Frontiers in Mathematics

Bruno Bianchini
Luciano Mari
Patrizia Pucci
Marco Rigoli

Geometric Analysis of Quasilinear Inequalities on Complete Manifolds

Maximum and Compact Support Principles and Detours on Manifolds

Birkhäuser

Birkhäuser

Frontiers in Mathematics

This series is designed to be a repository for up-to-date research results which have been prepared for a wider audience. Graduates and postgraduates as well as scientists will benefit from the latest developments at the research frontiers in mathematics and at the "frontiers" between mathematics and other fields like computer science, physics, biology, economics, finance, etc. All volumes are online available at SpringerLink.

More information about this series at http://www.springer.com/series/5388

Bruno Bianchini • Luciano Mari • Patrizia Pucci •
Marco Rigoli

Geometric Analysis of Quasilinear Inequalities on Complete Manifolds

Maximum and Compact Support Principles and Detours on Manifolds

Birkhäuser

Bruno Bianchini
Dipartimento di Matematica Pura e Applicata
Università degli Studi di Padova
Padova, Italy

Luciano Mari
Dipartimento di Matematica
Università degli Studi di Torino
Torino, Italy

Patrizia Pucci
Dipartimento di Matematica e Informatica
Università degli Studi di Perugia
Perugia, Italy

Marco Rigoli
Dipartimento di Matematica
Università degli Studi di Milano
Milano, Italy

ISSN 1660-8046 ISSN 1660-8054 (electronic)
Frontiers in Mathematics
ISBN 978-3-030-62703-4 ISBN 978-3-030-62704-1 (eBook)
https://doi.org/10.1007/978-3-030-62704-1

Mathematics Subject Classification: Primary 58J05, 53C42, 53C21; Secondary 35B08, 35B50, 35B53, 58J32, 58J65

This book is published under the imprint Birkhäuser, www.birkhauser-science.com, by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The use of typical tools from Analysis to shed light on geometric properties is at the core of modern Global Differential Geometry. The aim of the book is to discuss the influence of the structure of a manifold M on the qualitative behaviour of solutions to certain classes of elliptic PDEs arising in a Riemannian context. The differential inequalities that we shall consider are elliptic, possibly degenerate, with the left-hand side described by a quasilinear operator Δ_φ, shortly called the φ-Laplacian. Depending on the choice of φ, the class encompasses well-known examples, including, for instance, the p-Laplacian and the mean curvature operator. To illustrate our results, in the first two chapters we have chosen to discuss some geometric and physical problems where such inequalities naturally appear, and in what follows we shall often pay special attention to the mean curvature operator, which arises when considering the graph associated to a smooth function u on the manifold. This prototype case leads to investigate a large class of differential inequalities whose right-hand side depends not only on $x \in M$ and on the solution u but also on the gradient of u. Having motivated our study, a key part of the book deals with various generalized forms of the maximum principle for Δ_φ, which serve as a bridge to relate the Geometry of M to the analytical properties of the PDE under consideration. Our technical achievements will provide a bulk of flexible methods to describe qualitative properties of solutions to a variety of problems. In the last chapters, Liouville type theorems and compact support principles are investigated in detail, with special emphasis on the role played by the integrability requirements that, in the literature, are known as the Keller–Osserman conditions. Geometric applications, among others, touch upon Bernstein type theorems for graphs with prescribed mean curvature (notably, minimal or solitons for the mean curvature flow), Yamabe and capillarity equations. We refer to the "Contents" for a more exhaustive list.

The book presents the most recent points of view and trends in the field and is therefore to be considered at an advanced level suitable to researchers and senior post-graduate students with a solid knowledge in Differential Geometry and elliptic PDEs. Besides the presentation of new theorems, we collect and organize sharp refinements of some research results obtained by the authors and their collaborators in a period of about twenty years. In this respect, we mention the monographs of P. Pucci and J. Serrin [194], of S. Pigola, M. Rigoli and A.G. Setti [182], and of L.J. Alías, P. Mastrolia and M. Rigoli [6], which could serve as a starting introduction for the present work. In fact, we will consistently refer to them as well as to the references therein for the previous literature. Non-expert readers who are interested in the topic and have an eye on geometric applications can profit from the introductory chapters in [6]. The monograph aims to fulfil two purposes. First, to provide an exposition of some fundamental tools in Geometric Analysis and elliptic PDEs on complete manifolds, a subject that has become of general interest in the last decades. Second, to enable researchers to get familiar with the necessary results and concepts to proceed towards a further study of the specialized literature on the subject. We included an extensive and commented bibliography in the attempt to make it a reference book for future research.

We end by expressing our deep gratitude to the colleagues and students who made this book possible. Special thanks goes also to Sabrina Hoecklin from Springer Nature Switzerland AG, Birkhäuser, for her useful help at all stages of the editing process, and to Daniel Jagadisan, Sarah Annette Goob and Mahalakshmi Balamurugan for taking care of the production process.

Padova, Italy — Bruno Bianchini
Torino, Italy — Luciano Mari
Perugia, Italy — Patrizia Pucci
Milano, Italy — Marco Rigoli
August 2020

Contents

List of Symbols

The following symbols are frequently used along the manuscript:

$\mathbb{R}_0^+$	$\mathbb{R}_0^+ = [0, \infty)$;
$\mathbb{R}^+$	$\mathbb{R}^+ = (0, \infty)$;
$\mathbb{R}^m$	Flat m-dimensional Euclidean space;
$\mathbb{S}^m$	Unit sphere in $\mathbb{R}^{m+1}$ endowed by the induced Euclidean metric;
$\mathbb{H}^m$	Hyperbolic space of sectional curvature -1;
$\mathrm{d}\theta^2$	Round metric of curvature 1 on the unit sphere $\mathbb{S}^{m-1}$;
ω_{m-1}	Volume of the unit sphere $(\mathbb{S}^{m-1}, \mathrm{d}\theta^2)$;
$\mathbb{R} \times_h M$	Warped product of $\mathbb{R}$ and $(M, \langle\ ,\ \rangle)$ with metric $\mathrm{d}s^2 + h(s)^2\langle\ ,\ \rangle$;
$\mathrm{Lip}_{\mathrm{loc}}(\Omega)$	Locally Lipschitz functions on Ω;
Δ_p	p-Laplacian;
$\Delta_{p,q}$	(p, q)-Laplacian;
Δ_φ	φ-Laplacian;
$\mathcal{O}$	Origin of M;
$D_{\mathcal{O}}$	Maximal domain of normal coordinates centred at $\mathcal{O}$;
$\mathrm{cut}(\mathcal{O})$	Cut-locus of $\mathcal{O}$;
Ric	Ricci curvature;
Sec	Sectional curvature;
$\mathrm{Sec}_{\mathrm{rad}}$	Radial sectional curvature;
M_g	Radially symmetric model manifold with warping function g;
$\mathcal{G}$	Green kernel of Δ_p;
$\mathscr{G}^g(r)$	Green kernel of Δ_p on a model M_g;
$\mathrm{cap}_p(K)$	p-capacity of a set K;
ϱ	Fake distance from a given origin;
(FMP)	Finite maximum principle;
(WMP_∞)	Weak maximum principle at infinity;
(SMP_∞)	Strong maximum principle at infinity;
(OWMP_∞)	Open form of the weak maximum principle at infinity;
(L)	Liouville property;
(SL)	Strong Liouville property;

