.



The following properties can be identified from the examples:

- n-pointed stars exist for every n, which is greater than 4.
- For k you can use any number. You can get different star figures, if you use the following values in the drawing rule: k is at least 2, for even-numbered n use at most $\frac{n}{2} - 1$, for odd-numbered n use at most $\frac{n-1}{2}$.
  - In detail, the following applies for odd-numbered n: for $n = 5$ there is one star for $k = 2$; for n $= 7$ there are two stars, namely for $k = 2$ and for $k = 3$; for $n = 9$ there are three stars, namely for $k = 2$ for $k = 3$ and for $k = 4$; and so on.
  - In detail, the following applies for even-numbered n: for $n = 6$ there is one star for $k = 2$; for $n = 8$ there are two stars, namely for $k = 2$ and for $k = 3$; for $n = 10$ there are three stars, namely for $k = 2$, for $k = 3$ and for $k = 4$; and so on.
- If any vertex is determined as the beginning of a closed polygonal line with the number 0, then the line passes through the vertices with the numbers $0 - k - 2k - 3k - \cdots$, and similar as to a clock, the numbers are each reduced by n, when the multiple of k reaches or exceeds the number n.
- In every n-pointed star, there are further n-pointed stars inside for every possible $k > 2$.
- Some star figures can be drawn without lifting the pen; others consist of two or more polygons or star figures. In detail:
  - If k is a divisor of n, then the star consists of k polygons with e vertices, where $e = \frac{n}{k}$.

– If k and n have the common divisor g, then the n-pointed star is composed of g stars with $\frac{n}{g}$ vertices.
– If k and n are coprime, that is, if they only have the number 1 as a common divisor, the star can be drawn as a (single) closed polygonal line. Conversely, if a star can be drawn as a (single) closed polygonal line, then k and n are coprime.

Rule

Stars that can be Drawn as a Closed Polygonal Line

Regular n-pointed stars exist for all natural numbers n, k with $n > 4$ and $2 \leq k \leq \frac{n}{2} - 1$, if n is an even number, or $2 \leq k \leq \frac{n-1}{2}$, if n is an odd number.

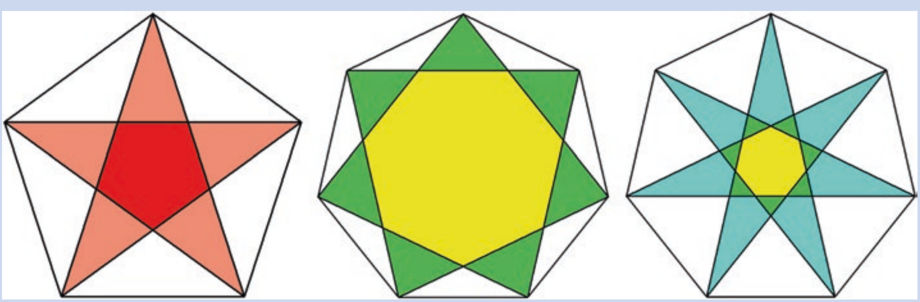
Then, and only then, the stars can be drawn as a closed polygonal line, if n and k are coprime. ◀

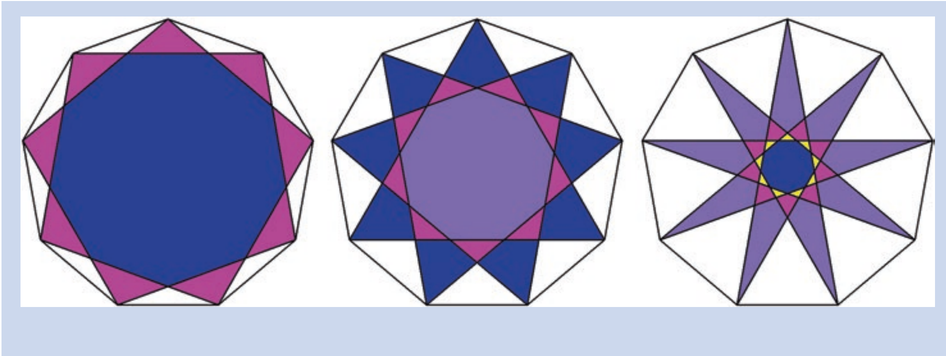
Since in regular n-pointed stars both the number of vertices n and the parameter k play an important role, they are often notated with the symbolic notation $\{n/k\}$, the so-called **Schläfli symbol** (named after the Swiss mathematician Ludwig Schläfli [1814–1895], who was particularly interested in regular polygons, polyhedrons and their generalization in higher dimensions).

