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Graphs and Combinatorial Optimization: from Theory to Applications

CTW2020 Proceedings

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Editors

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Preface

The Cologne-Twente Workshop (CTW) on Graphs and Combinatorial Optimization is a workshop series initiated by Ulrich Faigle, around the time he moved from the Twente University to the University of Cologne. After many CTW editions in Twente and Cologne, it was decided that CTWs were mature enough to move about: in 2004, the CTW was organized in Villa Vigoni (Menaggio, Como, Italy) by Francesco Maffioli (Politecnico di Milano) and Leo Liberti (CNRS-LIX). Since then, the CTW visited again Italy for three times, beyond France, Germany, the Netherlands, and Turkey. This edition is the first time that the CTW is organized with the contribution of CNR-IASI members (Claudio Gentile, Giuseppe Stecca, Paolo Ventura, Giovanni Rinaldi, and Fabio Furini) in addition to the members of University of Rome "Tor Vergata" (Andrea Pacifici), Roma Tre University (Gaia Nicosia), and CNRS & LIX Polytechnique Palaiseau (Leo Liberti).

Having been initially set up by discrete applied mathematicians, the CTW still follows the mathematical tradition. In this CTW edition (hereafter, CTW2020), for the first time we adopted two submission tracks: standard papers of at most 12 pages and traditional CTW extended abstracts of at most 4 pages.

This volume collects the standard papers that were submitted to the CTW2020. The papers underwent a standard peer review process performed by a Program Committee consisting of 30 members:[1] 17 CTW steering committee members and

[1]Ali Fuat Alkaya (Marmara U., Turkey), Christoph Buchheim (TU Dortmund, Germany), Francesco Carrabs (U. Salerno, Italy), Alberto Ceselli (U. Milano, Italy), Roberto Cordone (U. Milano, Italy), Ekrem Duman (Ozyegin U., Turkey), Yuri Faenza (Columbia U., USA), Bernard Gendron (IRO U. Montreal & CIRRELT, Canada), Claudio Gentile (CNR-IASI, Italy), Johann Hurink (U. Twente, The Netherlands), Ola Jabali (Politecnico di Milano, Italy), Leo Liberti (CNRS & LIX Polytechnique Palaiseau, France), Frauke Liers (FAU Erlangen-Nuremberg, Germany), Bodo Manthey (U. Twente, The Netherlands), Gaia Nicosia (U. Roma Tre, Italy), Tony Nixon (U. Lancaster, UK), Andrea Pacifici (U. Roma Tor Vergata, Italy), Ulrich Pferschy (U. Graz, Austria), Stefan Pickl (U. Bundeswehr München, Germany), Michael Poss (LIRMM U. Montpellier & CNRS, France), Bert Randerath (U. Koeln, Germany), Giovanni Righini (U. Milano, Italy), Heiko Roeglin (U. Bonn, Germany), Oliver Schaudt (RTWH Aachen U., Germany), Rainer Schrader (U. Koeln, Germany), Giuseppe Stecca (CNR-IASI, Italy), Frank Vallentin (U. Koeln, Germany),

13 guest members. PC members came from Italy, Germany, France, the USA, Canada, the Netherlands, the UK, Austria, and Turkey. We received 46 submissions of which we accepted 31 for publication in this volume with a rate of success of 67%.

The chapters of this volume present works on graph theory, discrete mathematics, combinatorial optimization, and operations research methods, with particular emphasis on coloring, graph decomposition, connectivity, distance geometry, mixed-integer programming, machine learning, heuristics, meta-heuristics, math-heuristics, and exact methods. Applications are related to logistics, production planning, energy, telecommunications, healthcare, and circular economy.

The scientific program of the CTW2020 includes the 31 standard papers in this volume, 33 extended abstracts, and two plenary invited talks. As usual for the CTW, extended abstracts were subject to a high acceptance level, allowing also papers presenting preliminary results with a particular accent to works presented by MScs, PhDs, or Postdocs. The traditional CTW extended abstracts will be published on the conference's website http://ctw2020.iasi.cnr.it, where also additional material collected during the conference will be posted.

We thank all the PC members and the subreviewers for the complex work performed to select the papers and to improve their quality considering also a possible second round of revision.

Following the CTW tradition, a special issue of Discrete Applied Mathematics (DAM) dedicated to this workshop and its main topics of interest will be edited.

Not every CTW edition features invited plenary speakers, but this one does. Two very well-known researchers accepted our invitation: Prof. Dan Bienstock (Columbia University) and Prof. Marco Sciandrone (University of Florence). Prof. Dan Bienstock works in many topics of Combinatorial Optimization, Integer and Mixed-Integer Programming, and Network Design. He is the author of many journal and conference papers and of two textbooks: "Electrical Transmission System Cascades and Vulnerability: An Operations Research Viewpoint," ISBN 978-1-611974-15-7, SIAM-MOS Series on Optimization (2015), and "Potential Function Methods for Approximately Solving Linear Programming Problems: Theory and Practice," ISBN 1-4020-7173-6, Kluwer Academic Publishers, Boston (2002). Prof. Marco Sciandrone works in Nonlinear Programming with a particular expertise in Machine Learning, Neural Networks, Multiobjective Optimization, and Nonlinear Approximation of Discrete Variables.

Finally, we thank AIRO for hosting this volume in its AIRO-Springer series. We thank both AIRO and CNR-IASI for their support to the realization of the conference.

Paolo Ventura (CNR-IASI, Italy), Maria Teresa Vespucci (U. Bergamo, Italy), and Angelika Wiegele (Alpen-Adria U. Klagenfurt, Austria).

This conference was originally supposed to take place in the wonderful Ischia island on 15–17 June, 2020. Due to the Covid-19 pandemic, we were first obliged to reschedule the conference in September 14–16, 2020, and then to move it online as the majority of conferences in 2020. Nevertheless, we very much hope you will all enjoy the CTW2020.

Rome, Italy	Claudio Gentile
Rome, Italy	Giuseppe Stecca
Rome, Italy	Paolo Ventura
September 2020	

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About the Editors

Claudio Gentile is a Research Director at the Institute of Systems Analysis and Computer Science "Antonio Ruberti" of the Italian National Research Council (CNR-IASI). From 2006 to 2015, he directed the CNR-IASI research unit "Control and Optimization of Complex Systems," and since 2016, he directs the CNR-IASI research unit "OPTIMA: Optimization and Discrete Mathematics." His main research interests are in Combinatorial Optimization, Polyhedral Theory for Linear and Nonlinear Mixed-Integer Programming Problems, Interior Point Methods with applications in Power Energy Production and Distribution, Logistics, Network Design, Staff Management, and Ship Scheduling. He is the author of many scientific publications among journal papers, book chapters, and articles in conference proceedings.

