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the Unione Matematica Italiana

7

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Luc Tartar

The General Theory of Homogenization

A Personalized Introduction

Springer

UMI

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Dedicated to Sergio SPAGNOLO

Helped with the insight of Ennio DE GIORGI, he was the first in the late 1960s to give a mathematical definition concerning homogenization in the context of the convergence of Green kernels: G-convergence.

to François MURAT

Starting from his discovery of a case of nonexistence of solutions for an optimization problem, in the spirit of the earlier work of Laurence YOUNG, which was not known in Paris, we started collaborating in the early 1970s and rediscovered homogenization in the context of optimal design problems, leading to a slightly more general framework: H-convergence and compensated compactness.

to Évariste SANCHEZ-PALENCIA

It was his work on asymptotic methods for periodically modulated media in the early 1970s that helped me understand that my joint work with François MURAT was related to questions in continuum mechanics, and this gave me at last a mathematical way to understand what I was taught in continuum mechanics and physics at École Polytechnique, concerning the relations between microscopic, mesoscopic, and macroscopic levels, without using any probabilistic ideas!

to Lucia

to my children
Laure, Michaël, André, Marta

to my grandchildren
Lilian, Lisa

and to my wife
Laurence

Preface

In 1993, from 27 June to 1 July, I gave ten lectures for a CBMS–NSF conference, organized by Maria SCHONBEK at UCSC,[1] Santa Cruz, CA. As I was asked to write lecture notes, I wrote the parts concerning homogenization and *compensated compactness* in the following years, but I barely started writing the part concerning *H-measures*.

In the fall of 1997, facing an increase in aggressiveness against me, I decided to put that project on hold, and I devised a new strategy to write lecture notes for the graduate courses that I was going to teach at CMU (*Carnegie Mellon University*),[2,3] Pittsburgh, PA. After doing so for the courses that I taught in the spring of 1999 and in the spring of 2000, I made the texts available on the web page of CNA (Center for Nonlinear Analysis at CMU). For the graduate course that I taught in the fall of 2001, I still needed to write the last four lectures, but I also prepared the last version of my CBMS–NSF course, from the summer of 1996, to make it also available on the web page of CNA, so that those who received a copy of various chapters would not be the only ones to know the content of those chapters that I wrote.

This led to a sharp increase of aggressiveness against me, so after putting my project on hold, I learned to live again in a hostile environment.

[1] Maria Elena SCHONBEK, Argentinean-born mathematician. She worked at Northwestern University, Evanston, IL, at VPISU (Virginia Polytechnic Institute and State University), Blacksburg, VA, at University of Rhode Island, Kingston, RI, at Duke University, Durham, NC, and she now works at UCSC (University of California at Santa Cruz), Santa Cruz, CA.

[2] Andrew CARNEGIE, Scottish-born businessman and philanthropist, 1835–1919. Besides endowing a technical school in Pittsburgh, PA, which became Carnegie Tech (Carnegie Institute of Technology) and then CMU (*Carnegie Mellon University*) after it merged in 1967 with the Mellon Institute of Industrial Research, he funded about three thousand public libraries, and those in United States are named *Carnegie libraries*.

[3] Andrew William MELLON, American financier and philanthropist, 1855–1937. He founded the Mellon Institute of Industrial Research in Pittsburgh, PA, which merged in 1967 with Carnegie Tech (Carnegie Institute of Technology) to form CMU (*Carnegie Mellon University*).

In the summer of 2002, I started revising my first two lecture notes by adding information about the persons whom I mention in the text, and for doing this I used footnotes, despite a warning by KNUTH [45][4] that footnotes tend to be distracting, but as he added "Yet Gibbon's *Decline and Fall* would not have been the same without footnotes,"[5] I decided not to restrain myself. I cannot say if my excessive use of footnotes resembles that of GIBBON, as I have not yet read *The History of the Decline and Fall of the Roman Empire* [34], but I wonder if the recent organized attacks on the western academic systems are following some of the reasons that GIBBON proposed for explaining the decline and the collapse of the mighty Roman empire.

Where should I publish my lecture notes once written? I found the answer in October 2002 at a conference at Accademia dei Lincei in Roma (Rome), Italy, when my good friends Carlo SBORDONE and Franco BREZZI mentioned their plan[6,7] to have a series of lecture notes at UMI (*Unione Matematica Italiana*), published by Springer.[8]

I submitted my first lecture notes for publication in the summer of 2004, but I took a long time before making the requested corrections, and they appeared only in August 2006 as volume 1 of the UMI Lecture Notes series [116], *An Introduction to Navier–Stokes Equation and Oceanography.*[9,10]

I submitted my second lecture notes for publication in August 2006, and they appeared in June 2007 as volume 3 of the UMI Lecture Notes series [117], *An Introduction to Sobolev Spaces and Interpolation Spaces.*[11]

I submitted my third lecture notes for publication in January 2007 and they appeared in March 2008 as volume 6 of the UMI Lecture Notes series [119], *From Hyperbolic Systems to Kinetic Theory, A Personalized Quest.*

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[4] Donald Ervin KNUTH, American mathematician, born in 1938. He worked at Caltech (California Institute of Technology), Pasadena, CA, and at Stanford University, Stanford, CA.

[5] Edward GIBBON, English historian, 1817–1877.

[6] Carlo SBORDONE, Italian mathematician, born in 1948. He works at Università degli Studi di Napoli Federico II, Napoli (Naples), Italy. He was president of UMI (Unione Matematica Italiana) from 2000 to 2006.

[7] Franco BREZZI, Italian mathematician, born in 1945. He works at Università degli Studi di Pavia, Pavia, Italy. He became president of UMI (Unione Matematica Italiana) in 2006.

[8] Julius SPRINGER, German publisher, 1817–1877.

[9] Claude Louis Marie Henri NAVIER, French mathematician, 1785–1836. He worked in Paris, France.

[10] Sir George Gabriel STOKES, Irish-born mathematician, 1819–1903. He worked in London and in Cambridge, England, holding the Lucasian chair (1849–1903).

[11] Sergei L'vovich SOBOLEV, Russian mathematician, 1908–1989. He worked in Leningrad, in Moscow, and in Novosibirsk, Russia. There is now a Sobolev Institute of Mathematics of the Siberian branch of the Russian Academy of Sciences, Novosibirsk, Russia. I first met Sergei SOBOLEV when I was a student, in Paris in 1969, and conversed with him in French, which he spoke perfectly (all educated Europeans at the beginning of the twentieth century learned French).