Suggestions for Reflection and for Investigations

A 1.1: Answer the following questions for $n = 13$, $n = 15$, and for $n = 18$ (that is, for an odd or even number of vertices): for which k (minimum and maximum value) do you get an n-pointed star? How many different star figures are possible? Which of the possible star figures can be drawn as a closed polygonal line, which consist of several stars, which of several polygons? Which numbers of vertices appear in the possible closed polygonal lines (start of lines at the vertex with number 0)?

A 1.2: In the following figures, areas of equal size are colored in the same way. How does the number of colors depend on the type of star, i.e. on the values for n and k?





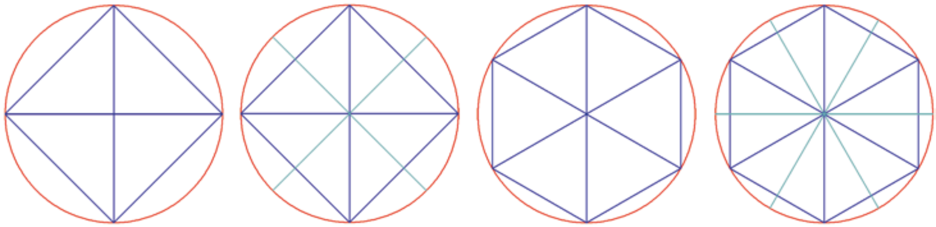
1.2 Drawing Stars

To draw a regular star with n vertices, you need to know how to draw a regular n-sided polygon.

Especially simple is the construction of a regular 4-sided figure (square) and a regular 6-sided figure (hexagon) as well as the regular polygons, each obtained by doubling the number of vertices from given regular n-sided figures:

- A regular 4-sided figure is obtained by drawing a circle of any radius r, selecting any point on the circle and drawing a straight line through the center of the circle until the circular line is intersected again. Then draw a perpendicular to this line through the center of the circle to get two more points of the 4-sided figure. These four points determine a square.
- A regular 6-sided figure is created by drawing a circle with an arbitrarily chosen radius r, then selecting any point on the circular line and from this point successively drawing lines of the length r on the circle. This construction is possible because the regular 6-sided figure consists of six equilateral triangles, i.e., the sides of the 6-sided figure are as long as the line segments which connect the vertices with the center of the circle (= radius of the circle).

If you draw a straight line from the center of the circle through each of the centers of the sides of the regular n-sided polygon, then the intersection points of these straight lines with the circular line are the additional vertices for the regular $2n$-sided polygon. In this way you will get out of the square a the regular 8-sided polygon, from the regular 6-sided polygon you will get the regular 12-sided polygon, and so on (see the following figures).



In general, that is, for any n, there are two possibilities:

- You start with a circle with radius r, which is drawn around a center point, and then draw the radius n-times from the center, changing the direction 360°/n each time. Figure 1.1 shows (for $n = 7$) not only the vertices but also the sides of the regular n-sided polygon and the altitudes of the resulting isosceles triangles. The n-pointed star is created when a starting point is connected with the k-next point according to the rules, and this procedure is then repeated n times.
- Alternatively, you can also start with one side of the n-sided polygon, that is, draw a line of length s, then change the direction in which you moved while drawing by the nth part of 360°, so that after repeating the process n times, you have made a total rotation of 360° and have arrived back at the starting point of the "walking tour."

There is a simple relationship between the circle radius r and the side length s of the regular n-sided polygon: two adjacent radii and one side of the n-sided polygon form an isosceles triangle, which is divided by the altitude h into two right-angled triangles.