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The Chromatic Polynomial of a Digraph

Winfried Hochstättler and Johanna Wiehe

Abstract An acyclic coloring of a digraph as defined by V. Neumann-Lara is a vertex-coloring such that no monochromatic directed cycles occur. Counting the number of such colorings with k colors can be done by counting so-called Neumann-Lara-coflows (NL-coflows), which build a polynomial in k. We will present a representation of this polynomial using totally cyclic subdigraphs, which form a graded poset Q. Furthermore we will decompose our NL-coflow polynomial, which becomes the chromatic polynomial of a digraph by multiplication with the number of colors to the number of components, using the geometric structure of the face lattices of a class of polyhedra that corresponds to Q. This decomposition leads to a representation using certain subsets of edges of the underlying undirected graph and will confirm the equality of our chromatic polynomial of a digraph and the chromatic polynomial of the underlying undirected graph in the case of symmetric digraphs.

Keywords Dichromatic number · Chromatic polynomial · Flow polynomial · Totally cyclic subdigraphs · Face lattice

1 Introduction

The notion of classic graph coloring deals with finding the smallest integer k such that the vertices of an undirected graph can be colored with k colors, where no two adjacent vertices share the same color. The chromatic polynomial counts those proper colorings a graph admits, subject to the number of colors. William T. Tutte developed a dual concept [17], namely his nowhere-zero flows (NZ-flows), which build a polynomial, the flow polynomial, too.

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We turn our attention to directed graphs, or digraphs for short. In 1982 Víctor Neumann-Lara [12] introduced the dichromatic number of a digraph D as the smallest integer k such that the vertices of D can be colored with k colors and each color class induces an acyclic digraph. This seems to be a reasonable generalization of the chromatic number since both numbers coincide in the symmetric case, where we have all arcs in both directions.

Moreover Neumann-Lara conjectured in 1985, that every orientation of a simple planar graph can be acyclically colored with two colors [13]. Regarding the dichromatic number this is not the only conjecture remaining widely open. Up to some relaxations, for instance Mohar and Li [10] affirmed the two-color-conjecture for planar digraphs of digirth four, it is known [4], that deciding whether an arbitrary digraph has dichromatic number at most two is NP-complete.

Although some progress has been made according thresholds (see e.g. [8]), even the complete case seems to be quite hard. To our knowledge it is not known how many vertices suffice to build a tournament which has dichromatic number five [14].

Nevertheless, Ellis and Soukup determined [6] thresholds for the minimum number of cycles, where reversing their orientation yields a digraph resp. tournament that has dichromatic number at most two.

Comparing the chromatic and the dichromatic number Erdős and Neumann-Lara conjectured [7] in 1979 that if the dichromatic number of a class of graphs is bounded, so is their chromatic number. While Mohar and Wu [11] considered the fractional chromatic number of linear programming proving a fractional version, this is another conjecture remaining unsolved.

With our work we hope to contribute to a better understanding of the dichromatic number. W. Hochstättler [9] developed a flow theory for the dichromatic number transferring Tutte's theory of NZ-flows from classic graph colorings. Together with B. Altenbokum [2] we pursued this analogy by introducing algebraic Neumann-Lara-flows (NL-flows) as well as a polynomial counting these flows. The formula we derived contains the Möbius function of a certain poset. Here, we will derive the values of the Möbius function by showing that the poset correlates to the face lattice of a polyhedral cone.

Probably, the chromatic polynomial of a graph is better known than the flow polynomial. Therefore, in this paper we consider the dual case of our NL-flow polynomial, the NL-coflow polynomial which equals the chromatic polynomial for the dichromatic number divided by the number of colors if the digraph is connected. We will present a representation using totally cyclic subdigraphs and decompose them to obtain an even simpler representation. In particular, it will suffice to consider certain subsets of edges of the underlying undirected graph.

Our notation is fairly standard and, if not explicitly defined, should follow the books of Bondy and Murty [5] for digraphs and Beck and Sanyal [3] for polyhedral geometry. Note that all our digraphs may have parallel and antiparallel arcs as well as loops if not explicitly excluded.

2 Definitions and Tools

Let G be a finite Abelian group and $D = (V, A)$ a digraph. Recall that a map $f : A \longrightarrow G$ is a flow in D, if it satisfies Kirchhoff's law of flow conservation

$$\sum_{a \in \partial^+(v)} f(a) = \sum_{a \in \partial^-(v)} f(a) \tag{1}$$

in every vertex $v \in V$, where $\partial^+(v)$ and $\partial^-(v)$ denote the set of outgoing resp. incoming arcs at v.

Analogously, a map $g : A \longrightarrow G$ is a coflow in D, if it satisfies Kirchhoff's law for (weak) cycles $C \subseteq A$

$$\sum_{a \in C^+} g(a) = \sum_{a \in C^-} g(a), \tag{2}$$

where C^+ and C^- denote the set of arcs in C that are traversed in forward resp. in backward direction.

Now let n be the number of vertices, m be the number of arcs and let M denote the totally unimodular $(n \times m)$-incidence matrix of D. While condition (1) is equivalent to the condition that the vector $f = (f(a_1), \ldots, f(a_m))^\top$ is an element of the null space of M, that is $Mf = 0$, condition (2) is equivalent to the condition that the vector $g = (g(a_1), \ldots, g(a_m))$ is an element of the row space of M, that is $g = pM$, for some $(1 \times n)$-vector $p \in G^{|V|}$.

Definition 1 A digraph $D = (V, A)$ is called *totally cyclic*, if every component is strongly connected. A *feedback arc set* of a digraph is a set $S \subseteq A$ such that $D - S$ is acyclic.

Definition 2 Let $D = (V, A)$ be a digraph and G a finite Abelian group. An *NL-G-coflow* in D is a coflow $g : A \longrightarrow G$ in D whose support contains a feedback arc set. For $k \in \mathbb{Z}$ and $G = \mathbb{Z}$, a coflow g is an *NL-k-coflow*, if

$$g(a) \in \{0, \pm 1, \ldots, \pm(k-1)\}\,, \text{ for all } a \in A,$$

such that its support contains a feedback arc set.

In order to develop a closed formula for the number of NL-G-coflows we use a generalization of the well-known inclusion-exclusion formula, the Möbius inversion.