(CSP)	Compact support principle;
(FE)	Feller property;
(KO_∞)	Keller–Osserman condition at infinity;
(KO_0)	Keller–Osserman condition at zero;
($\neg KO_\infty$)	Failure of the Keller–Osserman condition at infinity;
($\neg KO_0$)	Failure of the Keller–Osserman condition at zero;
($\mathcal{WS}$)	Weak Sard property.

1 Some Geometric Motivations

Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$, and let Δ_φ be a quasilinear operator depending on a real function φ, to be detailed in the next chapter. For instance, the family Δ_φ includes the Laplace–Beltrami operator Δ, defined with the sign agreement that

$$\Delta u = \operatorname{div}(\nabla u) = \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} \qquad \text{on } \mathbb{R}^m.$$

Since the birth of Geometric Analysis, several geometric applications lead to the study of differential inequalities of the type

$$\Delta_\varphi u \geq b(x) f(u) \qquad \text{on } M, \tag{1.1}$$

for given $b : M \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Here, u is a function encoding some property of the problem under consideration, and the determination of the qualitative properties of u can therefore be read in terms of constraints for the original geometric problem. Hence, it is fundamental to understand how the geometry of M influences the behaviour of u. By now, (1.1) is fairly well understood for most of the operators Δ_φ. A thorough investigation, with applications ranging from Lorentzian Geometry to the theory of Ricci and mean curvature flow solitons, to Kähler manifolds, Yang-Mills fields and minimal submanifolds, among many others, can be found in the monograph [182] and in the book [6]. However, some relevant problems are also characterized by the appearing of gradient terms in the right-hand side of (1.1), notably those related to the mean curvature operator

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right), \tag{1.2}$$

B. Bianchini et al., *Geometric Analysis of Quasilinear Inequalities on Complete Manifolds*, Frontiers in Mathematics, https://doi.org/10.1007/978-3-030-62704-1_1

and consequently to graphs with prescribed mean curvature. Despite the many works devoted to the study of (1.2) (some of them will be described in the next section), the existing literature is not sufficiently complete from the point of view that we are going to consider in this book. In particular, the results in the above mentioned references are not sufficient to provide a full picture of how geometry influences the behaviour of u. Although the mean curvature operator should be regarded as the main focus of the present work from the geometric point of view, the techniques that we present apply to a wide class of quasilinear elliptic inequalities of the type

$$\Delta_\varphi u \geq b(x) f(u) l(|\nabla u|) \qquad \text{on } M \tag{1.3}$$

for rather general b, f, l. This book can be considered as a natural continuation of [6, 182] as well as of [194], that are an invaluable source of material and whose results will be repeatedly commented herein. However, both our focus and our results significantly depart from those in [6, 182, 194], and in various instances, they give new insight even for equations of the type (1.1). It will be later underlined how the study of equations with a non-constant gradient term l will be also crucial to achieve sharp Liouville theorems for (1.4), where “apparently” no gradient term appears. This is remarkably the case of the capillarity equation, as we shall detail in Chap. 2. The purpose of this chapter is to motivate, via some natural examples, the study of (1.3) with a nontrivial gradient term.

1.1 Prescribing the Mean Curvature of a Graph

The mean curvature operator arises when considering the graph immersion

$$\varphi : M \to \mathbb{R} \times M, \qquad \varphi(x) = \big(u(x), x\big)$$

associated to a smooth function $u : M \to \mathbb{R}$. Precisely, we endow $\mathbb{R} \times M$ with the product metric $\mathrm{d}s^2 + \langle \, , \, \rangle$, and M with the induced graph metric

$$g = \varphi^*\big(\mathrm{d}s^2 + \langle \, , \, \rangle\big),$$

so that the image $\varphi(M)$ is isometric to (M, g). Then, the unnormalized mean curvature H of $\varphi(M)$ in the upward-pointing normal direction satisfies the differential equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) = H \qquad \text{on } M,$$

where the gradient ∇ and the divergence operator are those of the metric $\langle\ ,\ \rangle$. For instance, if $\varphi(M)$ is minimally immersed, that is, $H = 0$, the equation becomes

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{MSE}$$

and, already in the case M is the Euclidean plane $\mathbb{R}^2$, its study has brought to deep connections between Geometry and Analysis. We refer the reader to the old, but still actual, monumental work of J.C.C. Nitsche [172], to the book of E. Giusti [102] and to the informative survey of L. Simon [223], to appreciate the breadth of viewpoints and the depth of the techniques necessary to understand equation (MSE).

Hereafter, a solution of a differential equation or inequality is said to be *entire* if it is defined on the whole of M. In particular, differently from part of the existing literature, the term "entire" does not indicate a specific behaviour of u at infinity.