Therefore, the following applies to the half angle at the center:

$$\sin\left(\frac{180°}{n}\right) = \frac{s}{2r} \text{ and } \tan\left(\frac{180°}{n}\right) = \frac{s}{2h} \text{ and } \cos\left(\frac{180°}{n}\right) = \frac{h}{r}$$

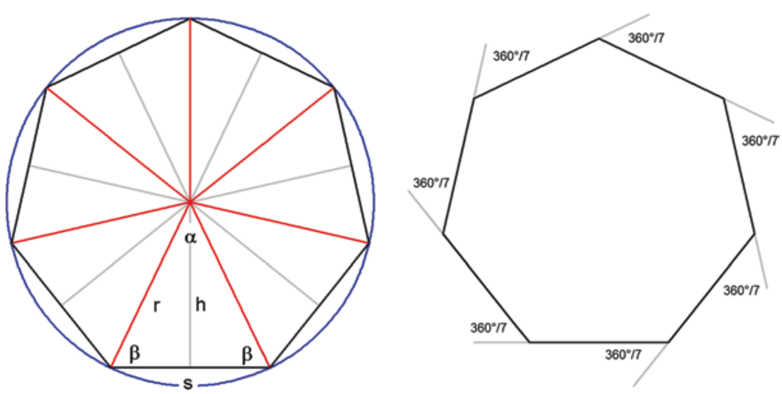


Fig. 1.1 Two of the ways to draw a regular 7-sided polygon

1.3 Diagonals in a Regular *n*-Sided Figure

In exploring the question which n-pointed stars are possible at all, it makes sense to draw a regular n-sided figure with all diagonals first and then, according to the instructions, mark the desired closed polygonal line for which the diagonals are used.

From each vertex of an n-sided figure you can draw line segments to the other vertices: 2 sides (to the two adjacent vertices) and $n - 3$ diagonals (to the remaining vertices).

The total number of diagonals in an n-sided polygon does not result directly from the product $n \cdot (n - 3)$ because with this method of counting each of the connecting lines is counted twice. Rather the following applies:

Rule

Number of Diagonals of an *n*-Sided Polygon

The number of diagonals in an n-sided polygon is equal to $\frac{1}{2} \cdot n \cdot (n - 3)$. ◄

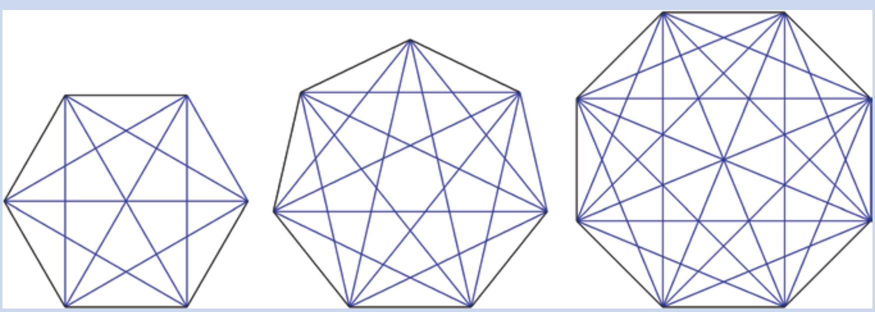
Examples for the Calculation of the Number of Diagonals

A regular 5-sided figure has $\frac{1}{2} \cdot 5 \cdot 2 = 5$ diagonals that form the regular 5-pointed star.

A regular 6-sided figure has $\frac{1}{2} \cdot 6 \cdot 3 = 9$ diagonals, but 3 of them only lead to the opposite point, so they are not suitable to draw a star. The remaining 6 diagonals form the 3 sides of the two equilateral triangles.

A regular 7-sided figure has $\frac{1}{2} \cdot 7 \cdot 4 = 14$ diagonals, of which 7 diagonals each form a polygonal line for the 7-pointed star with $k = 2$ or $k = 3$.

A regular 8-sided figure has $\frac{1}{2} \cdot 8 \cdot 5 = 20$ diagonals, of which 4 only lead to the opposite point, so they are not suitable to draw a star. In addition, two times four diagonals each form the two squares of which star {8/2} consists, so that 8 diagonals remain, which form the regular 8-pointed star {8/3}.