Definition 3 (See e.g. [1]) Let $(P, \leq)$ be a finite poset, then the *Möbius function* is defined as follows

$$\mu : P \times P \to \mathbb{Z},\ \mu(x, y) := \begin{cases} 0 & , \text{ if } x \not\leq y \\ 1 & , \text{ if } x = y \\ -\sum_{x \leq z < y} \mu(x, z) & , \text{ otherwise}. \end{cases}$$

Proposition 1 (See [1, 15]) *Let $(P, \leq)$ be a finite poset, $f, g : P \longrightarrow \mathbb{K}$ functions and μ the Möbius function. Then the following equivalence holds*

$$f(x) = \sum_{y \geq x} g(y),\ \textit{for all } x \in P \iff g(x) = \sum_{y \geq x} \mu(x, y) f(y),\ \textit{for all } x \in P.$$

With this so called *Möbius inversion from above* it will suffice to compute the number of G-coflows in some given minors B, which is $|G|^{\mathrm{rk}(B)}$, where $\mathrm{rk}(B)$ is the rank of the incidence matrix of $G[B]$ which equals $|V(B)| - c(B)$, i.e. the number of vertices minus the number of connected components of $G[B]$.

3 The NL-Coflow Polynomial

In this chapter we will define the NL-coflow polynomial, which counts the number of NL-G-coflows, using Möbius inversion. Therefor we need a specific partially ordered set. The following poset $(\mathscr{C}, \geq)$ with

$$\mathscr{C} := \left\{ A/C \mid \exists\, C_1, \ldots, C_r \text{ directed cycles, such that } C = \bigcup_{i=1}^{r} C_i \right\}$$

and

$$A / \bigcup_{j \in J} C_j \geq A / \bigcup_{i \in I} C_i :\Leftrightarrow \bigcup_{j \in J} C_j \subseteq \bigcup_{i \in I} C_i,$$

will serve our purpose. Note that in case D is strongly connected, A is the unique minimum of this poset.

Definition 4 Let $D = (V, A)$ be a digraph and μ the Möbius function of $\mathscr{C}$. Then the *NL-Coflow Polynomial* of D is defined as

$$\psi_{NL}^{D}(x) := \sum_{Y \in \mathscr{C}} \mu(A, Y) x^{\mathrm{rk}(Y)}.$$

The dual version of Theorem 3.5 in [2] reveals the following.

Theorem 1 *The number of NL-G-coflows of a digraph D depends only on the order k of G and is given by $\psi_{NL}^{D}(k)$.*

Proof Using Proposition 1 with $f_k, g_k : \mathscr{C} \to \mathbb{Z}$, such that $f_k(Y)$ indicates all G-coflows and $g_k(Y)$ all NL-G-coflows in $D[Y]$, it suffices to show that

$$f_k(Z) = \sum_{\substack{Y \leq Z \\ Y \in \mathscr{C}}} g_k(Y) \tag{3}$$

holds for all $Z \in \mathscr{C}$. Then we obtain

$$\psi_{NL}^{D}(k) = g_k(A) = \sum_{\substack{Y \leq A \\ Y \in \mathscr{C}}} \mu(A, Y) f_k(Y) = \sum_{Y \in \mathscr{C}} \mu(A, Y) k^{\mathrm{rk}(Y)},$$

since the number of G-coflows on $D[Y]$ is given by $k^{\mathrm{rk}(Y)}$.

Concerning (3) let $Z \in \mathscr{C}$ and φ be a G-coflow on $D[Z]$. With d we denote the number of directed cycles in $D[Z]$ and set

$$Y := Z / \bigcup_{i=1}^{d} \{C_i \mid C_i \text{ is a directed cycle in } D[Z] \text{ and } \forall c \in C_i : \varphi(c) = 0\}.$$

Then clearly $Y \in \mathscr{C}$ and $\varphi|_Y$ is an NL-G-coflow on $D[Y]$.

The other direction is obvious since every NL-G-coflow g on $D[Y]$ with $Y \in \mathscr{C}$ can be extended to a G-coflow $\tilde{g}$ on $D[Z]$, setting $\tilde{g}(a) := 0_G$ for all $a \in Z - Y$. □

3.1 Totally Cyclic Subdigraphs

Since many unions of directed cycles determine the same strongly connected subdigraph it suffices to consider all totally cyclic subdigraphs which turn out to form a graded poset.

Lemma 1 *The poset*

$$Q := \{B \subseteq A \mid D[B] \textit{ is totally cyclic subdigraph of } D\},$$

ordered by inclusion, is a graded poset with rank function rk_Q *and its Möbius function alternates in the following fashion:*

$$\mu_Q(\emptyset, B) = (-1)^{\mathrm{rk}_Q(B)}.$$

Proof Let M be the totally unimodular $(n \times m)$-incidence matrix of D. We will show that the face lattice of the polyhedral cone PC described by

$$\begin{pmatrix} M \\ -M \\ -I \end{pmatrix} x \leq 0,$$

corresponds to Q.

Since M is totally unimodular all extreme rays of PC are spanned by integral points. It follows that every totally cyclic subdigraph can be represented by a face of PC, where an arc $1 \leq i \leq m$ exists iff for the corresponding entry $x_i > 0$ holds.

Thus the elements of the face lattice of PC coincide with the elements of our poset and so do the Möbius functions. Well-known facts from topological geometry which can be found for instance in Corollary 3.3.3 and Theorem 3.5.1 in [3] yield that Q is a graded poset and

$$\mu_Q(\emptyset, B) = (-1)^{\dim(B)+1}\chi(B) = (-1)^{\mathrm{rk}_Q(B)}\chi(B),$$

where χ denotes the reduced Euler characteristic, which equals one in this case, since the faces of PC build non-empty closed polytopes (see e.g. Thm. 3.4.1 in [3]). □

Theorem 2 *Let D be a digraph and $(Q, \subseteq)$ the poset defined above. Then the NL-coflow polynomial of D is given by*

$$\psi_{NL}^D(x) = \sum_{B\in Q}(-1)^{\mathrm{rk}_Q(B)}x^{\mathrm{rk}(A/B)}.$$

Proof With Lemma 1 we immediately obtain:

$$\psi_{NL}^D(x) = \sum_{Y\in\mathscr{C}}\mu(A, Y)x^{\mathrm{rk}(Y)} = \sum_{B\in Q}\mu_Q(\emptyset, B)x^{\mathrm{rk}(A/B)} = \sum_{B\in Q}(-1)^{\mathrm{rk}_Q(B)}x^{\mathrm{rk}(A/B)}.$$

□

It is well known that coflows and colorings are in bijection, once the color of some vertex in each connected component has been chosen. As a consequence we have the following corollary, where $c(D)$ denotes the number of connected components in D.