In the summer of 2007, it was time for me to think again about my CBMS–NSF course. Because I already wrote lecture notes on how homogenization appears in optimal shape design [111] for lectures given during a CIME–CIM summer school, organized by Arrigo CELLINA and António ORNELAS,[12,13] in Tróia, Portugal, in June 1998, I wrote this book in a different way, describing how my ideas in homogenization were introduced during my quest for understanding more about continuum mechanics and physics, so that chapters follow a loose chronological order.

As in my preceding lecture notes, I use footnotes for giving some biographical information about people related to what I mention, and in the text I use the first name of those whom I met. In my third lecture notes, I started putting at the end of each chapter the additional footnotes that are not directly related to the text but expand on some information found in previous footnotes; in this book, instead of presenting them in the order where the names appeared, I organized the additional footnotes in alphabetical order.

When one misses the footnote containing the information about someone, a chapter of biographical information at the end of the book permits one to find where the desired footnote is.

I may be wrong about some information that I give in footnotes, and I hope to be told about my mistakes, and that is true about everything that I wrote in the book, of course!

I want to thank my good friends Carlo SBORDONE and Franco BREZZI for their support, in general, and for the particular question of the publication of my lecture notes in a series of *Unione Matematica Italiana*.

I want to thank *Carnegie Mellon University* for according me a sabbatical period in the fall of 2007, and *Politecnico di Milano* for its hospitality during that time, at it was of great help for concentrating on my writing programme.

I want to thank *Université Pierre et Marie Curie* for a 1 month invitation at *Laboratoire Jacques-Louis Lions*,[14,15] in May/June 2008, as it was during

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[12] Arrigo CELLINA, Italian mathematician, born in 1941. He works at Università di Milano Bicocca, Milano (Milan), Italy.

[13] António COSTA DE ORNELAS GONÇALVES, Portuguese mathematician, born in 1951. He works in Évora, Portugal.

[14] Pierre CURIE, French physicist, 1859–1906, and his wife Marie SKŁODOWSKA-CURIE, Polish-born physicist, 1867–1934, received the Nobel Prize in Physics in 1903 in recognition of the extraordinary services they have rendered by their joint research on the radiation phenomena discovered by Professor Henri BECQUEREL, jointly with Henri BECQUEREL. Marie SKŁODOWSKA-CURIE also received the Nobel Prize in Chemistry in 1911 in recognition of her services to the advancement of chemistry by the discovery of the elements radium and polonium, by the isolation of radium, and the study of the nature and compounds of this remarkable element. They worked in Paris, France. Université Paris VI, Paris, is named after them, UPMC (Université Pierre et Marie Curie).

[15] Jacques-Louis LIONS, French mathematician, 1928–2001. He received the Japan Prize in 1991. He worked in Nancy and in Paris, France; he held a chair (analyze

this period that I wrote the last chapters of the book. I want to thank François MURAT[16] for his hospitality during my visits to Paris for almost 20 years and for his unfailing friendship for almost 40 years.

I could not publish my first three lecture notes and start the preparation of this fourth book without the support of Lucia OSTONI. I want to thank her for much more than providing the warmest possible atmosphere during my stays in Milano, because she gave me the stability that I lacked so much during a large portion of the last 30 years, so that I now feel safer for resuming my research, whose main goal is to give a sounder mathematical foundation to twentieth century continuum mechanics and physics.

Milano, June 2008

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PS: (Pittsburgh, August 2008) Although I finished writing the book at the end of June, while I was in Milano attending the last meeting of Instituto Lombardo before the summer, I still had to check the chapter on notation and to create an index, and while doing that, I realized that I should explain my choices in a better way, in particular the subject of Chap. 1.

My general goal is to understand in a better way the continuum mechanics and the physics of the twentieth century, that is, the questions where small scales appear, plasticity and turbulence on the one hand, atomic physics and phase transitions on the other, and I think that the General Theory of Homogenization (GTH) as I developed it is crucial for starting in the right direction, but as there are a few dogmas to change, if not to discard completely, in continuum mechanics and in physics, I need to explain why the difficulties are similar to those that appeared in religions, where the deadlocks still remain.

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mathématique des systèmes et de leur contrôle, 1973–1998) at Collège de France, Paris. The laboratory dedicated to functional analysis and numerical analysis which he initiated, funded by CNRS (Center National de la Recherche Scientifique) and UPMC (Université Pierre et Marie Curie), is now named after him, LJLL (Laboratoire Jacques-Louis Lions). He was my teacher at École Polytechnique in Paris in 1966–1967; I did research under his direction until my thesis in 1971.

[16] François MURAT, French mathematician, born in 1947. He works at CNRS (Centre National de la Recherche Scientifique) and UPMC (Université Pierre et Marie Curie), in LJLL (Laboratoire Jacques-Louis Lions), Paris, France.

Describing my family background and my studies is a way to answer the question that should be asked in the future: among those who realized at the end of the twentieth century that some of the dogmas in continuum mechanics and physics had to be discarded as wrong and counter-productive, what explains how they could start thinking differently? Should I say that I do not know who else but myself fits in this category? I expect that by telling this story, more will be able to follow a path similar to mine in the future, that is, there will be more mathematicians interested in the other sciences than mathematics!

Because I use the words parables and gospels in the first sentence of Chap. 1, some may stop reading the book, but in the second sentence I explain why parables are like general theorems, and by the end of the second footnote at the bottom of the first page, one will already learn that I am no longer a Christian, so that any misunderstanding about my intentions should result from the prejudices of the reader against religions, which is not a scientific attitude, and at the end of the book it should be clear that many "scientists" behaved in the recent past like religious fundamentalists.

What I advocate is for all to use their brain in a critical way!

Additional footnotes: BECQUEREL,[17] DUKE,[18] Federico II,[19] LUCAS H.,[20] NOBEL,[21] STANFORD.[22]

Detailed Description of Contents

a.b: refers to Corollary, Definition, Lemma, or Theorem # b in Chap. # a, while (a.b) refers to Eq. # b in Chap. # a.

Chapter 1: Why Do I Write?
About my sense of duty.

Chapter 2: A Personalized Overview of Homogenization I
About my understanding of homogenization in the 1970s.