A basic question related to (MSE) is whether there exist entire solutions possibly with some restrictions such as, for instance, a controlled growth at infinity or the boundedness on one side. Classically, the problem has first been considered in Euclidean space $\mathbb{R}^m$, and the efforts to solve it favoured the flourishing of Geometric Analysis. In 1915, S. Bernstein [18] proved that the only entire solutions of (MSE) over $\mathbb{R}^2$ are affine functions (a topological gap in his highly nontrivial argument has been pointed out and corrected in [125, 159]). Various other proofs later appeared in the literature, [76, 97, 117, 160, 171, 188, 223], each one inferring the validity of Bernstein's theorem from some peculiarities of $\mathbb{R}^2$ and thus not extendible to higher dimensions. We suggest to consult Farina's survey [83] for more details. The first proof to allow for a higher-dimensional generalization has been given by W.H. Fleming [99], and since then, in few years, fundamental contributions of E. De Giorgi ([68, 69], $m = 3$), F. Almgren ([8], $m = 4$) and J. Simons ([224], $m \leq 7$) extended the validity of Bernstein theorem up to dimension $m = 7$. In 1969, E. Bombieri, De Giorgi and E. Giusti [30] gave the first example of entire minimal graph over $\mathbb{R}^m$ for $m \geq 8$ (further examples were later given by L. Simon [222]), thus leading to the following remarkable result:

$$\text{each minimal graph over } \mathbb{R}^m \text{ is affine} \qquad \Longleftrightarrow \qquad m \leq 7. \tag{$\mathscr{B}1$}$$

If the graph is subjected to some a priori bound, then further tools are available and more rigidity is expected. We mention that

($\mathscr{B}2$) by work of Bombieri, de Giorgi and M. Miranda [29], positive solutions of (MSE) over $\mathbb{R}^m$ are constant for every $m \geq 2$;

($\mathscr{B}3$) coupling [29] with works of De Giorgi [68, 69] and J. Moser [163], every entire solution of (MSE) over $\mathbb{R}^m$ with at most linear growth is affine.

We also mention that an entire solution of (MSE) having $m-7$ derivatives bounded on one side is necessarily affine. This result was shown by A. Farina [85], improving on [28, 84].

If $M \neq \mathbb{R}^m$, the set of solutions of (MSE) may drastically change. For instance, if M is the hyperbolic space $\mathbb{H}^m$ of curvature -1, then (MSE) has plenty of bounded solutions: for every ϕ continuous on the boundary at infinity $\partial_\infty \mathbb{H}^m \approx \mathbb{S}^{m-1}$, there exists u solving (MSE) on $\mathbb{H}^m$ and approaching ϕ at infinity. A proof of this result is contained in [169] for $m = 2$, and in [77] for larger m. Up to defining $\partial_\infty M$ appropriately, the same Plateau's problem at infinity can be considered on any Cartan–Hadamard manifold, that is, on any complete, simply connected manifold whose sectional curvature satisfies

$$\mathrm{Sec} \leq 0 \qquad \text{on } M.$$

Since the problem is solvable on $\mathbb{H}^m$ but not on $\mathbb{R}^m$ because of ($\mathscr{B}2$), it is natural to wonder about the sharp thresholds on Sec, the sectional curvature, that guarantee its solvability. The question has recently been addressed in a series of works [39–41, 44, 123, 210]. The picture is indeed subtle, and in general a negative upper bound on Sec does not suffice to guarantee that the Plateau's problem has a solution. A thorough discussion of the above results would lead us a bit too far from the main focus of this introductory chapter, so we refer the interested reader to the above mentioned works, as well as to the surveys [21, 114]. Similarly, it is natural to consider the following

Question Which geometric conditions on M can guarantee the validity of results analogous to those in ($\mathscr{B}1$), ($\mathscr{B}2$) and ($\mathscr{B}3$)?

To present, a thorough answer to the above question is still far from reaching, and for instance we are aware of no results addressing ($\mathscr{B}1$) on manifolds different from $\mathbb{R}^m$. In view of the structure theory initiated by J. Cheeger and T. Colding in [46–49], it is reasonable to hope that results similar to those holding in $\mathbb{R}^m$ could be obtainable on manifolds with non-negative Ricci curvature:

$$\mathrm{Ric} \geq 0.$$

Nevertheless, a complete solution of the question would be relevant even on manifolds with $\mathrm{Sec} \geq 0$. The guess might also be motivated by the fact that (MSE) rewrites as

$$\Delta_g u = 0 \qquad \text{on } M,$$

with Δ_g the Laplace–Beltrami operator of the graph metric g, and by the theory of harmonic functions on manifolds with non-negative Ricci or sectional curvature. Nevertheless, by Gauss equation it should be stressed that no control on any of these curvatures on the graph (M, g) would easily follow from a control on the corresponding one on the base manifold $(M, \langle\, , \rangle)$, so the analogy does not allow to directly apply techniques from the theory of harmonic functions to the realm of minimal graphs. To the best of our knowledge, some interesting results regarding ($\mathscr{B}2$) and ($\mathscr{B}3$) are known under both Ricci and sectional curvature requirements, see [44, 73, 212]. Very recently in [59, 72], the authors independently obtained the following extension of ($\mathscr{B}2$) without the need of sectional curvature bounds:

Theorem 1.1 ([59, 72]) *If M is a complete manifold with* $\mathrm{Ric} \geq 0$*, then every positive solution of* (MSE) *is constant.*

A complete solution to the above question is likely to significantly improve the current understanding of the geometry of the mean curvature operator on Riemannian manifolds.

Together with the minimal case, it is of interest to study graphs whose mean curvature H is a prescribed function of the height, possibly via a non-homogeneous weight, namely we investigate solutions to

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = b(x)f(u) \tag{1.4}$$

on M, for some $f \in C(\mathbb{R})$ and $0 < b \in C(M)$. Equation (1.4) models hypersurfaces that, on every $\Omega \Subset M$, are stationary for the functional

$$u \mapsto \int_\Omega \left[\sqrt{1+|\nabla u|^2} + b(x)F(u)\right]\mathrm{d}x,$$

where

$$F(t) = \int_0^t f(s)\mathrm{d}s, \tag{1.5}$$

with respect to variations compactly supported inside Ω. A notable example is that of the *capillarity equation*, for which $b \equiv 1$ and $f(t) = \kappa t$ for some constant $\kappa > 0$. In this case, the graph of u describes the surface of a fluid, and the equation arises from balancing forces of tension (proportional to the mean curvature of the capillarity surface) with the weight of the fluid supported. The physical constant κ is positive or negative depending on whether the surface is an upper or lower boundary of the fluid. Various aspects of capillarity surfaces were studied in detail in R. Finn's book [96].

In Euclidean space and if f is non-decreasing, the behaviour of solutions of (1.4) is well understood thanks to the following beautiful Liouville theorem, due to V. Tkachev [225] for $b \equiv 1$ and later improved by Y. Naito and H. Usami [168] and J. Serrin [219], see also [83, Thm. 10.4].

Theorem 1.2 ([168, 219, 225]) *Assume that*

$$b(x) \geq C\big(1+|x|\big)^{-\mu} \qquad \text{on } \mathbb{R}^m,$$

for some constants $C > 0$ and $\mu < 1$. If $u \in C^2(\mathbb{R}^m)$ solves (1.4) with f non-decreasing and $f \not\equiv 0$, then u is constant.