Corollary 1 *The chromatic polynomial of a digraph D is given as*

$$\chi(D, x) = x^{c(D)}\cdot\psi_{NL}^D(x) = \sum_{B\in Q}(-1)^{\mathrm{rk}_Q(B)}x^{\mathrm{rk}(A/B)+c(D)}.$$

4 Decomposing the NL-Coflow Polynomial

In the following we will put our previous results into the setting of polyhedral geometry. There we will find a way to compound some of the objects considered, which will, going back to graph theory, decompose the NL-coflow polynomial such that only certain subsets of edges of the underlying undirected graph need to be considered.

More precisely, fixing the support, implying a fixed exponent in our polynomial, we will show that all existing totally cyclic orientations correlate to the face lattice of some usually unbounded polyhedron. This will yield a relation between the above mentioned poset Q and the maximal faces of a class of polyhedra to be defined in the following. Using the geometric structure of those polyhedra we can contract the corresponding order complex and, by correlating the corresponding Möbius functions, obtain an even simpler representation of the NL-coflow polynomial and therefore of the chromatic polynomial of arbitrary digraphs.

Let $D = (V, A)$ be a digraph, $G = (V, E)$ its underlying undirected graph with $|V| = n$ and $|E| = m$. For $\emptyset \neq B \subseteq E$ a *partial orientation* $\mathscr{O}(B)$ is an orientation of a subset $B' \subseteq B$ of the edges, where the remaining edges in $B \setminus B'$ are considered as pair of antiparallel arcs, called digons. We say a partial orientation is *totally cyclic* if the corresponding induced digraph is. Once the support is fixed, there is a unique inclusionwise maximal partial orientation, denoted with $\bar{\mathscr{O}}(B)$, where we have as many digons as possible.

A flow $x = (\overrightarrow{x}, \overleftarrow{x})^\top \in \mathbb{R}^{2m}$ on D is related to a partial orientation $\mathscr{O}(B)$ by orienting only the edges with $x_i \neq 0$.

Let M be the totally unimodular incidence $(n \times m)$-matrix of the subgraph induced by $\emptyset \neq B \subseteq E$. Then $x \in \mathbb{R}^{2m}$ is a flow iff $(M, -M)x = 0$ holds.

Now, consider the following system

$$\left.\begin{aligned} (M, -M)(\overrightarrow{x}, \overleftarrow{x})^\top &= 0 \\ \overrightarrow{x_i} + \overleftarrow{x_i} &\geq 1 \quad \forall 1 \leq i \leq m \\ \overrightarrow{x_i} &= 0 \quad \text{if } \overrightarrow{i} \notin A \text{ but } \overleftarrow{i} \in A \\ \overleftarrow{x_i} &= 0 \quad \text{if } \overleftarrow{i} \notin A \text{ but } \overrightarrow{i} \in A \\ \overrightarrow{x}, \overleftarrow{x} &\geq 0. \end{aligned}\right\} (P)$$

We denote the polyhedron described above with P and take a look at its vertices, which are the solutions of the program (P), in the first place.

Lemma 2 *Let $x = (\overrightarrow{x}, \overleftarrow{x})^\top$ be a solution of (P). Then a solution $y = (\overrightarrow{y}, \overleftarrow{y})^\top$ of (P) exists with* $\operatorname{supp}(y) \subseteq \operatorname{supp}(x)$ *and $\overrightarrow{y_a} = \overleftarrow{y_a} = \frac{1}{2}$, if a is a bridge and* $\min\{\overrightarrow{y_a}, \overleftarrow{y_a}\} = 0$*, otherwise.*

Proof Let y be a solution with minimal support such that the corresponding partial orientation contains a minimum number of directed cycles.

Let $1 \leq \overrightarrow{a} \leq m$. If a is a bridge, then $y_{\overrightarrow{a}} = y_{\overleftarrow{a}}$ has to hold since otherwise the flow condition would be violated. In the other case assume that $y_{\overrightarrow{a}} \geq y_{\overleftarrow{a}} > 0$. Let $\overrightarrow{a} = (v, w)$ and $C := \{\overrightarrow{a}, b_0, b_1, \dots, b_k\}$ be a directed cycle. After reassigning

$$\begin{aligned} \tilde{y}_{\overrightarrow{a}} &:= 1 + y_{\overrightarrow{a}} - y_{\overleftarrow{a}} \geq 1, \\ \tilde{y}_{\overleftarrow{a}} &:= y_{\overleftarrow{a}} - y_{\overleftarrow{a}} = 0, \\ \tilde{y}_b &:= y_b + 1, \forall b \in C \setminus \{a\} \\ \tilde{y}_c &:= y_c, \qquad \text{otherwise,} \end{aligned}$$

the flow condition still holds in v:

$$\begin{aligned}\sum_{i\in\partial^+(v)} \tilde{y}_i &= \sum_{\substack{i\in\partial^+(v)\\ i\neq\overrightharpoon{a}}} y_i + 1 + y_{\overrightharpoon{a}} - y_{\overleftharpoon{a}} = \sum_{i\in\partial^+(v)} y_i + 1 + y_{\overrightharpoon{a}} - y_{\overleftharpoon{a}} - y_{\overrightharpoon{a}}\\ &= \sum_{i\in\partial^-(v)} y_i + 1 - y_{\overleftharpoon{a}} = \sum_{\substack{i\in\partial^-(v)\\ i\neq\overleftharpoon{a},i\neq b_k}} \tilde{y}_i + 1 + y_b = \sum_{i\in\partial^-(v)} \tilde{y}_i,\end{aligned}$$

as well as in w:

$$\begin{aligned}\sum_{i\in\partial^+(w)} \tilde{y}_i &= \sum_{\substack{i\in\partial^+(w)\\ i\neq\overleftharpoon{a},i\neq b_0}} y_i + y_b + 1 = \sum_{i\in\partial^+(w)} y_i - y_{\overleftharpoon{a}} + 1\\ &= \sum_{\substack{i\in\partial^-(w)\\ i\neq\overrightharpoon{a}}} \tilde{y}_i + y_{\overrightharpoon{a}} + 1 - y_{\overleftharpoon{a}} = \sum_{i\in\partial^-(w)} \tilde{y}_i.\end{aligned}$$

Thus the solution $\tilde{y}$ yields a contradiction to y having minimal support. □

As a result of the preceding lemma, the vertices $\mathscr{V}$ of P are totally cyclic subdigraphs, where the only remaining digons are bridges.