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[17] Antoine Henri BECQUEREL, French physicist, 1852–1908. He received the Nobel Prize in Physics in 1903, in recognition of the extraordinary services he has rendered by his discovery of spontaneous radioactivity, jointly with Pierre CURIE and Marie SKŁODOWSKA-CURIE. He worked in Paris, France.

[18] Washington DUKE, American industrialist, 1820–1905. Duke University, Durham, NC, is named after him.

[19] Friedrich VON HOHENSTAUFEN, German king, 1194–1250. Holy Roman Emperor, as Friedrich II, 1220–1250. He founded the first European state university in 1224, in Napoli (Naples), Italy, where he is known as Federico secondo, and Università degli Studi di Napoli is named after him.

[20] Henry LUCAS, English clergyman and philanthropist, 1610–1663. The Lucasian chair in Cambridge, England, is named after him.

[21] Alfred Bernhard NOBEL, Swedish industrialist and philanthropist, 1833–1896. He created a fund to be used as awards for people whose work most benefited humanity.

[22] Leland STANFORD, American businessman, 1824–1893. Stanford University is named after him (as is the city of Stanford, CA, where it is located).

Chapter 3: A Personalized Overview of Homogenization II
About my understanding of homogenization after 1980.

Chapter 4: An Academic Question of Jacques-Louis Lions
Studying in Paris in the late 1960s, the question of J.-L. Lions (4.1)–(4.3), the counter-example of Murat (4.4)–(4.6); 4.1: the basic one-dimensional homogenization lemma (4.7)–(4.9), a natural relaxation problem (4.10)–(4.14); 4.2: characterization of sequential weak $\star$ limits (4.15)–(4.18), around the ideas of L.C. Young.

Chapter 5: A Useful Generalization by François Murat
Research and development, technical ability, a two-dimensional problem (5.1); 5.1: layering in x_1 for the special case in $\mathbb{R}^2$ (5.2)–(5.4); 5.2: layering in x_1 for the symmetric elliptic case in $\mathbb{R}^N$ (5.5)–(5.12); 5.3: layering in x_1 for the not necessarily symmetric or elliptic case in $\mathbb{R}^N$ (5.13)–(5.16).

Chapter 6: Homogenization of an Elliptic Equation
Distinguishing the G-convergence of Spagnolo, the H-convergence of Murat and myself, and the Γ-convergence of De Giorgi; 6.1: G-convergence (6.5) and (6.6), the work of Spagnolo (6.1)–(6.4) and (6.7)–(6.10), V-ellipticity and norm (6.11); 6.2: abstract weak convergence of $(T^m)^{-1}$ (6.12); 6.3: $\mathcal{M}(\alpha,\beta;\Omega)$ (6.13) and (6.14); 6.4: H-convergence; 6.5: $\mathcal{M}(\alpha,\beta;\Omega)$ is sequentially closed for H-convergence (6.15)–(6.22), computing a convex hull for obtaining bounds (6.32)–(6.29); 6.6: a lower semi-continuity result in the symmetric case (6.30) and (6.31); 6.7: lower and upper bounds in the symmetric case (6.32).

Chapter 7: The Div–Curl Lemma
7.1: A case where coefficients are products (7.1)–(7.3); 7.2: the div–curl lemma (7.4)–(7.8); 7.3: a counter-example for $\int_\omega (E^n, D^n)\,dx$ (7.9)–(7.14), a generalization of Robbin using the Hodge theorem; 7.4: the necessity of (E,D) (7.15)–(7.18), a generalization of Murat to the L^p setting (7.19)–(7.21), a generalization of Hanouzet and Joly, a fake generalization.

Chapter 8: Physical Implications of Homogenization
About conjectures and theorems, my approach to different scales based on weak convergences, what internal energy is, and the defects of the second principle of thermodynamics, homogenization of first-order equations is important for turbulence and quantum mechanics, the nonexistent "particles" of quantum mechanics and the defects of the Boltzmann equation, the div–curl lemma in electrostatics, errors about effective coefficients in the literature, the div–curl lemma in electricity, and in equipartition of energy (8.1)–(8.4).

Chapter 9: A Framework with Differential Forms
9.1: The generalization of Robbin using the Hodge theorem, electrostatics with differential forms (9.1)–(9.5), the Maxwell–Heaviside equation with differential forms (9.6)–(9.20), a question about the Lorentz force and the motion of charged particles (9.21)–(9.23).

Chapter 10: Properties of H-Convergence

The danger of applying a general theory to too many examples, the ability with abstract concepts, my method of oscillating test functions (10.1)–(10.6); 10.1: the uniform bound for A^{eff} (10.7)–(10.12); 10.2: transposition in H-convergence (10.13)–(10.18); 10.3: independence from boundary conditions (10.19)–(10.21); 10.4: convergence up to the boundary for some variational inequalities (10.22)–(10.35); 10.5: local character of H-convergence (10.36)–(10.42); 10.6: a result of De Giorgi and Spagnolo (10.43)–(10.47); 10.7: preserving order by H-convergence (10.48)–(10.51); a counter-example of Marcellini (10.52)–(10.55); 10.8: perturbation of $\mathcal{M}(\alpha,\beta;\Omega)$ (10.56) and (10.57); 10.9: estimating $||A^{\text{eff}} - B^{\text{eff}}||$ for perturbations (10.58)–(10.68); 10.10: C^k and analytic dependence upon a parameter.

Chapter 11: Homogenization of Monotone Operators

An analogue of V-ellipticity for monotone operators (11.1)–(11.3); 11.1: the class $\mathcal{M}on(\alpha,\beta;\Omega)$ (11.4); 11.2: homogenization for $\mathcal{M}on(\alpha,\beta;\Omega)$ (11.5)–(11.17), a nonlinear analogue of symmetry (11.18); 11.3: homogenization of k-monotone and cyclically monotone operators in $\mathcal{M}on(\alpha,\beta;\Omega)$ (11.19)–(11.22); 11.4: an analogue of a result of De Giorgi and Spagnolo (11.23)–(11.34); 11.5: lower and upper bounds (11.35)–(11.42).

Chapter 12: Homogenization of Laminated Materials

12.1: The general one-dimensional case (12.1)–(12.9), an interpretation from electricity, using physical models in mathematics and drawings in geometry; 12.2: an application of the div–curl lemma (12.10); 12.3: sequences not oscillating in (x,e) (12.11) and (12.12), proofs of 12.2 and of 12.3 (12.13)–(12.19), a hyperbolic situation (12.20), my general method for laminated materials (12.21)–(12.33); 12.4: correctors for laminated materials (12.34)–(12.40).