Suggestions for Reflection and for Investigations

A 1.3: Determine the number of diagonals for $n = 9$ to $n = 12$ in the regular n-sided polygon. Which of these diagonals are needed for drawing n-pointed stars? Generalize these statements about diagonals and stars for an even and odd number of vertices.

In the regular 5-sided figure (pentagon), all diagonals have the same length. If you connect the end points of a diagonal to the center of the circle, an isosceles triangle with base d and two legs of the length r is formed. Since the diagonals connect one vertex of the regular 5-sided figure with the second next vertex, the size of the angle δ at the center of the circle is equal to $2 \cdot \frac{360^\circ}{5}$ that is, the size of half the angle is equal to $2 \cdot \frac{180^\circ}{5} = 72^\circ$.

Therefore applies to the diagonals in the regular 5-sided figure:

$$\sin\left(\frac{2 \cdot 180^\circ}{5}\right) = \frac{\frac{d}{2}}{r}, \text{ that is } d = 2r \cdot \sin\left(\frac{2 \cdot 180^\circ}{5}\right).$$

In general, for the diagonals in any regular n-sided polygon, which connect one vertex with the second next vertex, the length of the diagonal d_2 is given as:

$$d_2 = 2r \cdot \sin\left(\frac{2 \cdot 180^\circ}{n}\right)$$

In the case of diagonals connecting one vertex with the third next vertex, the angle δ at the center of an isosceles triangle changes accordingly to $3 \cdot \frac{360^\circ}{n}$, that is, half the angle to $3 \cdot \frac{180^\circ}{n}$. Therefore, the following applies:

$$d_3 = 2r \cdot \sin\left(\frac{3 \cdot 180^\circ}{n}\right)$$

Formula

Length of the Diagonals of a Regular *n*-Sided Polygon

In general, for the length d_k of a diagonal, that connects a vertex with the k-next vertex of a regular n-sided polygon and that lies opposite to the angle $\delta = k \cdot \frac{360^\circ}{n}$, the following applies:

$$d_k = 2r \cdot \sin\left(\frac{k \cdot 180^\circ}{n}\right) \tag{1.1}$$

By means of formula (1.1), the total length of the closed polygonal line which forms the regular n-pointed star can then be calculated, see also Table 1.1 below. ◄

Table 1.1 Angular sizes and line lengths for regular *n*-pointed stars

Star type $\{n/k\}$	Number of polygonal lines	Center angle δ_k (opposite to the diagonal d_k) $\delta_k = k \cdot \frac{360°}{n}$ (°)	Angle ε at the "tip" (°)	Total length of all lines of the star $n \cdot 2r \cdot \sin\left(\frac{k \cdot 180°}{n}\right)$
{5/2}	1	144	36	9.51 · r
{6/2}	2	120	60	10.39 · r
{7/2}	1	102.86	77.14	10.95 · r
{7/3}	1	154.29	25.71	13.65 · r
{8/2}	2	90	90	11.31 · r
{8/3}	1	135	45	14.78 · r
{9/2}	1	80	100	11.57 · r
{9/3}	3	120	60	15.59 · r
{9/4}	1	160	20	17.73 · r
{10/2}	2	72	108	11.76 · r
{10/3}	1	108	72	16.18 · r
{10/4}	2	144	36	19.02 · r
{11/2}	1	65.45	114.55	11.89 · r
{11/3}	1	98.18	81.82	16.63 · r
{11/4}	1	130.91	49.91	20.01 · r
{11/5}	1	163.64	16.36	21.78 · r
{12/2}	2	60	120	12 · r
{12/3}	3	90	90	16.97 · r
{12/4}	4	120	60	20.78 · r
{12/5}	1	150	30	23.18 · r

1.4 Vertex Angle in a Regular *n*-Pointed Star

At the vertices of the regular *n*-pointed stars, there are angles that depend on the values for *n* and *k*. These are easy to determine by applying the so-called **inscribed angle theorem**. The theorem deals with the central angle above a chord and the associated inscribed angle (peripheral angle) above it. The theorem states that all peripheral angles above a chord are equal. The central angle is twice as large as the periphal angles.