In particular, the result guarantees that the only solution of the capillarity equation on the entire $\mathbb{R}^m$ is $u \equiv 0$, and guarantees that entire graphs with constant mean curvature H must be minimal. This last property of $\mathbb{R}^m$ was first proved by E. Heinz [118] if $m = 2$, and by S.S. Chern [54] and H. Flanders [98] for general m, see also improvements by I. Salavessa [213]. Condition $f \not\equiv 0$ is necessary, as the example of affine functions shows. As we shall detail later, the restriction $\mu < 1$ is sharp.

It is instructive to observe that the proof of Theorem 1.2 splits into two parts:

(a) first, the author shows the identity $f(u) \equiv 0$ on $\mathbb{R}^m$, thus u solves (MSE);
(b) then, since $f \not\equiv 0$ and is non-decreasing, u must be bounded from one side, so the constancy of u follows from ($\mathscr{B}2$) above.

Let us comment on the above two steps with a simple, heuristic argument: hereafter, we define

$$u^* \doteq \sup_M u, \qquad u_* \doteq \inf_M u.$$

Suppose that M is compact, and that u solves (1.4). Evaluating the equation at a point x_0 where u achieves its maximum u^*, and recalling that the mean curvature operator is elliptic, we deduce $f(u^*) \le 0$. Similarly, evaluating at a minimum point gives $f(u_*) \ge 0$. Since f is non-decreasing, we therefore deduce $f(u) \equiv 0$ on M. Step (a) can therefore be obtained as a consequence of a maximum principle for u. Assume for convenience that $b \equiv 1$. If M is non-compact and we assume that u is bounded, even though u may fail to attain its maximum or minimum, the same conclusion $f(u) \equiv 0$, with the same argument, can be achieved provided that there exist sequences $\{x_k\}$, $\{y_k\}$ of points of M such that

$$\begin{aligned} u(x_k) &\to u^*, \qquad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)(x_k) < \tfrac{1}{k}, \\ u(y_k) &\to u_*, \qquad \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)(y_k) > -\tfrac{1}{k}. \end{aligned}$$

The existence of $\{x_k, y_k\}$ is not guaranteed on each manifold, but depends on the geometry of M at infinity: if such sequences can be found for each bounded $u \in C^2(M)$, then we say that the mean curvature operator satisfies the (weak) maximum principle at infinity. Maximum principles at infinity have first been introduced in the celebrated papers by H. Omori [174] and S.T. Yau ([239] and [50], the second with S.Y. Cheng), for the Laplace–Beltrami operator, and proved to be an essential tool in Geometric Analysis.

Their use, in suitable forms, is ubiquitous in the present monograph, and identifying the optimal thresholds for their validity is one of the central problems to reach sharp existence-nonexistence results for equations like (1.4) and many more. In the setting of Theorem 1.2, however, the function u is not known to be bounded a priori, so one needs further arguments.

The Seeming Lack of a Keller–Osserman Condition

Even though the actual proof of Step (a), for Theorem 1.2, is rather special, it is convenient for the moment to think that the strategy to show $f(u) \equiv 0$ proceeds via the following substeps:

- first, prove that a solution of (1.4) is bounded;
- then, show that a maximum principle at infinity holds for the mean curvature operator on the underlying manifold.

In fact, this will be a general strategy applied to classes of PDEs that we are going to study, that includes (1.4). It is illustrative to compare Theorem 1.2, say in the case $b \equiv 1$, to the situation occurring for solutions of

$$\Delta u = f(u) \qquad \text{on } \mathbb{R}^m, \tag{1.6}$$

with $f \in C(\mathbb{R})$. It is known that positive solutions of $\Delta u = 0$ are still constant, that is, the corresponding of Step (b) holds; indeed, by Yau's theorem [239], positive solutions of $\Delta u = 0$ are constant on every complete manifold M with non-negative Ricci curvature. However, it is false that every solution of (1.6) enjoys $f(u) \equiv 0$ on $\mathbb{R}^m$: for instance, by a result of D. Fisher-Colbrie and R. Schoen [97], and W.F. Moss and J. Piepenbrink [165], there are plenty of positive solutions of

$$\Delta u = \kappa u \qquad \text{on } \mathbb{R}^m$$

for constants $\kappa > 0$. As we show in a moment, the choice $f(t) = \kappa t$ is sharp for the existence of nontrivial solutions. In two seminal works, J.B. Keller and R. Osserman [135, 175] independently studied conditions on f to guarantee that solutions of (1.6) satisfy L^∞ bounds which, in many instances, make them trivial. Their investigation arose in connection to the type problem for Riemann surfaces [175] and to uniqueness problems for charged fluids in a container [134]. If $f(t) > 0$ for $t >> 1$, both of the authors identified the integrability requirement

$$\frac{1}{\sqrt{F}} \in L^1(\infty), \tag{1.7}$$

where F is defined in (1.5), that hereafter will be called a Keller–Osserman condition. Note that f satisfies (1.7) if, loosely speaking, it grows faster than linearly at infinity; for instance, $f(t) = t^\sigma$ satisfies (1.7) if and only if $\sigma > 1$. Under the validity of (1.7) and assuming that

$$f \text{ is non-decreasing on } \mathbb{R},$$

the authors prove that a solution of the inequality

$$\Delta u \geq f(u) \qquad \text{on } \mathbb{R}^m$$

must satisfy

$$u^* < \infty, \qquad f(u^*) \leq 0. \tag{1.8}$$

Furthermore, if (1.7) fails (the non-decreasing assumption on f is dropped here), the authors constructed a radial, positive unbounded solution of (1.6) on $\mathbb{R}^m$. It should be stressed that claim (1.8) is a reformulation of the results in [135, 175] in a form more suited to our purposes. For convex f, a characterization of solutions to (1.6) depending on the validity of (1.7) was previously obtained by E.K. Haviland [110]. Claim (1.8) easily implies the following Liouville theorem via a "reflection" argument that we shall repeatedly use hereafter. Assume that

$$tf(t) > 0 \qquad \text{on } \mathbb{R}\backslash\{0\},$$

and that Keller–Osserman conditions hold both at ∞ and $-\infty$, that is,

$$\frac{1}{\sqrt{F}} \in L^1(\infty) \cap L^1(-\infty), \tag{1.9}$$

(note that, in our assumptions, F is positive on $\mathbb{R}\backslash\{0\}$). Then, the only solution u of (1.6) is $u \equiv 0$. Indeed, Claim (1.8) implies that $f(u^*) \leq 0$, so $u^* \leq 0$. On the other hand, $\bar{u} \doteq -u$ solves

$$\Delta \bar{u} = \bar{f}(\bar{u}) \qquad \text{on } \mathbb{R}^m, \text{ where } \bar{f}(t) \doteq -f(-t),$$

and the integrability at $-\infty$ in (1.9) guarantees the validity of (1.7) with $\bar{f}$ replacing f. Thus, $\bar{f}(\bar{u}^*) \leq 0$, equivalently $f(u_*) \geq 0$. Concluding, $u^* \leq 0 \leq u_*$, that is, $u \equiv 0$.