Chapter 13: Correctors in Linear Homogenization

13.1: The general correctors (13.1)–(13.13); 13.2: a first type of lower-order terms (13.14)–(13.17); 13.3: a second type of lower-order terms (13.18)–(13.28); 13.4: some weak limits are better than expected (13.29)–(13.41); 13.5: correctors for 13.3 (13.42)–(13.49); 13.6: a third type of lower-order terms (13.50)–(13.52), the case of periodic data (13.53)–(13.55); 13.7: correctors in the periodic case (13.56)–(13.61).

Chapter 14: Correctors in Nonlinear Homogenization

Remarks on nonlinear elasticity; 14.1: the formula for laminated materials in $\mathcal{M}on(\alpha,\beta;\Omega)$ (14.1)–(14.11); 14.2: correctors for laminated materials in $\mathcal{M}on(\alpha,\beta;\Omega)$ (14.12)–(14.15); 14.3: correctors for $\mathcal{M}on(\alpha,\beta;\Omega)$ (14.16)–(14.27); 14.4: weak limits of $|grad(u_m)|^2$ (14.28)–(14.31), an application (14.32)–(14.35), the formula for laminated materials in nonlinear elasticity (14.36)–(14.40).

Chapter 15: Holes with Dirichlet Conditions

15.1: Homogenization for holes with Dirichlet conditions and data bounded in $L^2(\Omega)$ (15.1)–(15.10); 15.2: constants in the Poincaré inequality (15.11)–(15.13); 15.3: Homogenization for holes with Dirichlet conditions and data bounded in $H^{-1}(\Omega)$ (15.14)–(15.18); 15.4: a lemma involving the volume of the hole in a period cell (15.19)–(15.29); 15.5: the convergence of a rescaled sequence in the periodic case (15.30)–(15.40); 15.6: correctors in the periodic case (15.41)–(15.54), the convergence of the Stokes equation to the Darcy law according to Ené and Sanchez-Palencia.

Chapter 16: Holes with Neumann Conditions

Hypotheses on the holes (16.1)–(16.4); 16.1: using the extensions to prove convergence (16.5)–(16.9); 16.2: passing to the limit in a variational equation (16.10) and (16.11); 16.3: homogenization for holes with Neumann conditions (16.12)–(16.29), remarks on the periodic case (16.30)–(16.33).

Chapter 17: Compensated Compactness

The evolution of the ideas of Murat and myself; 17.1: a necessary condition for sequential weak lower semi-continuity (17.1)–(17.12); 17.2: a necessary condition for sequential weak continuity (17.13) and (17.14); 17.3: quadratic forms satisfying $Q(\lambda) \geq 0$ for all $\lambda \in \Lambda$ (17.15)–(17.38); 17.4: quadratic forms satisfying $Q(\lambda) = 0$ for all $\lambda \in \Lambda$ (17.39) and (17.40), examples (17.41)–(17.43), the general characteristic set $\mathcal{V}$ (17.44); 17.5: necessary conditions of higher-order (17.45)–(17.50); 17.6: a condition motivated by a result of Šverák (17.51) and (17.52).

Chapter 18: A Lemma for Studying Boundary Layers

The importance of asking questions, setting of the problem asked by J.-L. Lions (18.1)–(18.9); 18.1: my generalization of the Lax–Milgram lemma (18.10)–(18.14), my construction of M (18.15)–(18.18); 18.2: applying my abstract lemma (18.19)–(18.30); 18.3: my more general approach based on the Lax–Milgram lemma (18.31)–(18.45).

Chapter 19: A Model in Hydrodynamics

Explaining my model (19.1) and (19.2); 19.1: homogenization of my model (19.3)–(19.26), the case $div(w^n) = 0$ (19.27)–(19.29), a hint about H-measures, defects of kinetic theory.

Chapter 20: Problems in Dimension $N = 2$

Characterizing mixtures of two isotropic conductors (20.1)–(20.3), a preceding interaction between mathematics and physics, distinguishing conjectures and theorems, an observation of J. Keller (20.4)–(20.7); 20.1: $\frac{(A^n)^T}{\det(A^n)}$ H-converges to $\frac{(A^{\text{eff}})^T}{\det(A^{\text{eff}})}$ (20.8); 20.2: $\det(A^n) = \kappa$ implies $\det(A^{\text{eff}}) = \kappa$ (20.9); 20.3: $\tau_P(M) = (-c\,R_{\pi/2} + d\,M)(a\,I + b, R_{\pi/2}M)^{-1}$ defines a group homomorphism if $\det(P) = a\,d - b\,c > 0$ (20.10) and (20.11); 20.4: $\tau_P(A^n)$ H-converges to $\tau_P(A^{\text{eff}})$ (20.12)–(20.18), the Beltrami equation (20.19); 20.5:

writing the Beltrami equation as a system (20.20)–(20.24); 20.6: a characterization of symmetric M with $M_{1,1}, M_{2,2} > 0$, $\det(M) = 1$ and $\mathrm{Trace}(M) \geq 2$ (20.25) and (20.26); 20.7: the formula for laminated materials uses an inversion (20.27)–(20.32); 20.8: closed discs inside the closed unit disc are stable by H-convergence (20.33)–(20.36), the conjecture of Mortola and Steffé.

Chapter 21: Bounds on Effective Coefficients

Using symmetries; 21.1: change of variable (21.1)–(21.6), equations which are not frame indifferent; 21.2: $\mathcal{B}(\theta)$, $\mathcal{H}(\theta)$, $\mathcal{K}(\theta)$ (21.7)–(21.10), the intuition about defining $\mathcal{K}(\theta)$; 21.3: basic estimates (21.11)–(21.13); 21.4: generating bounds using correctors (21.14)–(21.23), a choice of functionals based on compensated compactness (21.24)–(21.27); 21.5: a result in linear algebra (21.28); 21.6: a general lower bound (21.29)–(21.32); 21.7: a general upper bound (21.33)–(21.36), more general functionals (21.37)–(21.39); 21.8: mixtures of two isotropic conductors (21.40)–(21.48), the conjectured bounds of Hashin and Shtrikman, a result of Francfort and Murat and myself.