To describe the polyhedron completely we take a look at the recession cone

$$\begin{aligned}rec(P) &= \{y \in \mathbb{R}^{2m} \mid \forall c \in P,\ \forall\lambda \geq 0 : c + \lambda y \in P\}\\ &= P(A, 0)\\ &= Cone\left(\{y \in \mathbb{R}^{2m} \mid y \text{ is directed cycle}\}\right).\end{aligned}$$

Thus we have $P = Conv(\mathscr{V}) + Cone\left(\{y \in \mathbb{R}^{2m} \mid y \text{ is directed cycle}\}\right)$.

In the following we would like to correlate the elements of our poset Q to the face lattice of P, where maximal and minimal elements, $\hat{1}$ and $\hat{0}$, are adjoined and the corresponding Möbius function is denoted with μ_P.

Since there may be several faces corresponding to the same element of Q we define a closure operator on the set of faces $cl : \mathscr{F} \to \overline{\mathscr{F}}$ as follows, where $\mathrm{eq}(F)$ is the set of constraints in (P) where equality holds:

$$\begin{aligned}cl(F) = F_{max} &:= \bigvee\{\tilde{F} \mid \mathrm{supp}(\tilde{F}) = \mathrm{supp}(F)\}\\ &= \{x \in P \mid \mathrm{supp}(F_{max}) = \mathrm{supp}(F), \mathrm{eq}(F_{max}) \text{ is minimal}\},\end{aligned}$$

where $\vee$ is the join of all faces with equal support in the face lattice.

This function is well-defined since the dimension of every face is bounded by $2m$ and F_{max} is uniquely determined since the join is. It is also easy to check that cl is indeed a closure operator.

Now we can identify the maximal faces with the elements of Q by either forgetting the values of a flow or by first taking an arbitrary flow $x \in \mathbb{R}_{+}^{2m}$ satisfying $\overrightarrow{x} + \overleftarrow{x} \geq 1$, that lives on some face F_x and then taking its closure operator $cl(F_x)$.

As a result the Möbius function of $\mathscr{F}$ behaves for $x, y \in P$ as follows (see Prop. 2 on p. 349 in [15]):

$$\sum_{\substack{z \in P \\ cl(F_z)=cl(F_y)}} \mu_P(F_x, F_z) = \begin{cases} \mu_{\bar{\mathscr{F}}}(cl(F_x), cl(F_y)) & \text{, if } F_x = cl(F_x) \\ 0 & \text{, if } F_x \subset cl(F_x) \end{cases}.$$

This is why we will simply write $\mu_P(B, B')$ instead of $\mu_{\bar{\mathscr{F}}}(cl(F_x), cl(F_y))$ for flows x, y on $B, B' \in Q$. Also we identify $\hat{0}$ with $\emptyset$ and $\hat{1}$ with $\bar{\mathscr{O}}(B)$, respectively.

Examining the polyhedron P we find three cases which determine the structure and therefore the Möbius function of the face lattice:

1. There is exactly one vertex v in P.
   1.1 There are no further faces in P including v, i.e. $\dim(P) = 0$.
   1.2 There are further faces in P including v, so P is a pointed cone and $\dim(P) \geq 1$.
2. There are at least two vertices in P.

Note that all cases are mutually exclusive and complete since every P has at least one vertex.

Lemma 3 *Let $\emptyset \neq X \in \mathscr{F}$ be a face of P. Then*

$$\mu_P(\emptyset, X) = \begin{cases} -1 & \textit{if } \dim(X) = 0, \\ (-1)^{\mathrm{rk}_P(X)} & \textit{in cases } 1.1 \textit{ and } 2, \\ 0 & \textit{in case } 1.2. \end{cases}$$

Proof If X is a vertex, then $\dim(X) = 0$ and

$$\mu_P(\emptyset, X) = -\mu_P(\emptyset, \emptyset) = -1 = (-1)^{\mathrm{rk}_P(X)}.$$

For the other cases we will use Theorem 3.5.1 and Corollary 3.3.3 in [3]:

$$\mu_P(\emptyset, X) = (-1)^{\dim(X)+1} \chi(X) = (-1)^{\mathrm{rk}_P(X)} \chi(X),$$

where χ denotes the reduced Euler characteristic.

1.2 Since there is only one vertex, every face of dimension greater 0 builds a pointed cone. Proposition 3.4.9 in [3] yields that $\chi(X) = 0$.
2. Since there are at least two vertices, there are also some faces including them. Those form non-empty closed polytopes with $\chi(X) = 1$ (see Thm. 3.4.1 in [3]).

□

Comparing the Möbius functions of P and Q we find the following relation, where $cr(B) = |B| - |V(B)| + c(B)$ denotes the corank and $\beta(B)$ the number of bridges in the graph induced by $B \subseteq E$.

Lemma 4 *Let $\emptyset \neq B \subseteq E$ and $\mathscr{O}(B)$ be a totally cyclic partial orientation of B, then*

$$\mu_Q(\emptyset, \mathscr{O}(B)) = (-1)^{cr(B)+\beta(B)+1}\mu_P(\emptyset, \mathscr{O}(B))$$

holds, if $\mu_P(\emptyset, X)$ alternates, i.e. in cases 1.1, 2 and if $\dim(X) = 0$*, where $X \in \mathscr{F}$ is the maximal face corresponding to $\mathscr{O}(B)$. Otherwise (in case 1.2) we find*

$$\sum_{\substack{\mathscr{O}(B)\subseteq A \\ tot.cyclic}} \mu_Q(\emptyset, \mathscr{O}(B)) = 0.$$

Proof If both Möbius functions alternate it suffices to consider elements $\mathscr{O}(B) \subseteq A$ where $\mathrm{rk}_P(\mathscr{O}(B))$ is minimal. In this case $\mu_P(\emptyset, \mathscr{O}(B)) = -1$ and we are left to verify

$$\mu_Q(\emptyset, \mathscr{O}(B)) = (-1)^{cr(B)+\beta(B)}.$$

We prove the statement by induction over the number of edges in B. The base cases can be easily checked. Deleting one edge $d \in B$ yields the following two cases:

1. d is a bridge.
   Then $\mathrm{rk}_Q(B-d) = \mathrm{rk}_Q(B)-1$, $cr(B-d) = cr(B)$ and $\beta(B-d) = \beta(B)-1$.
2. d is not a bridge.
   Then $\mathrm{rk}_Q(B-d) = \mathrm{rk}_Q(B)-1$, $cr(B-d) = cr(B)-1$ and $\beta(B-d) = \beta(B)$.