Figure 1.2 shows the symmetric case of the theorem; for a general proof of the theorem look at the references.

If two adjacent vertices of a regular *n*-sided figure are connected to each other, then the central angle belonging to the side of the *n*-sided figure is equal to $\frac{360°}{n}$; the corresponding peripheral angles are equal to $\frac{180°}{n}$.

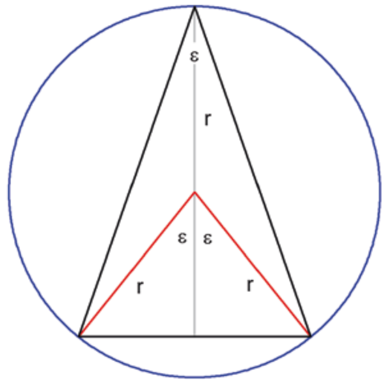


Fig. 1.2 Relationship between the center angle and the peripheral angle in a symmetric triangle

If you connect a vertex of a regular n-sided figure with the second next vertex, then the central angle belonging to this diagonal d_2 is twice as large as $\frac{360°}{n}$ thus equal to $\frac{720°}{n}$ and the corresponding peripheral angles are equal to $\frac{360°}{n}$.

In general:

Rule

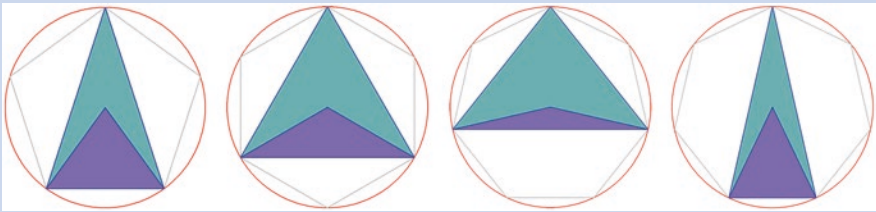
Central Angles and Peripheral Angles Over a Chord in Regular *n*-Sided Polygons

If you connect a vertex of a regular n-sided polygon with the k-next vertex, then the angle at the center of this diagonal d_k is k-times as big as $\frac{360°}{n}$; the corresponding peripheral angles are equal to $k \cdot \frac{180°}{n}$. ◀

Examples of the Angles in the Vertices of Regular n-Pointed Stars

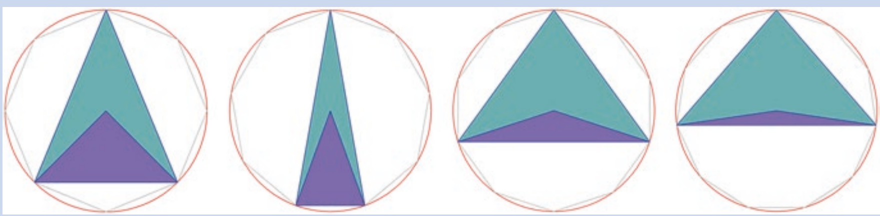
- With the regular 5-pointed star the vertex is "above" one side of the 5-sided figure. Therefore, the angle ε at the vertex is half the angle at the center of the regular 5-sided figure. Since the angle at the center has an angular size of $\frac{360°}{5} = 72°$, the angle at the vertex of the regular 5-pointed star is $\varepsilon = \frac{180°}{5} = 36°$ - see the first of the following figures.
- In the regular 6-pointed star, the vertex is also "above" a diagonal of the 6-sided figure, which connects one vertex with the second-next. Therefore the angle ε is half as large as the corresponding central angle, that is, half as large as $2 \cdot \frac{360°}{6}$, that is $\varepsilon = 60°$, see the second of the following figures.
- With the regular 7-pointed star {7/2} the vertex is also "above" a diagonal of the 7-sided figure, which connects one vertex with the third next vertex. Therefore, the angle ε is half as large as the corresponding central angle, namely half the size of $3 \cdot \frac{360°}{7}$, that is $\varepsilon \approx 77.14°$.

On the other hand, with the star {7/3} the point is "above" a diagonal of the 7-sided figure, which connects one vertex with the next vertex. Therefore, the point angle ε is half as large as the corresponding central angle, namely half as large as $1 \cdot \frac{360°}{7}$, that is $\varepsilon \approx 25.71°$, see the third and fourth of the following figures.