Chapter 22: Functions Attached to Geometries

Some approaches are not homogenization; 22.2: same geometries for two materials M^1, M^2 and defining $F(\cdot, M^1, M^2)$ (22.1)–(22.5), special cases (22.6) and (22.7); 22.2: numerical range (22.8); 22.3: its convexity (22.9)–(22.11); 22.4: the cases of $F(\cdot, M^1, M^2)$ and $\big(F(\cdot, M^1, M^2)\big)^{-1}$ (22.12)–(22.14); 22.5: a more precise result (22.15) and (22.16), transposed and complex conjugate (22.17); 22.6: using order (22.18); 22.7: Pick functions and Herglotz functions; 22.8: Herglotz functions (22.19)–(22.24); 22.9: Pick functions (22.25)–(22.29), using the constraints $g(1) = 1$ and $g'(1) = 1 - \theta$ (22.30)–(22.33); 22.10: generalizing 20.1 (22.34); 22.11: $\frac{F^T}{det(F)}$ for $N = 2$ (22.35) and (22.36), the reiteration formula in the simple case (22.37); 22.12: the reiteration formula in the general case (22.38)–(22.41), remarks about percolation.

Chapter 23: Memory Effects

Observations of physical phenomena and conjectures about equations used as models, why the second principle is wrong, why an experiment of spectroscopy is related to effective equations with nonlocal effects, a too-general question (23.1), my simplified model (23.2)–(23.4); 23.1: the Laplace transform of the kernel (23.5)–(23.7); 23.2: solving (23.5) by convolutions (23.8)–(23.15), looking in the correct family of equations in the linear case, my first approach using only convolutions (23.16)–(23.21); 23.3: my solution using Pick functions (23.22)–(23.25); 23.4: characterizing the Radon measure defining the kernel (23.26)–(23.31), a possible origin of irreversibility, the case where the kernel is a finite combination of exponentials (23.32)–(23.35), a different way to write the effective equation (23.36) and (23.37).

Chapter 24: Other Nonlocal Effects

Time-dependent coefficients and a nonlinear equation (24.1)–(24.4), an approach by perturbation (24.5)–(24.9); 24.1: the expansion of the kernel

(24.10)–(24.31), a degenerate elliptic problem (24.32)–(24.34); 24.2: its effective equation (24.35)–(24.44), a different way to write the effective equation (24.45) and (24.46), a study by Amirat, Hamdache, and Ziani (24.47) and (24.48); 24.3: their effective equation (24.49)–(24.56), their different way to write the effective equation (24.57) and (24.58), models for a porous medium, a nonlinear model (24.59)–(24.62), its perturbation expansion (24.63)–(24.77), analogy with Feynman diagrams, a truncated expansion (24.78)–(24.80).

Chapter 25: The Hashin–Shtrikman Construction

25.1: Equivalent media (25.1) and (25.2); 25.2: the Hashin–Shtrikman coated spheres (25.3)–(25.8); 25.3: using a Vitali covering (25.9)–(25.17); 25.4: coated spheres give optimal bounds (25.18)–(25.21); 25.5: $g''(1)$ for binary mixtures (25.22)–(25.24), a remark of Bergman on cubic symmetry; 25.6: optimal values of Pick functions for Taylor expansion at order 2 (25.25)–(25.31), giving the Hashin–Shtrikman bounds in "dimension" d (25.32) and (25.33); 25.7: bounds for a Pick function g when $z\,g\big(\frac{1}{z}\big)$ is a Pick function (25.34)–(25.39), a remark of Milton for ternary mixtures; 25.8: a Riccati equation for general coated spheres (25.40)–(25.44), properties of Riccati equations (25.45)–(25.48); 25.9: generalizing the remark of Milton to arbitrary proportions (25.49)–(25.58); 25.10: the optimal radial construction (25.59)–(25.63).

Chapter 26: Confocal Ellipsoids and Spheres

Confocal ellipsoids (26.1); 26.1: derivatives of an implicit function (26.2)–(26.8); 26.2: particular solutions for isotropic materials in the confocal ellipsoids geometry (26.9)–(26.15); 26.3: a Riccati equation for general confocal ellipsoids (26.16)–(26.20); 26.4: its explicit solution for a binary mixture in the coated ellipsoid case (26.21)–(26.25), why I use old methods of explicit solutions, the difficulty of learning some fields of mathematics by lack of scientific behavior of the specialists, some of the useless fashions that I witnessed, the defect of not mentioning the names of those who had the ideas and of advertising things which are wrong; 26.5: the Schulgasser construction for the radial case (26.26)–(26.30); 26.6: extension by Francfort and myself to the confocal ellipsoids case (26.31)–(26.37); 26.7: solving (26.26); 26.8: solving (26.31), (26.38)–(26.40), can the Schulgasser construction improve 25.10? (26.41); 26.9: it does not (26.42)–(26.51), a two-dimensional case of Gutiérrez, Murat, Weiske and myself (26.52)–(26.62); 26.10: a corresponding Riccati equation (26.63) and (26.64), discussion of the result (26.65)–(26.71), a result of Francfort and myself about the natural character of confocal ellipsoids (26.72)–(26.84).

Chapter 27: Laminations Again, and Again

27.1: A formula for laminated materials (27.1); 27.2: a result of Braidy and Pouilloux (27.2)–(27.5), disadvantage of being shown a line of proof, my writing the general formula for laminations as a differential equation (27.6)–(27.13); 27.3: my formula for repeating laminations (27.14)–(27.16); 27.4: identifying a term in (27.14) and (27.17), my use of relaxation techniques for ordinary differential equations (27.18)–(27.20); 27.5: my direct method

(27.21)–(27.28); 27.6: the inverse after adding a rank one matrix (27.29); 27.7: my formula for laminating m materials in one direction (27.30)–(27.36).