Using the induction hypothesis we find in both cases

$$(-1)^{\mathrm{rk}_Q(B)} = (-1)^{\mathrm{rk}_Q(B-d)+1} \overset{IH}{=} (-1)^{cr(B-d)+\beta(B-d)+1} = (-1)^{cr(B)+\beta(B)}.$$

Otherwise, i.e. case 1.2 due to Lemma 3, we have exactly one vertex and some faces containing it. The number of these faces is determined by the number of digons in

$\bar{\mathscr{O}}(B)$, which we denote with d. Then we have

$$\sum_{\substack{\mathscr{O}(B)\subseteq A\\ tot.cyclic}} \mu_Q(\emptyset, \mathscr{O}(B)) = -\binom{d}{0} + \binom{d}{1} - \binom{d}{2} + \ldots \pm \binom{d}{d} = -\sum_{k=0}^{d} (-1)^k \binom{d}{k} = 0.$$

□

The key point is the following lemma, where the contraction finally takes place.

Lemma 5 *Let $\emptyset \neq B \subseteq E$. Then*

$$\sum_{\emptyset \neq X \subseteq \bar{\mathscr{O}}(B)} \mu_P(\emptyset, X) = -1.$$

Proof Since P is obviously unbounded and has at least one vertex, Corollary 3.4.10 in [3] yields that P has reduced Euler characteristic zero. Consequently the corresponding Möbius function $\mu_P(\emptyset, \bar{\mathscr{O}}(B))$, which is the reduced Euler characteristic (see Prop. 3.8.6 in [16]), equals zero, too. As a result,

$$0 = \mu_P(\emptyset, \bar{\mathscr{O}}(B)) = -\sum_{\emptyset \subseteq X \neq \bar{\mathscr{O}}(B)} \mu_P(\emptyset, X) = -1 - \sum_{\emptyset \neq X \subseteq \bar{\mathscr{O}}(B)} \mu_P(\emptyset, X)$$

holds. □

Combining the last two lemmas we find two different kinds of compression: In cases 1.1 and 2 it suffices to count the element having minimal support due to Lemma 5 and in case 1.2 all totally cyclic partial orientations sum up to zero due to Lemma 4. The following observation translates these cases from polyhedral language into graph theoretical properties.

Definition 5 Let $D = (V, A)$ be a totally cyclic digraph. A digon $d \subseteq A$ is called *redundant* for cyclicity if $D - d$ is still totally cyclic.

Note that every bridge is redundant for cyclicity. Fig. 1 shows a digon that is redundant but not a bridge.

Lemma 6 *Case 1.2 does not hold true if and only if there exists a digon in $\bar{\mathscr{O}}(B)$ that is redundant for cyclicity but not a bridge, or every digon in $\bar{\mathscr{O}}(B)$ is a bridge.*

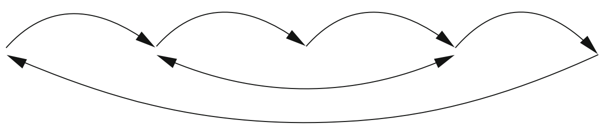

Fig. 1 A digon that is redundant for cyclicity

Proof First we proof the following equivalence:

> There are at least two vertices in P if and only if there is a digon in $\bar{\mathscr{O}}(B)$ that is redundant for cyclicity but not a bridge.

Let e be a digon in $\bar{\mathscr{O}}(B)$ that is redundant but not a bridge, then $\bar{\mathscr{O}}(B) - \overleftarrow{e}$ and $\bar{\mathscr{O}}(B) - \overrightarrow{e}$ contain vertices including $\overleftarrow{e}$, resp. $\overrightarrow{e}$ which hence are two different vertices in P. For the other direction take vertices $v \neq w$ in P. Then $v \cup w$ is a face in P including a digon e that is no bridge. Assume e is not redundant, then $\bar{\mathscr{O}}(B) - \overleftarrow{e}$ or $\bar{\mathscr{O}}(B) - \overrightarrow{e}$ could not have been totally cyclic and so one of the vertices v or w.

Consequently case 1.2 does not hold true iff there is a digon that is redundant but not a bridge (case 2) or, if there is only one vertex in P, then there are no further faces including it, which means that every digon in $\bar{\mathscr{O}}(B)$ is a bridge (case 1.1). □

This leads to the following main result of this paper, a representation of the NL-coflow polynomial for arbitrary digraphs, where we sum only over certain subsets of the edges of the underlying undirected graph.

Theorem 3 *Let $D = (V, A)$ be a digraph and $G = (V, E)$ its underlying undirected graph. Then*

$$\psi_{NL}^{D}(x) = \sum_{B \in TC} (-1)^{|B|} x^{\tilde{c}(B) - c(D)}$$

holds, where $\tilde{c}(B)$ counts the components in the spanning subgraph of G with edge set B and TC includes all $B \subseteq E$ which admit a totally cyclic partial orientation $\mathscr{O}(B)$ in A such that $\bar{\mathscr{O}}(B)$ has no digons but bridges or $\bar{\mathscr{O}}(B)$ has a digon that is redundant but not a bridge.

Proof Instead of counting totally cyclic subdigraphs one can count totally cyclic partial orientations of a fixed underlying subgraph. Thus the preceding lemmas yield

$$\begin{aligned}
\psi_{NL}^{D}(x) &= \sum_{\substack{X \subseteq A \\ tot.cyclic}} \mu_Q(\emptyset, X) x^{\mathrm{rk}(A/X)} \\
&= \sum_{B \subseteq E} \sum_{\substack{\mathscr{O}(B) \\ tot.cyclic}} \mu_Q(\emptyset, \mathscr{O}(B)) x^{\mathrm{rk}(A/B)} \\
&= \sum_{\emptyset \neq B \subseteq E} \sum_{\substack{\mathscr{O}(B) \\ tot.cyclic}} \mu_Q(\emptyset, \mathscr{O}(B)) x^{\mathrm{rk}(A/B)} + x^{-c(D)} \\
&\overset{\text{Lemma } 4}{=} \sum_{\substack{\emptyset \neq B \subseteq E \\ (*)}} \sum_{\substack{\mathscr{O}(B) \\ tot.cyclic}} (-1)^{cr(B) + \beta(B) + 1} \mu_P(\emptyset, \mathscr{O}(B)) x^{\mathrm{rk}(A/B)} + x^{-c(D)}
\end{aligned}$$

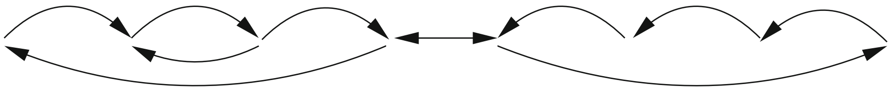

Fig. 2 A totally cyclic orientation that is not considered in TC

$$\overset{\text{Lemma 5}}{=} \sum_{\substack{\emptyset \neq B \subseteq E \\ (*)}} (-1)^{cr(B)+\beta(B)} x^{\mathrm{rk}(A/B)} + x^{-c(D)}$$

$$= \sum_{\substack{B \subseteq E \\ (*)}} (-1)^{cr(B)+\beta(B)} x^{n-|V(B)|+c(B)-c(D)}.$$