Turning to the mean curvature operator, in Theorem 1.2 no growth condition of Keller–Osserman type is required on f, a fact that shows a further, striking difference with the Laplacian. In the next section and in Chap. 10, we aim to clarify the reasons for their absence, and to investigate the behaviour of solutions of (1.4) on more general manifolds.

As we shall see, the picture will be quite interesting and Keller–Osserman conditions will appear again in a somehow hidden way for Eq. (1.4), as soon as the volume of geodesic balls centred at a fixed origin of M grows faster than polynomially. The case of the capillarity equation reveals quite instructive.

Geodesic Graphs in $\mathbb{R} \times_h M$

Although sufficiently general to be analytically challenging, ambient manifolds of the type $\mathbb{R} \times M$ leave aside various cases of interest, notably that of the hyperbolic space $\mathbb{H}^{m+1}$. For this reason, it is useful to consider ambient manifolds that can be written as the warped product

$$\bar{M}^{m+1} = \mathbb{R} \times_h M^m,$$

that is topologically $\mathbb{R} \times M$ endowed with the Riemannian metric

$$(\ ,\ ) = \mathrm{d}s^2 + h(s)^2 \langle\ ,\ \rangle,$$

for some positive $h \in C^\infty(\mathbb{R})$. Recall that the hyperbolic space $\mathbb{H}^{m+1}$ of constant sectional curvature -1 admits the following three different representations as a warped product of the above type:

- if we remove a fixed origin o, and we endow $\mathbb{H}^{m+1} \backslash \{o\}$ with polar coordinates $(r, \theta) \in \mathbb{R}^+ \times \mathbb{S}^m$, we can write

$$\mathbb{H}^{m+1} = \mathbb{R}^+ \times_{\sinh s} \mathbb{S}^m,$$

  where $\mathbb{S}^m$ is the round sphere of curvature 1;
- consider the upper half-space model

$$\mathbb{H}^{m+1} = \Big\{ (x_0, x) \in \mathbb{R} \times \mathbb{R}^m \ : \ x_0 > 0 \Big\}$$

  with metric

$$(\ ,\ ) = \frac{1}{x_0^2}\Big(\mathrm{d}x_0^2 + \langle\ ,\ \rangle_{\mathbb{R}^m} \Big).$$

  With the change of variables $s = -\log x_0$ we express $(\ ,\ )$ as the warped product

$$\mathbb{H}^{m+1} = \mathbb{R} \times_{e^s} \mathbb{R}^m, \qquad (\ ,\ ) = \mathrm{d}s^2 + e^{2s} \langle\ ,\ \rangle_{\mathbb{R}^m},$$

  whose slices $\{s = \text{const}\}$ are horospheres;

– similarly, we can view $\mathbb{H}^{m+1}$ as a warped product

$$\mathbb{H}^{m+1} = \mathbb{R} \times_{\cosh s} \mathbb{H}^m, \qquad (\ ,\ ) = \mathrm{d}s^2 + \cosh^2 s \langle\ ,\ \rangle_{\mathbb{H}^m}$$

along totally umbilical hyperspheres, that in the upper half-space model correspond to Euclidean spheres having the same $(m-1)$-dimensional sphere as a trace on the boundary at infinity $\{x^0 = 0\}$.

Given a function $v : M \to \mathbb{R}$, one can consider the graph

$$\Sigma^m = \Big\{(s, x) \in \mathbb{R} \times M, \ \ s = v(x)\Big\}.$$

Note that $X = h(s)\partial_s$ is a conformal field with geodesic flow lines, as a direct computation gives

$$\bar{\nabla}_v X = h'(s)v \qquad \forall\, v \in T\bar{M},$$

with $\bar{\nabla}$ the connection on $\bar{M}$. For this reason, we call Σ a *geodesic graph*. We let Φ_t be the flow of X, and note that the flow parameter satisfies

$$t = \int_0^s \frac{\mathrm{d}\sigma}{h(\sigma)}, \qquad t\ :\ \mathbb{R} \to t(\mathbb{R}) = I. \tag{1.10}$$

We let $s(t)$ be its inverse. For convenience, we express the prescribed mean curvature equation in terms of the function $u(x) = t(v(x))$. Let ∇ be the connection on $(M, \langle\ ,\ \rangle)$. If H is the normalized mean curvature of Σ with respect to the upward pointing unit normal

$$\nu = \frac{1}{\lambda(u)\sqrt{1 + |\nabla u|^2}}\Big(\partial_t - (\Phi_u)_*\nabla u\Big), \tag{1.11}$$

then a computation in [62] shows that $u : M \to I$ solves

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= m\lambda(u)H + m\frac{\lambda_t(u)}{\lambda(u)}\frac{1}{\sqrt{1+|\nabla u|^2}} \\ &= mh(v)H + \frac{mh'(v)}{\sqrt{1+|\nabla u|^2}} \qquad \text{on } M, \end{aligned} \tag{1.12}$$

where λ_t is the derivative of λ with respect to t, and h' is the derivative of h with respect to s. In particular, entire minimal graphs satisfy

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{mh'(v)}{\sqrt{1+|\nabla u|^2}} \qquad \text{on } M, \tag{1.13}$$

that belongs to the family described in (1.3). The behaviour of solutions to (1.12), of course, depends on the relation between the geometry of M and the growth of H, h'. In the hyperbolic space $\mathbb{H}^{m+1}$, the existence of graphs with constant mean curvature (CMC) along horospheres or hyperspheres has been considered in a paper by M.P. Do Carmo and H.B. Lawson [75], who proved the following remarkable result:

Theorem 1.3 *Let $\Sigma^m \to \mathbb{H}^{m+1}$ be a graph over a horosphere or a hypersphere M. Suppose that Σ has constant mean curvature $H \in [-1, 1]$. Then, the following occurs:*

(i) if M is a horosphere, then $H = \pm 1$ and Σ is a horosphere;
(ii) if M is a hypersphere, then $H \in (-1, 1)$ and Σ is a hypersphere.