Chapter 28: Wave Front Sets, H-Measures

Singular support of L. Schwartz, wave front set of Hörmander (28.1)–(28.6), propagation of microlocal regularity is not propagation of singularities, oscillations, and concentration effects (studied in a microlocal way) are more important in continuum mechanics and physics than singularities, were H-measures known before I introduced them?, the intuition about H-measures (28.7)–(28.11), $\mathbb{S}^{N-1}$ is a simple way to talk about a quotient space; 28.1: operators M_b and P_a (28.13) and (28.14), using the Plancherel formula (28.15) and (28.16); 28.2: a first commutation lemma (28.17) and (28.18); 28.3: a homogeneous of degree 0; 28.4: $a(s\,\xi, s^2\tau) = a(\xi,\tau)$ for $s > 0$ (28.19)–(28.25), using results of Coifman, Rochberg, and Weiss; 28.5: existence of scalar H-measures (28.26)–(28.29), vectorial H-measures (28.30), scalar first-order equation (28.31) and (28.32); 28.7: the localization principle (28.33)–(28.38); 28.8: scalar first-order equation (28.39) and (28.40); 28.9: gradients (28.41)–(28.43); 28.10: wave equation (28.44)–(28.48); 28.11: compensated compactness with variable coefficients (28.49)–(28.53); 28.12: symbols (28.54) and (28.55), examples (28.56)–(28.58); 28.13: symbol of a product (28.59); 28.14: weak $\star$ limit of $S_1 U_k^m \overline{S_2 U_\ell^m}$ (28.60), periodically modulated sequences (28.61) and (28.62); 28.15: the H-measure it defines (28.63)–(28.66); 28.16: the H-measure for a concentration effect at a point (28.67)–(28.69); 28.18: the necessity of some convergences in $H^{-1}_{loc}(\Omega)$ strong (28.70)–(28.73).

Chapter 29: Small-Amplitude Homogenization

Two approximations from Landau and Lifshitz (29.1) and (29.2), my interpretation using small-amplitude homogenization (29.3)–(29.11); 29.1: the correction in γ^2 uses H-measures (29.12)–(29.26), the injectivity of a mapping, my model of Chap. 19 (29.27) and (29.28); 29.2: expressing M^{eff} with H-measures (29.29)–(29.38); 29.3: density in x of the projection of H-measures for sequences in L^p (29.39) and (29.40); 29.4: application to the Taylor expansion of $F(\cdot, M^1, M^2)$ on the diagonal (29.41) and (29.42); 29.5: H-measures associated to characteristic functions (29.43) and (29.44).

Chapter 30: H-Measures and Bounds on Effective Coefficients

Description of the general method (30.1)–(30.12) and (30.13)–(30.17); 30.1: notation $\langle\!\langle \mu, Q(x,\xi,U)\rangle\!\rangle$; 30.2: a lower bound (30.18)–(30.24); 30.3: a consequence (30.25)–(30.28), the case of binary mixtures (30.29) and (30.30); 30.4: an upper bound (30.31)–(30.43); 30.5: a consequence (30.44)–(30.47), the case of binary mixtures (30.48) and (30.49).

Chapter 31: H-Measures and Propagation Effects

How conserved quantities hide at mesoscopic level, an error of thermodynamics, how waves carry conserved quantities around; 31.1: a second commutation lemma (31.1)–(31.8); 31.2: the Poisson bracket (31.9); 31.3: the second commutation lemma with standard operators (31.10), an improved regularity hypothesis uses a result of Calderón, differences between localization and propagation (31.11)–(31.14); 31.4: the scalar first-order hyperbolic case (31.15)–(31.27), how to generalize the result to more general systems (31.28)–(31.36); 31.5: the scalar wave equation (31.37)–(31.48), differences between propagation of H-measures and geometrical optics, a question of smoothness of the coefficients (31.49), the question of initial data, transformation of H-measures under local diffeomorphisms (31.50)–(31.53).

Chapter 32: Variants of H-Measures

My idea for introducing one characteristic length (32.1) and (32.2); 32.1: it gives H-measures independent of x_{N+1}, the idea of semi-classical measures of P. Gérard (32.3); 32.2: semi-classical measures for one-dimensional oscillations (32.4)–(32.6); 32.3: and its H-measures (32.7) and (32.8); 32.4: a commutation lemma (32.9)–(32.11); 32.5: two compactifications; 32.6: H-measure on the compactification (32.12)–(32.14), a mistake of P.-L. Lions and Paul, the Wigner transform (32.15)–(32.17), the idea of P.-L. Lions and Paul using the Wigner transform (32.18), my approach with P. Gérard using two-point correlations (32.19)–(32.33), more general equations for the localization principle (32.34); 32.7: the localization principle away from 0 (32.35)–(32.42); 32.8: its implication at ∞ (32.43) and (32.44); 32.9: another form of the localization principle at ∞ (32.45)–(32.50), a computation with P. Gérard on a sequence with two characteristic lengths (32.51)–(32.60), an intuitive explanation with beats, puzzling facts about spectroscopy, the approach of P. Gérard for deriving equations for the two-point correlation measures for the Schrödinger equation (32.61)–(32.68), for the heat equation (32.69)–(32.74), a computation with P. Gérard for k-point correlation measures for the heat equation (32.75)–(32.77), the case of variable coefficients (32.78)–(32.83), a computation of P. Gérard on how the Lorentz force appears from the Dirac equation with a large mass term, my research programme.

Chapter 33: Relations Between Young Measures and H-Measures

Why Young measures cannot see differential equations and cannot characterize microstructures; 33.1: laminating m materials in one direction at order γ^2 (33.1) and (33.2); 33.2: H-measures associated to characteristic functions (33.3)–(33.6), a model from micromagnetism, mixing r materials (33.7)–(33.10); 33.3: a first type of construction (33.11)–(33.18); 33.4: a second type of construction (33.19)–(33.39), an analogy with matrices of inertia (33.40)–(33.46); 33.5: admissible decompositions (33.47) and (33.48); 33.6: a third type of construction (33.49)–(33.54); 33.7: H-measures constructed by lamination (33.55)–(33.63); 33.8: a generalization (33.64); 33.9: a first type of construction (33.65); 33.10: a second type of construction (33.66)–(33.70); 33.11: a third type of construction (33.71)–(33.78); 33.12: sequences corresponding to a given Young measure and satisfying some particular differential system.

Chapter 34: Conclusion

My early difficulties about reading and writing, splitting some chapters into two parts, remarks on homogenization in optimal design, adapted microstructures for heat conduction and elasticity, remarks about three-point correlations, the difficulty of discovering useful generalizations, why periodicity assumptions are not so useful, when does the frequency of light play a role, the geometrical theory of diffraction (GTD) of Keller, about Bloch waves and the Bragg law for X-ray diffraction, about concentration effects, beyond partial differential equations and GTH.