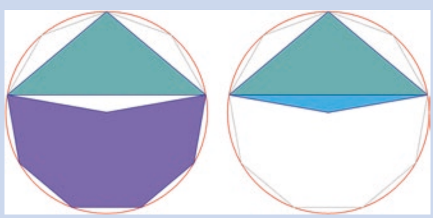


Suggestions for Reflection and for Investigations

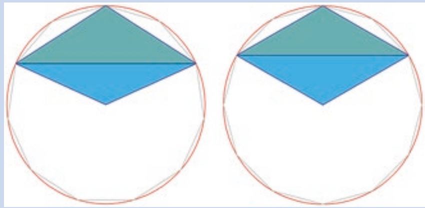
A 1.4: Using the 8-, 9-, 10-, or 12-pointed stars shown in the figure, consider which are the angular sizes in the vertices of the *n*-pointed stars.



A 1.5: One of the regular 9-pointed stars has a central angle greater than 180°. Use the following two figures to explain how the angle in the vertex is calculated here.



A 1.6: The following regular stars also have a central angle that is greater than 180°. In each case, explain how the angles in the vertices are calculated.



On the basis of the examples, it can be assumed that there is a simple relationship between the angle ε in the vertex and the angle at the center δ_k above the diagonals, namely $\varepsilon = 180° - \delta_k$, see the following table.

Star type {*n*/*k*}	Center angle δ_k (opposite to the diagonal d_k)	Angle ε (at the "tip")
{5/2}	144°	36°
{6/2}	120°	60°
{7/2}	102.86°	77.14°
{7/3}	154.29°	25.71°

Figure 1.3 shows that this is true: the vertex is determined by two diagonals, of which each has the central angle δ_k. According to Sect. 1.3 this angle can be calculated as $\delta_k = k \cdot \frac{360°}{n}$. For the base angles γ of the associated isosceles triangles, the following applies, due to the angle sum in the triangle, $2\gamma + \delta_k = 180°$.

But since the vertex angle ε consists of twice the angle γ, the proposition applies $\varepsilon + \delta_k = 180°$.

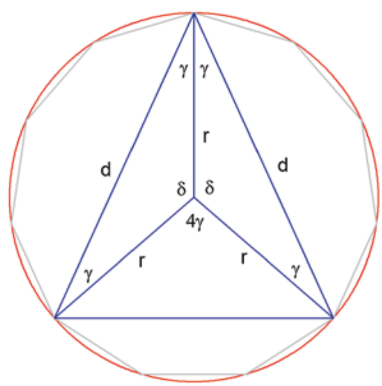


Fig. 1.3 To determine the angle $\varepsilon = 2\gamma$ at the vertex of a regular n-sided polygon

Rule

Size of the Vertex Angles in Regular *n*-Pointed Stars

For the vertex angle ε of a regular n-pointed star of the type $\{n/k\}$ the following applies:

$$\varepsilon = 180^\circ - \frac{k \cdot 360^\circ}{n}$$

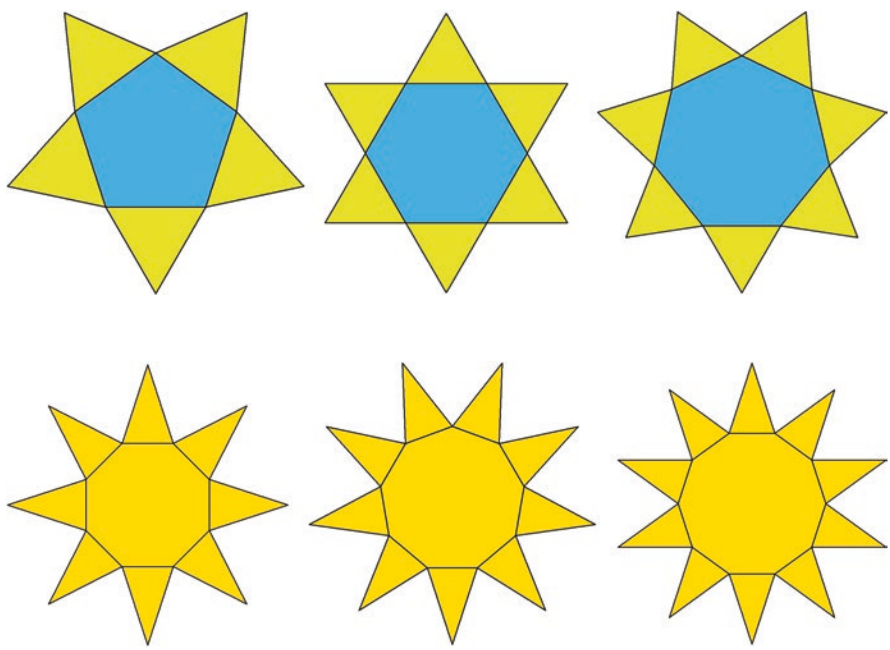
Inside a star of type $\{n/k\}$ further n-pointed stars $\{n/m\}$ appear with $1<m<k$. At the very center of a regular star there is also a regular n-sided figure, for whose interior angles α applies: $\alpha = 180^\circ - \frac{360^\circ}{n}$.

So you can apply the formula for calculating ε also to the case $k = 1$ and mark regular n-sided figures with the Schläfli symbol $\{n/1\}$.

The results so far are shown in Table 1.1. ◄

1.5 Compounded *n*-Pointed Stars

In principle, you can also create regular n-pointed stars by first creating a regular n-sided polygon, and then drawing isosceles triangles above the sides of the polygon. In the following figures, equilateral and *golden* triangles, respectively have been placed on the sides of a regular 5, 6, and 7-sided figure. (Isosceles triangles with a base angle of 72° are called golden triangles).



Suggestions for Reflection and for Investigations

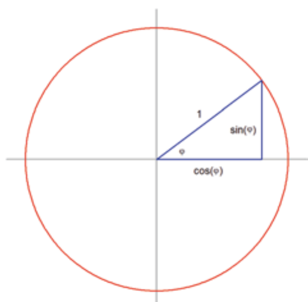
A 1.7: Prove the proposition: all regular n-pointed stars of the type $\{n/2\}$ can be interpreted as compounded n-pointed stars.

1.6 Regular *n*-Sided Figures in the Complex Plane

Section 1.2 explained how to draw regular n-sided polygons. No coordinate system is required for these drawings.

In complex analysis, one often uses representations based on the so-called **complex plane** (also called “Argand diagram” named after the French amateur mathematician Jean-Robert Argand, 1768–1822). This is a two-dimensional coordinate system in which the real part of a complex number is plotted in horizontal direction and the imaginary part in vertical direction.

Complex numbers $z = x + i \cdot y$ are defined in the coordinate system of the complex plane as points with the coordinates (x, y) (see Fig. 1.4).



The points (x, y) of the unit circle, that is, a circle with the radius 1, satisfy the equation $x^2 + y^2 = 1$ – according to the Pythagorean theorem. If you name the angle between the ray, leading from the center to a point on the unit circle, and the x-axis with φ, then each point can also be described by the coordinates $(\cos(\varphi), \sin(\varphi))$.

Fig. 1.4 Stamps of the postal service of the Federal Republic of Germany (“Deutsche Bundespost”) on C. F. Gauss and the complex plane (in Germany named as “Gauss’sche Zahlenebene”)

An equation of the form $z^n = 1$ is called as **cyclotomic polynomial equation**. According to the the **fundamental theorem of algebra**, such an equation has exactly n solutions in the set of complex numbers. In the complex plane, the solutions of the cyclotomic polynomial equation form the vertices of a regular n-sided figure (hence the name for the equation).