In other words, in the above setting the graph function is constant. Note that, differently from the Euclidean case, no restriction appears on the dimension m. The result applies in fact to the much more general setting of properly embedded hypersurfaces, but its proof, relying on the moving plane method, is not suited to ambient manifolds with few isometries or to graphs with non-constant mean curvature, and calls for new ideas and techniques. It is natural to ask for which classes of M and h a result like Theorem 1.3 holds for CMC graphs in $\mathbb{R} \times_h M$.

Mean Curvature Flow Solitons

A further example leading to an equation of the type (1.3) is that of solitons for the mean curvature flow. We recall that a smooth map

$$\varphi : [0, T) \times \Sigma^m \to \bar{M}^{m+1}$$

with $\varphi_t = \varphi(t, \cdot)$ an immersion for each t, is said to be a mean curvature flow (shortly, MCF) if

$$\frac{\partial \varphi}{\partial t} = \overrightarrow{H}_t,$$

where $\overrightarrow{H}_t$ is the unnormalized mean curvature vector of φ_t. The MCF φ is called a soliton if there exists a conformal vector field Y on $\bar{M}$ such that

$$\varphi_t(M) = \Psi_{\tau(t)}(\varphi_0(M)) \qquad \text{for every } t \in [0, T),$$

where Ψ_τ is the flow of Y and $\tau(t)$ is a time reparametrization. In other words, we require

$$\varphi(t, x) = \Psi\big(\tau(t), \eta(t, x)\big) \qquad \forall\, (x, t) \in \Sigma \times [0, T),$$

with η the flow of some time-dependent tangential vector field on M. Differentiating at $t = 0$, a soliton satisfies the identity $\vec{H}_0 = \tau'(0)Y^\perp$, where $\perp$ is the orthogonal projection on the normal bundle. Up to rescaling Y, we can assume that $\tau'(0) = 1$, obtaining

$$\vec{H} = Y^\perp. \tag{1.14}$$

Solitons in $\mathbb{R}^{m+1}$ with respect to the homothetically shrinking and to the translating vector fields Y give rise, respectively, to classical self-shrinkers and self-translators, that model the singularities developed under the MCF (cf. [149]). Bernstein type theorems for shrinkers that are graphs over $\mathbb{R}^m$ have been proved in [80, 234], and for translators in [15]. Although solitons in more general ambient spaces can no longer describe the blow-up picture near a singularity, nevertheless they are still relevant since they act as barriers for the MCF evolution. Suppose that Σ is a geodesic graph, with graph function $u : M \to I$ along the lines of the conformal field $X = \partial_t$, and note that $\vec{H} = mH\nu$. If the soliton field Y coincides with $\pm X$, Eq. (1.12) specifies to

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) &= \left[\frac{m\lambda_t(u) \pm \lambda^3(u)}{\lambda(u)}\right]\frac{1}{\sqrt{1+|\nabla u|^2}} \\ &= \frac{mh'(v) \pm h^2(v)}{\sqrt{1+|\nabla u|^2}}. \end{aligned} \tag{1.15}$$

In particular, the equation for a self-translator in $\mathbb{R}^{m+1}$ that is a translating graph in the vertical direction is

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{\pm 1}{\sqrt{1+|\nabla u|^2}}.$$

Equidistant Graphs in $M \times_h \mathbb{R}$

Together with geodesic graphs in $\mathbb{R} \times_h M$, it is natural to study graphs in products $M \times_h \mathbb{R}$ endowed with the metric

$$(\,,\,) = \langle\,,\,\rangle + h(x)^2 \mathrm{d}s^2,$$

for some $0 < h \in C^\infty(M)$. In this case, $X = \partial_s$ is Killing but not parallel, unless h is constant. Curves $\{x = \bar{x}\}$ for $\bar{x}$ constant have the property that the intersection of any two of them with a slice $\{s = \text{const}\}$ is a pair of points whose distance does not depend on the slice, and for this reason graphs of $u : M \to \mathbb{R}$ will be named *equidistant graphs*. For instance, $\mathbb{H}^{m+1}$ admits two such warped product decompositions, according to whether X has one or two fixed points at infinity: the first can be obtained by isolating the coordinate

$s = x_m$ in the upper half-space model, leading to

$$\mathbb{H}^{m+1} = \mathbb{H}^m \times_h \mathbb{R} \qquad \text{with } h(x_0, \ldots, x_{m-1}) = \frac{1}{x_0}; \tag{1.16}$$

the second can be written as

$$\mathbb{H}^{m+1} = \mathbb{H}^m \times_{\cosh r} \mathbb{R}, \qquad (\, ,\,) = \langle\, ,\, \rangle_{\mathbb{H}^m} + \big(\cosh^2 r(x)\big) \mathrm{d}s^2, \tag{1.17}$$

with $r : \mathbb{H}^m \to \mathbb{R}$ the distance from a fixed origin in $\mathbb{H}^m$, and corresponds, in the upper half-space model, to the fibration of $\mathbb{H}^{m+1}$ via Euclidean lines orthogonal to the totally geodesic hypersphere $\{x_0^2 + |x|^2 = 1\}$. Having defined the normal direction

$$\nu = \frac{1}{h\sqrt{1 + h^2|\nabla u|^2}} \Big(\partial_t - h^2 (\Phi_u)_* \nabla u\Big),$$

a computation in [61] shows that u solves

$$\operatorname{div}\left(\frac{h\nabla u}{\sqrt{1 + h^2|\nabla u|^2}}\right) = mH - \left\langle \frac{h\nabla u}{\sqrt{1 + h^2|\nabla u|^2}}, \frac{\nabla h}{h} \right\rangle \qquad \text{on } M. \tag{1.18}$$

If we consider the conformal deformation

$$\widehat{\langle\, ,\, \rangle} = h^{-2} \langle\, ,\, \rangle,$$

and we denote with $\|\cdot\|$, D and $\widehat{\operatorname{div}}$, respectively, the norm, connection and divergence in the metric $\widehat{\langle\, ,\, \rangle}$, then (1.18) is equivalent to

$$\widehat{\operatorname{div}}_h \left(\frac{Du}{\sqrt{1 + \|Du\|^2}}\right) = mHh^2 \qquad \text{on } \big(M, \widehat{\langle\, ,\, \rangle}\big), \tag{1.19}$$

where $\widehat{\operatorname{div}}_h$ is the following weighted divergence:

$$\widehat{\operatorname{div}}_h Y = h^{m-1} \widehat{\operatorname{div}}\big(h^{1-m} Y\big).$$