35: Biographical Information

Basic biographical information for people whose name is associated with something mentioned in the book.

36: Abbreviations and Mathematical Notation

References

Index

Contents

Chapter 1
Why Do I Write?

I often quote the parable of talents from the gospels.[1] Parables are like general theorems, and they can be transmitted by people who do not necessarily understand all the various applications of the teaching: if after stating a general theorem one gives an example, the weak students only understand the example while the bright students foresee that the theorem applies to many situations. The gospels repeatedly show that the disciples of Jesus of Nazareth did not understand what the parables were about,[2] as they often asked for examples. The parables of talents which appear in Matthew 25:14–30 and Luke 19:12–27 differ, but the scenario is that a master left for a long trip and gave various amounts, five talents, two talents, one talent, to three of his servants, and when he came back he asked them to report about what they did. The servant who received five talents made them fructify and earned five more, the servant who received two talents earned two more, and they

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[1] The four evangelists, Matthew, Mark, Luke, and John, are not very well known. Matthew supposedly was a tax collector, chosen by Jesus to be one of his 12 disciples. Mark supposedly was a hellenist, converted by Peter. Luke supposedly was a physician, converted by Paul. John supposedly was a disciple of John the Baptist, who became one of the 12 disciples of Jesus.

[2] Jesus of Nazareth is believed by Christians to be the (unique) son of God, and the messiah whom Jews were waiting for, hence its title Christ, which probably has that meaning in Greek. Of course, I consider that he was only human, and I often refer to him as the Teacher. According to the gospels, he practised meditation past the point where one can do miracles, but without using that power for a personal advantage. He was executed by the Romans, probably because some of his followers believed him to be the messiah whom Jews were waiting for, and whom they expected to put an end to the Roman occupation. Oriental religions mention that after death the bodies of people who are extremely advanced on the spiritual path may shrink, and even dissolve completely, sometimes leaving hair and nails, an effect called "rainbow body"; could it be the reason why the body of Jesus could not be found?

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were both praised, with the same words.[3] The servant who received one talent said that he was afraid to lose it, and he buried it into the ground, so that he only gave back the initial amount, and he was punished. The versions in the gospels were probably distorted from an original teaching,[4] which I believe is reported in an apocryphal gospel, which has a fourth servant who also received one talent, and this servant tried to make it fructify but he lost it; however, in the end this servant was not punished, and again it was the servant who did not try to use his talent who was punished. Obviously, either the disciples of Jesus or those to whom they told the story could not understand why the servant who lost his talent was not punished, so they took him out of the parable, probably because they thought that the parable was about money, but that interpretation using money is a dull one, and cannot be the meaning intended by the Teacher, of course!

Although the talent in the parable was a unit of money (probably like a pound of silver), I interpret it as a gift for something useful, like mathematics, and my interpretation is that we are not the creators of our brains, and anyone who received a very efficient brain is bound to be successful and he/she should not be proud about that, but anyone who misuses his/her talent should be castigated, and that applies to the very bright mathematicians who do not attack difficult problems and settle for more elementary ones (for them), in order to be praised for solving many of these easy problems, instead of trying

[3] My father pointed out to me that the praises are identical, so that the servant who was given five talents and earned five more was not considered more worthy than the servant who was given two talents and earned two more!

[4] I once told my father, who was a Protestant minister, that I did not think that Jesus existed, and that it did not matter because only his teachings are important, but he disagreed, because he believed in resurrection. Many years after, in reading magazines published by BAS (Biblical Archaeology Society), Washington, DC, either *BAR* (*Biblical Archaeology Review*) or *BR* (*Bible Review*), I learned that the first version of the gospel of Mark, which is the earliest of the four gospels, ended after the women found the tomb empty, and a sequel was written a few centuries after (obviously for making it conform to the other gospels, which were written afterwards, and talked about resurrection), and I checked that my father knew about that. For me it is the sign that the gospel of Mark was written before the dogma of resurrection was invented, and propagated by Paul, who I think was the real inventor of Christianity. However, I finally thought that Jesus existed, arguing that the reporting of parables by the evangelists show a superior Teacher who was trying to transmit a deep message to uneducated students, and I see the fact that the evangelists transmitted us the information that the disciples of Jesus did not understand his teachings as a sign that they did not invent the whole story. Actually, if some teachings must be transmitted orally, and without too much distortion, by people who do not really understand what the teachings are about, then one is bound to invent using parables for transmitting the teachings. Of course, distortions occurred later, and I imagine that the original version of the gospel of Mark contained what the Teacher taught and how he died, without trying to interpret how his body could dissolve.

to open new doors. There should be no shame for failing to open a closed door behind which something very interesting is supposed to be found, but one should be able to show that one made efforts in reasonable directions; one should also give advice to those who also plan some attempts, explaining what was tried before, and possibly why it did not work.

I tried to follow these general principles, and as *I was lucky to study in Paris in the mid 1960s, in a special scientific environment* that is almost impossible to reproduce nowadays, I feel the need to *explain what I was taught and what knowledge I added by my own research work*, not so much because it is my own but because it should help the young researchers for avoiding the long and useless meanderings that many others are still following.[5] Also, *I witnessed the behavior of a few famous mathematicians*, and I met many in person, which was the initial reason why I wanted to share some biographical information about them, but then I tried to find biographical information concerning those whom I did not meet, usually for the obvious reason that they lived in a different period. I obtained much of my information by searching the Internet, but not everything comes from such a reliable source as MacTutor, http://www-history.mcs.st-andrews.ac.uk/, the web site from University of St Andrews, St Andrews, Scotland, which is dedicated to history of mathematics, and some of my information coming from other sources could be slightly wrong, and I expect every interested reader to tell me about my mistakes. I am not interested in the actual citizenship of the people whom I mention, but sometimes they were born in a different country than the one where they worked, and my point is to show that exchanges between countries and continents play a role in the creation and dissemination of knowledge. My hope is that this biographical information will help give a more global picture about how *science progresses by the work of many, coming from different times, different places, and different cultures.*[6]

[5] There could be psychological reasons why many continue on a path which was already shown to be wrong, but a few have political reasons to mislead these researchers whom I am trying to educate.