Condition $(*)$ means, that we sum over all $B \subseteq E$ having a totally cyclic partial orientation $\mathscr{O}(B) \subseteq A$, where case 1.2 is not true. Due to Lemma 6 this situation occurs if and only if $\bar{\mathscr{O}}(B)$ has no digons but bridges, or there exists a digon that is redundant but not a bridge. Clearly, $n - |V(B)| + c(B) = \tilde{c}(B)$ holds, and we are left to verify

$$(-1)^{cr(B)+\beta(B)} = (-1)^{|B|}.$$

This can be done by induction. Deleting a bridge $d \in B$ yields $cr(B-d) = cr(B)$ and $\beta(B-d) = \beta(B) - 1$ while deleting a non-bridge yields $cr(B-d) = cr(B) - 1$ and $\beta(B-d) = \beta(B)$. In both cases we find

$$(-1)^{cr(B)+\beta(B)} = (-1)^{cr(B-d)+\beta(B-d)+1} \overset{IH}{=} (-1)^{|B-d|+1} = (-1)^{|B|}. \qquad \square$$

Note that TC includes all $B \subseteq E$ which admit a totally cyclic partial orientation $\mathscr{O}(B)$ in A, but not those, where $\bar{\mathscr{O}}(B)$ includes a digon that is no bridge and no digon is redundant unless it is a bridge in $\bar{\mathscr{O}}(B)$ (Fig. 2).

5 Symmetric Digraphs

Considering symmetric digraphs $D = (V, A)$, it is obvious that the NL-coflow polynomial equals the chromatic polynomial $\chi(G, x)$ of the underlying undirected graph $G = (V, E)$ divided by the number of colors since both polynomials count the same objects. Using Theorem 3 we find an alternative proof of this fact, where the chromatic polynomial is represented by (see [5])

$$\chi(G, x) = \sum_{B \subseteq E} (-1)^{|B|} x^{\tilde{c}(B)}.$$

Corollary 2 *Let $D = (V, A)$ be a symmetric digraph and $G = (V, E)$ its underlying undirected graph. Then the following holds*

$$\psi_{NL}^{D}(x) = \chi(G, x) \cdot x^{-c(G)}.$$

Proof In a symmetric digraph every edge is a digon, so for every subset $B \subseteq E$ there exists a totally cyclic partial orientation $\mathscr{O}(B)$. Furthermore, if $cr(D) = 0$, every digon is a bridge and if $cr(D) \geq 1$ there exists a cycle of length ≥ 3 in D where every digon is redundant but no bridge. □

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On List k-Coloring Convex Bipartite Graphs

Josep Díaz, Öznur Yaşar Diner, Maria Serna, and Oriol Serra

Abstract List k-Coloring (LI k-COL) is the decision problem asking if a given graph admits a proper coloring compatible with a given list assignment to its vertices with colors in $\{1, 2, \ldots, k\}$. The problem is known to be NP-hard even for $k = 3$ within the class of 3-regular planar bipartite graphs and for $k = 4$ within the class of chordal bipartite graphs. In 2015 Huang, Johnson and Paulusma asked for the complexity of LI 3-COL in the class of chordal bipartite graphs. In this paper, we give a partial answer to this question by showing that LI k-COL is polynomial in the class of convex bipartite graphs. We show first that biconvex bipartite graphs admit a multichain ordering, extending the classes of graphs where a polynomial algorithm of Enright et al. (SIAM J Discrete Math 28(4):1675–1685, 2014) can be applied to the problem. We provide a dynamic programming algorithm to solve the LI k-COL in the class of convex bipartite graphs. Finally, we show how our algorithm can be modified to solve the more general LI H-COL problem on convex bipartite graphs.

Keywords List coloring · Convex bipartite · Biconvex bipartite graphs

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1 Introduction

A *coloring* of a graph $G = (V, E)$ is a map $c : V \to \mathbb{N}$. A coloring is *proper* if no two adjacent vertices are assigned the same color. If there is a proper coloring of a graph that uses at most k colors, then we say that G is *k-colorable*, and that c is a *k-coloring* for G. The coloring problem COL asks for a given graph $G = (V, E)$, and a positive integer k, whether there is a k-coloring for G or not. When k is fixed, we have the k-COLORING problem.

A list assignment $L : V \to 2^{\mathbb{N}}$ is a map assigning a set of positive integers to each vertex of G. Given G and L, the List Coloring problem LICOL asks for the existence of a proper coloring c that obeys L, i.e., each vertex receives a color from its own list. If the answer is positive, G is said to be *L-colorable*. Variants of the problem are defined by bounding the total number of available colors or by bounding the list size. In LIST k-COLORING (LI k-COL), $L(v) \subseteq \{1, 2, \ldots, k\}$ for each $v \in V$. Thus, there are k colors in total. On the other hand, in k-LIST COLORING (k-LICOL) each list L has size at most k. In this case, the total number of colors can be larger than k.

Precoloring Extension, PREXT, is a special case of LICOL and a generalization of COL. In PREXT all of the vertices in a subset W of V are previously colored; and the task is to extend this coloring to all of the vertices. If, in addition, the total number of colors is bounded, say by k, then it is called the k-Precoloring Extension, k-PREXT. k-COL is clearly a special case of k-PREXT, which in turn is a special case of LI k-COL. Refer to [16] for a chart summarizing these relationships.

For general graphs COL and its variants LICOL and PREXT are NP-complete; see [14, 24]. Most of their variants are NP-complete even when the parameter k is fixed for small values of k: k-COL, k-LICOL, LI k-COL and k-PREXT are NP-complete when $k \geq 3$ [29]; and they are polynomially solvable when $k \leq 2$ [13, 38].

Concerning the complexity of these problems in graph classes, COL is solvable in polynomial time for perfect graphs [18] whereas LICOL is NP-complete when restricted to perfect graphs and many of its subclasses, such as split graphs, bipartite graphs [28] and interval graphs [2]. On the other hand, LICOL is polynomially solvable for trees, complete graphs and graphs of bounded treewidth [23]. Refer to Tuza [37], and more recently to Paulusma [33] for related surveys.