The French mathematician Abraham de Moivre (1667–1754), who lived in exile in England, discovered that for every complex number $z = \cos(\varphi) + i \cdot \sin(\varphi)$ and for each natural number n the following equation applies:

Formula

Theorem of Moivre

$$[\cos(\varphi) + i \cdot \sin(\varphi)]^n = \cos(n \cdot \varphi) + i \cdot \sin(n \cdot \varphi) \tag{1.2}$$

Therefore the following applies for every angle $\varphi = k \cdot \frac{360°}{n}$ with $k = 0, 1, 2, \ldots, n-1$:

$$\left[\cos\left(k \cdot \frac{360°}{n}\right) + i \cdot \sin\left(k \cdot \frac{360°}{n}\right)\right]^n = \cos(k \cdot 360°) + i \cdot \sin(k \cdot 360°) = 1 + i \cdot 0 = 1$$

That means that the n complex numbers $z_k = \cos\left(k \cdot \frac{360°}{n}\right) + i \cdot \sin\left(k \cdot \frac{360°}{n}\right)$ satisfy the equation $z^n = 1$. ◀

Formula

Solutions of the Cyclotomic Polynomial Equation

The n solutions of the cyclotomic polynomial equation $z^n = 1$ have the form

$$z_k = \cos\left(k \cdot \frac{360°}{n}\right) + i \cdot \sin\left(k \cdot \frac{360°}{n}\right),$$

where $k = 0, 1, 2, \ldots, n-1$.

The n solutions can be found by drawing n rays from the origin of the coordinate system with an angle φ with $\varphi = k \cdot \frac{360°}{n}$ and determining their points of intersection with the unit circle.

In special cases, the solutions of the cyclotomic polynomial equation can also be determined using elementary algebraic methods, that is, without using trigonometric functions. This is illustrated by the examples for $n = 3$, 4, and 5. ◀

Example 1: Solution of the Equation $x^3 = 1$

Using Trigonometric Functions:

The cubic equation $z^3 = 1$ has the three solutions:

$$\begin{aligned}
z_0 &= \cos(0°) + i \cdot \sin(0°) = 1 \\
z_1 &= \cos(120°) + i \cdot \sin(120°) = -\tfrac{1}{2} + i \cdot \tfrac{\sqrt{3}}{2} \\
z_2 &= \cos(240°) + i \cdot \sin(240°) = -\tfrac{1}{2} - i \cdot \tfrac{\sqrt{3}}{2}
\end{aligned}$$

Using Algebraic Methods:

The cubic equation $x^3 = 1$ has only one real-valued solution, namely $x_1 = 1$. This solution is represented in the complex plane by the point (1, 0).

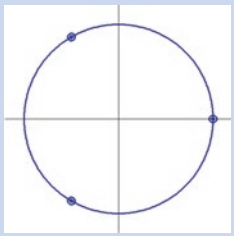
Since $x_1 = 1$ is a solution, the division of terms $\frac{x^3-1}{x-1}$ can be performed without remainder. This leads to the quadratic equation

$$x^2 + x + 1 = 0 \Leftrightarrow \left(x + \tfrac{1}{2}\right)^2 = -\tfrac{3}{4},$$

which has two complex solutions, namely

$$x_2 = -\tfrac{1}{2} + i \cdot \tfrac{\sqrt{3}}{2} \text{ and } x_3 = -\tfrac{1}{2} - i \cdot \tfrac{\sqrt{3}}{2}.$$

In the complex plane, these two solutions are drawn as points with the coordinates $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$.



Example 2: Solution of the Equation x^4=1

Using Trigonometric Functions:

The 4th degree equation $z^4 = 1$ has the four solutions:

$$\begin{aligned} z_0 &= \cos(0°) + i \cdot \sin(0°) = 1 \\ z_1 &= \cos(90°) + i \cdot \sin(90°) = i \\ z_2 &= \cos(180°) + i \cdot \sin(180°) = -1 \\ z_3 &= \cos(270°) + i \cdot \sin(270°) = -i \end{aligned}$$

Using Algebraic Methods:

The 4th degree equation $x^4 = 1$ has two real-valued solutions, namely $x_1 = 1$ and $x_2 = -1$. These solutions are represented in the complex plane by the points (1, 0) and (−1, 0).

Since $x_1 = 1$ and $x_2 = -1$ are solutions, the division of terms $\frac{x^4-1}{x^2-1}$ can be performed without remainder.

This leads to the quadratic equation $x^2 + 1 = 0$, which has two complex solutions: $x_3 = i$ and $x_4 = -i$.