In particular, if φ is a soliton for $Y = \pm X$, then (1.19) becomes

$$\widehat{\operatorname{div}}_h \left(\frac{Du}{\sqrt{1 + \|Du\|^2}}\right) = \frac{\pm h(x)^3}{\sqrt{1 + \|Du\|^2}} \qquad \text{on } \big(M, \widehat{\langle\, ,\, \rangle}\big). \tag{1.20}$$

Up to replacing the Riemannian divergence with a weighted one, (1.19) and (1.20) belong to the family of equations that are considered in the present book. Indeed, the proof of some of our major results proceed verbatim in the case of a weighted divergence in the left-hand side of (1.3). However, to apply our main nonexistence results we shall need that the metric $\widehat{\langle\, , \,\rangle}$ be complete, which impose restrictions on h and excludes graphs in $\mathbb{H}^{m+1}$ along the two decompositions mentioned above. In fact, Rigidity does not hold for such graphs in $\mathbb{H}^{m+1}$, at east in the CMC case: the Plateau's problem at infinity for graphs with constant $H \in (0, 1)$ is always solvable, by work of P. Guan and J. Spruck [108] for (1.17), and J. Ripoll and M. Telichevesky [211] for (1.16). Nevertheless, our theorems apply when $h \in L^\infty(M)$, notably for h vanishing at infinity, that is analytically the most subtle case of (1.20). Existence for the prescribed mean curvature equation on more general products $M \times_h \mathbb{R}$ is studied in [42].

1.2 A Problem from the Theory of Stochastic Control

Equation (1.12) for geodesic graphs with prescribed mean curvature suggests to consider more general problems of the type

$$\Delta_\varphi u \geq b(x) f(u) + \bar{b}(x)\bar{f}(u)\bar{l}(|\nabla u|), \tag{1.21}$$

for suitable $b, \bar{b}$ positive on M, $f, \bar{f} \in C(\mathbb{R})$ and $\bar{l} \in C(\mathbb{R}_0^+)$, where we set $\mathbb{R}_0^+ \doteq [0, \infty)$. Here, Δ_φ denotes a general quasilinear operator that depends on an increasing function φ on $\mathbb{R}_0^+$, whose properties will be listed in the next chapter. For suitable φ, the family includes the mean curvature operator a well as the Laplace–Beltrami and the p-Laplacian

$$\Delta_p u \doteq \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \qquad p > 1.$$

As a matter of fact, a prototype case appears in Stochastic Control Theory, and a detailed description of the model can be found in J.-M. Lasry and P.-L. Lions [140]. It can be roughly summarized as follows: given an open subset $\Omega \subset \mathbb{R}^m$, and given a continuous function $a : \Omega \to \mathbb{R}^m$, the state of a controlled system is assumed to be a diffusion process X_t valued in $\overline{\Omega}$ that satisfies the stochastic PDE

$$\mathrm{d}X_t = a(X_t)\mathrm{d}t + \mathrm{d}\mathscr{B}_t,$$

where $\mathscr{B}_t$ is a Brownian motion and $a(X_t)$ models a feedback control. The function a is assumed to lie a suitable class $\mathscr{A} \subset C(\Omega, \mathbb{R}^m)$ to guarantee that X_t is valued in Ω almost surely, so typically $|a(x)| \to \infty$ on $\partial\Omega$. In this case, no boundary conditions have to be imposed on $\partial\Omega$. For $b \in C(\overline{\Omega}) \cap L^\infty(\Omega)$, one can consider a cost function $J : \Omega \times \mathscr{A} \to \mathbb{R}$ given by

$$J(x, a) = \mathbb{E}\left[\int_0^\infty \left(\frac{1}{q}|a(X_t)|^q - b(X_t)\right) e^{-\lambda t}\mathrm{d}t\right]$$

Here, $\lambda > 0$ represents a discount factor, and $q > 1$. In view of Bellman's dynamic programming principle, the Bellman function

$$u(x) = \inf_{a \in \mathscr{A}} J(x, a) \qquad \forall\, x \in \Omega$$

solves

$$\Delta u = \frac{1}{p}|\nabla u|^p + \lambda u + b(x) \qquad \text{on } \Omega, \text{ where } p = \frac{q}{q-1}. \tag{1.22}$$

(at least heuristically: as pointed out in [140], the restrictions on a, X_t require nontrivial arguments to justify (1.22)). The case $q \geq 2$, that is, a running cost blowing up fast at infinity, is particularly interesting, and corresponds to $1 < p \leq 2$. It is also meaningful to consider the problem set in the entire space $\Omega = \mathbb{R}^m$, and in this case to consider *large* solutions, i.e. solutions satisfying $u(x) \to \infty$ as $|x| \to \infty$. Needless to say, the case of manifold-valued processes X is also natural, so is the study of (1.22) on Riemannian manifolds. Note that (1.22) implies an inequality of the type (1.21) under various reasonable assumptions on b, λ. In particular, the ergodic limit $\lambda \to 0$ leads to the study of

$$\Delta u = b(x) + \frac{1}{p}|\nabla u|^p \qquad \text{on } M.$$

As another example, assume $b \geq 0$ on $\mathbb{R}^m$ and $\lambda \geq \|b\|_\infty$, in which case

$$\Delta u \geq \frac{1}{p}|\nabla u|^p + b(x)(u+1)$$

is matched on the set $\{u > 0\}$ (open, if we assume $u \in C(\mathbb{R}^m)$). In particular, choosing $\varepsilon > 0$ and $f \in C(\mathbb{R})$ with $f(t) = t+1$ for $t > \varepsilon$, $f \equiv 0$ on $(-\infty, 0)$ and $0 \leq f(t) \leq t+1$ on $\mathbb{R}$, $v \doteq u_+$ solves

$$\Delta v \geq b(x) f(v) + \frac{1}{p}|\nabla v|^p \qquad \text{on } M.$$

The existence and nonexistence problem on the entire $\mathbb{R}^m$ for solutions to

$$\Delta u = b(x) f(u) + c|\nabla u|^p, \qquad \text{with } c > 0,\ p > 0,$$

where $0 < b \in C(\mathbb{R}^m)$, $f \in C(\mathbb{R})$, was studied by A.V. Lair and A.W. Wood [139], M. Ghergu and V. Radulescu [103], F. Toumi [227] and R. Filippucci, P. Pucci and M. Rigoli [94]. On complete manifolds, it was addressed in [157]. Although the main results of the present book are restricted to inequalities of the type

$$\Delta_\varphi u \geq b(x) f(u) l(|\nabla u|),$$

we mention that most of them can be applied in the more general setting of (1.21). However, a direct study of (1.21) would have lead to much more involved results, making hardly readable our attempt to describe in detail the influence of geometry on the behaviour of solutions.