For small values of k, Jansen and Scheffler [23] have shown that 3-LICOL is NP-complete when restricted to complete bipartite graphs and cographs, as observed in [15]. Kratochvíl and Tuza [27] showed that 3-LICOL is NP-complete even if each color appears in at most three lists, each vertex in the graph has degree at most three and the graph is planar. 3-PREXT is NP-complete even for 3-regular planar bipartite graphs and for planar bipartite graphs with maximum degree 4 [7].

For fixed $k \geq 3$, LI k-COL is polynomially solvable for P_5-free graphs [20]. Note that chordal bipartite graphs contain P_5-free graphs, but P_6 free graphs are incomparable with chordal bipartite graphs [35]. LI 3-COL is polynomial for P_6-free graphs [6] and for P_7-free graphs [3]. Computational complexity of LI 3-COL for P_8-free bipartite graphs is open [3]. Even the restricted case of LI 3-COL for P_8-free chordal bipartite graphs is open. Golovach et al. [16] give a survey that summarizes the results for Li k-Col on H-free graphs in terms of the structure of H.

PrExt problem is solvable in linear time on P_5-free graphs; and it is NP-complete when restricted to P_6-free chordal bipartite graphs [22]. 3-PrExt is NP-complete even for planar bipartite graphs [26], even for those having maximum degree 4 [7]. Recall that PrExt generalizes k-PrExt and Li k-Col generalizes k-PrExt. But there is no direct relation between PrExt and Li k-Col [16].

Coloring problems can be placed in the more general class of H-coloring problems. Given two graphs G and H, a function $f : V(G) \to V(H)$ such that $f(u)$ and $f(v)$ are adjacent in H whenever u and v are adjacent in G is called a graph homomorphism from G to H. For a fixed graph H and for an input G, the H-coloring problem, H-Col asks whether there is a G to H homomorphism. In the list H-coloring problem, Li H-Col, each vertex of the input graph G is associated with a list of vertices of H, and the question is whether a G to H homomorphism exists that maps each vertex to a member of its list. Observe that Li H-Col is a generalization of Li k-Col. The complexities of the H-coloring and list H-coloring problems for arbitrary input graphs are completely characterized in terms of the structure of H, see Nešetřil and Hell [19].

Although intensive research on this subject has been undertaken in the last two decades, there are still numerous open questions regarding computational complexities on LiCol and its variants when they are restricted to certain graph classes. Huang et al. [21] proved that Li 4-Col is NP-complete for P_8-free chordal bipartite graphs and 4-PrExt is NP-complete for P_{10}-free chordal bipartite graphs. They further pose the problem on the computational complexity of the Li 3-Col and 3-PrExt on chordal bipartite graphs. Here Li k-Col and k-PrExt on convex bipartite graphs, a proper subclass of chordal bipartite graphs, are studied for fixed k, and a partial answer to this question is given. Figure 1 summarizes the related results. Note that, here by Li k-Col it is assumed that k is fixed.

A bipartite graph $G = (X \cup Y, E)$ is convex if it admits an ordering on one of the parts of the bipartition, say X, such that the neighbours of each vertex in Y are consecutive in this order. If both color classes admit such an ordering the graph is called biconvex bipartite (see Sect. 2 for formal definitions). Chordal bipartite graphs contain convex bipartite graphs properly. Convex bipartite graphs contain as a proper subclass biconvex bipartite graphs, which contain bipartite permutation graphs properly. More information on these classes can be found in Spinrad [35] and in Brandstädt et al. [4].

Enright et al. [12] have shown that Li k-Col is solvable in polynomial time when restricted to graphs with all connected induced subgraphs having a multichain ordering. They apply this result to permutation graphs and interval graphs. Here, we show that connected biconvex graphs also admit a multichain ordering, implying a polynomial time algorithm for Li k-Col on this graph class.

From the point of view of parameterized complexity, treewidth can be computed in polynomial time on chordal bipartite graphs [25]. Li k-Col can be solved in polynomial time on chordal bipartite graphs with bounded treewidth [9, 23], which includes chordal bipartite graphs of bounded degree [30]. Li k-Col is polynomial

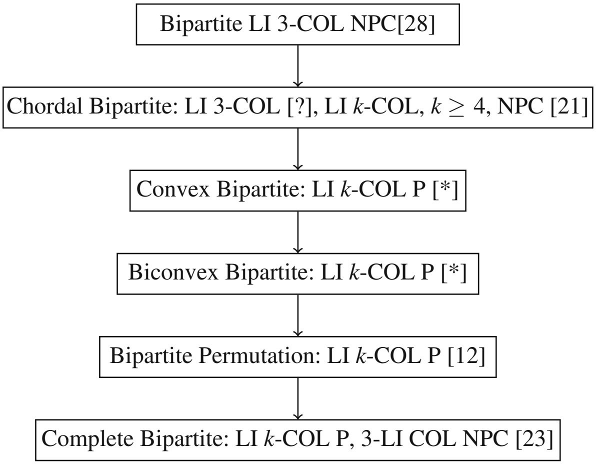

Fig. 1 Chart for known complexities for LiCol and its variants for chordal bipartite graphs and its subclasses, for $k \geq 3$. The complexity results marked with [*] is the topic of this paper, while [?] stands for open cases. Results without reference are trivial. P stands for Polynomial and NPC for NP-complete

for graphs of bounded cliquewidth [8]. Note that convex bipartite graph contains graphs with unbounded treewidth as well as graphs with unbounded cliquewidth.

The paper is organized as follows. In Sect. 2, we give the necessary definitions. In Sect. 3, we show that connected biconvex bipartite graphs admit multichain ordering. In Sect. 4, we show that, for fixed k, Li k-Col is polynomially solvable when it is restricted to convex bipartite graphs. Then, we show how to extend this result to Li H-Col. For an extended version of this paper the reader may refer to [10].

2 Preliminaries

We consider finite simple graphs $G = (V, E)$. For terminology refer to Diestel [11]. An edge joining non adjacent vertices in the cycle, C_n, is called a *chord*. A graph G is *chordal* if every induced cycle of length $n \geq 4$ has a chord. *Chordal bipartite graphs* are bipartite graphs in which every induced $C_n, n \geq 6$ has a chord. This graph class is introduced by Golumbic and Gross [17]. Chordal bipartite graphs may contain induced C_4, so they do not constitute a subclass of chordal graphs but it is a proper subclass of bipartite graphs. Chordal bipartite graphs can be recognized in polynomial time [32].

A bipartite graph is represented by $G = (X \cup Y, E)$, where X and Y form a bipartition of the vertex set into stable sets. An ordering of the vertices X in a bipartite graph $G = (X \cup Y, E)$ has the *adjacency property* (or the ordering is