2 An Overview of Our Results

The study of differential inequalities of the type

$$\operatorname{div}\mathcal{A}(x, u, \nabla u) \geq \mathcal{B}(x, u, \nabla u) \tag{2.1}$$

on Euclidean space $\mathbb{R}^m$ is a classical subject, and a great deal of work has been devoted to the analysis of the qualitative properties of solutions. The literature is vast, and, as said in Chap. 1, we restrict ourselves to the special case

$$\mathcal{B}(x, u, \nabla u) = b(x) f(u) l(|\nabla u|), \tag{2.2}$$

for continuous b, f, l. With no claim of completeness, we quote [14, 66, 86, 87, 92, 95, 105, 144, 162, 195], and for similar inequalities in the sub-Riemannian setting of Carnot groups, [5, 33–35, 63, 147]. The results in the references above will be related to those in our work in a more precise way later on.

To the best of our knowledge, only a few authors have analysed the influence of geometry on the behaviour of solutions of (2.1), (2.2) in a general setting, for instance see [6, 147, 157, 198], leaving however the picture still fragmentary, especially in case where l in (2.2) is a non-constant function. As one of the main purposes of the present work, we aim to give a detailed account of how geometry comes into play at the global level. Nevertheless, some interesting questions and problems remain open, and will be specified in due course in the book.

B. Bianchini et al., *Geometric Analysis of Quasilinear Inequalities on Complete Manifolds*, Frontiers in Mathematics, https://doi.org/10.1007/978-3-030-62704-1_2

2.1 Setting and Main Properties Under Investigation

From now on, we let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension $m \geq 2$. We shall assume throughout this book that M is non-compact. To avoid excessive technicalities, while still keeping a good amount of generality, we study the following subclass of (2.1): we consider a quasilinear operator Δ_φ, called the φ-Laplacian, weakly defined by

$$\Delta_\varphi u = \mathrm{div}\left(\frac{\varphi(|\nabla u|)}{|\nabla u|}\nabla u\right),$$

where we assume

$$\varphi \in C(\mathbb{R}^+_0), \quad \varphi(0) = 0, \quad \varphi(t) > 0 \ \text{ on } \mathbb{R}^+; \tag{2.3}$$

hereafter, $\mathbb{R}^+_0 = [0, \infty)$ and $\mathbb{R}^+ = (0, \infty)$. Different choices of φ give rise to well-known operators including, for instance, those presented in the examples below.

Example 2.1 The p-Laplace operator Δ_p for $p > 1$, where $\varphi(t) = t^{p-1}$:

$$\Delta_p u = \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

Besides its importance in Physics and Image Processing, the p-Laplacian is also interesting since it allows to efficiently bridge between two relevant operators obtained formally as (suitably normalized) limits as $p \to 1$ and $p \to \infty$:

$$\Delta_1 u \doteq \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right), \qquad \Delta_\infty u \doteq \nabla^2 u\left(\frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right).$$

The 1-Laplacian $\Delta_1 u$ describes the mean curvature of the level set $\{u = \text{const}\}$ in the normal direction $-\nabla u/|\nabla u|$, and is therefore important in the study of level set methods for the weak definition of geometric flows driven by the mean curvature. In fact, approximation with the p-Laplacian works well, for instance, to construct solutions of the inverse mean curvature flow, see [137, 156, 164]. On the other hand, the (normalized) ∞-Laplacian $\Delta_\infty u$ is tightly related to the metric geometry of the underlying manifold, and indeed the geodesic completeness of a Riemannian metric can be detected in terms of a potential theory for the ∞-Laplacian (see [153, 154]).

Example 2.2 The mean curvature operator, for which $\varphi(t) = t(1 + t^2)^{-1/2}$.

Example 2.3 The operator of exponentially harmonic functions, where $\varphi(t) = t\exp\big(t^2\big)$, was introduced in [79, 81]. The operator has connections with the nonlinear Hodge–De Rham theory developed in [221], and has interesting applications in gas dynamics: indeed, following [221], if u represents the velocity potential of a compressible fluid, and if the density of the fluid is assumed to be a positive function $\varphi(|\nabla u|)/|\nabla u|$ of the speed $|\nabla u|$, in view of the motion and continuity equations, and because of Bernoulli's law, u satisfies

$$\Delta_\varphi u = 0.$$

The ellipticity condition $\varphi' > 0$ characterizes the flow to be subsonic. A prototype case is that of the polytropic flow, corresponding to the choice

$$\varphi(t) = t\left(1 - \frac{\gamma - 1}{2}t^2\right)^{\frac{1}{\gamma-1}}$$

for a given adiabatic constant $\gamma > 1$, that is subsonic whenever

$$t < \sqrt{\frac{2}{\gamma + 1}}.$$

In this case, our techniques can be applied provided that we know, a priori, that $|\nabla u|^2$ is less than the sonic value $2/(\gamma + 1)$. More details on the physical interpretation can be found in [19].

Example 2.4 The (p, q)-Laplacian

$$\Delta_{p,q}u = \Delta_p u + \Delta_q u,$$

associated to $\varphi(t) = t^{p-1} + t^{q-1}$ with $1 < p < q$. This operator was independently introduced by V.V. Zhikov [240] and P. Marcellini [152], and can be regarded as a model example related to functionals with non-standard growth conditions. In particular, Zhikov introduced the operator in his study of anisotropic materials, homogenization and elasticity, cf. also [241]. The (p, q)-Laplacian also appears in models for quantum and plasma physics, and in chemical reaction design: for instance, in [71], G.H. Derrick proposed to use operators like $\Delta_{p,q}$ in an attempt to overcome a problem arising in the description of elementary particles by means of static solutions with finite energy to the generalized Klein–Gordon equation. The problem has been tackled in [16], see also [53].

We focus our attention on the differential inequalities

$$\begin{aligned} &(P_\geq) \qquad \Delta_\varphi u \geq b(x)f(u)l(|\nabla u|) \\ &(P_\leq) \qquad \Delta_\varphi u \leq b(x)f(u)l(|\nabla u|) \\ &(P_=) \qquad \Delta_\varphi u = b(x)f(u)l(|\nabla u|) \end{aligned} \tag{2.4}$$