[6] With the help of MacTutor (http://www-history.mcs.st-andrews.ac.uk/), one can look at mathematicians from the past (including some astronomers, and some philosophers), and one finds that 75% of the (86) names of people born before 500 are Greek (and 12% are Chinese, and 8% are Indian), that 70% of the (7) names of people born between 500 and 750 are Indian (and 15% are Chinese, and 15% are European), that 80% of the (36) names of people born between 750 and 1,000 are Arabic (and 20% are Indian), that 44% of the (32) names of people born between 1,000 and 1,250 are European (and 25% are Arabic, 19% are Chinese, and 12% are Indian), and that 69% of the (46) names of people born between 1,250 and 1,500 are European (and 15% are Arabic, and 12% are Indian). It is an interesting fact that there are no Greek names after 500, and that from 1,400 to 1,500 almost all names are European. One may deduce that the development of mathematics (or more generally of all sciences) was not independent of economical, political, and religious factors in the past. Therefore, one should stay alert about counteracting the bad tendencies which are observed nowadays.

I consider mathematics as a part of a big puzzle, certainly quite an important piece of science, and I learned about the interplay of various scientific fields of research, a little more than most mathematicians, and this did not just happen by chance.

In the mid 1960s, I succeeded at exams which gave me the possibility to study either at *École Normale Supérieure* or at *École Polytechnique*, Paris, France. I wanted to do something useful, and no one told me that *mathematics can be useful for something else than teaching mathematics*, but I thought that engineers were doing useful things, and this idea led me to choose to study at École Polytechnique, which is not actually an engineering school, as I only understood much later. I did not know what the work of an engineer is, and no one in my family knew about that either, so I took my decision alone; after 1 year, Laurent SCHWARTZ gave an evening talk,[7] on the *role and duties of scientists*, and he mentioned that engineers do a lot of administration, and that made me understand that I needed to change my orientation, and I decided to do research in mathematics, possibly with an applied twist, in agreement with my original choice. After studying at École Polytechnique, it became clear that this choice gave me an enormous advantage on the majority of mathematicians, because of what I studied outside mathematics.

During the first year I learned about classical mechanics, which is an eighteenth century point of view of mechanics, based on ordinary differential equations; during the second year I learned about continuum mechanics, which is a nineteenth century point of view of mechanics, based on partial differential equations; I did hear a little about a twentieth century point of view of mechanics, which included questions about turbulence and plasticity, the latter being the research topic of the teacher, Jean MANDEL,[8] but a few years after I discovered that the mathematical tools for that point of view did not exist yet, and my research work (after my thesis) transformed into developing a new mathematical approach for that.

Studying analysis with Laurent SCHWARTZ [86], and numerical analysis with Jacques-Louis LIONS, who became my thesis advisor, was the best preparation for hearing about all the mathematical tools in partial differential equations which were used for understanding continuum mechanics and

[7] Laurent SCHWARTZ, French mathematician, 1915–2002. He received the Fields Medal in 1950 for his work in functional analysis. He worked in Nancy, in Paris, at École Polytechnique, which was first in Paris (when he was my teacher in 1965–1966 [86]), and then in Palaiseau, and at Université Paris 7 (Denis Diderot), Paris, France.

[8] Jean MANDEL, French mathematician, 1907–1982. He worked in Saint-Étienne and in Paris, France. He was my teacher for the course of continuum mechanics at École Polytechnique in 1966–1967 in Paris [58].

physics [52] (although neither of them was really interested in mechanics or physics), before the introduction of the ideas that I started developing in the mid 1970s.

I also learned about classical physics, special relativity, quantum mechanics and statistical physics, but with teachers who often gave the impression that they did not know how to disentangle mathematics and physics, and I thought later that it could be the result of an infamous classification by COMTE,[9] a French philosopher, who studied at École Polytechnique for 1 year, and obviously valued abstraction so much that he put mathematics above all other sciences,[10] before astronomy,[11] physics, chemistry, biology, in this order. One needs different abilities for becoming a good mathematician, a good physicist, a good chemist, or a good biologist, and it is not wise to disparage others because they possess an ability that one has not, so I find quite silly, if not completely ridiculous, to imagine a linear order between various fields, inside or outside science, whatever its definition is.[12] Nowadays, there are many people who lack the abilities for mathematics, like the sense of abstraction for example, and they would choose another field more suited to their interests and abilities, were it not for this unnatural attraction created by the Comte classification, or other silly reasons.

My teacher in probability was not good, and as the teacher of statistical physics gave me a bad impression too, I was bound to distrust any probabilistic model for linking different phenomena, and I was glad to discover in the early 1970s that I could avoid probabilities altogether for relating what happens at different scales, and use various types of weak convergence instead; finding this was not only due to some joint work that I did with François MURAT [93], generalizing some earlier work of Sergio SPAGNOLO [89, 90],[13] helped with the insight of Ennio DE GIORGI [22],[14] but also to some particular applications of Évariste SANCHEZ-PALENCIA [81, 82],[15] which helped

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[9] Auguste COMTE, French philosopher, 1798–1857. He worked in Paris, France.

[10] Mathematics is one of the sciences, and the sentence "mathematics and science" was probably coined by experts in sabotage.

[11] This explains why nowadays, many of those who chose to study physics because they thought that they were not good enough for studying mathematics end up in astrophysics.

[12] It is precisely that mistake which makes weaker people in one group believe that they are worth much more than stronger people in another group, a disease which grew too much in our times, and which is called *racism*!

[13] Sergio SPAGNOLO, Italian mathematician, born in 1941. He works at Università degli Studi di Pisa, Pisa, Italy.

[14] Ennio DE GIORGI, Italian mathematician, 1928–1996. He received the Wolf Prize in 1990, for his innovating ideas and fundamental achievements in partial differential equations and calculus of variations, jointly with Ilya PIATETSKI-SHAPIRO. He worked at Scuola Normale Superiore, Pisa, Italy.

[15] Enrique Évariste SANCHEZ-PALENCIA, Spanish-born mathematician, born in 1941. He works at CNRS (Centre National de la Recherche Scientifique) and UPMC

me understand this new point of view, and this could only happen because I was interested in understanding continuum mechanics and physics, of course! Peter LAX later observed that the idea that some numerical schemes only converge in a weak topology was used before,[16] by VON NEUMANN,[17] but it does not seem that VON NEUMANN thought of changing the way one looks at physics, in the manner that I developed.[18]

Understanding better a subject is both an intellectual advantage and a social disadvantage, because one quickly finds oneself isolated, among a majority who prefers to continue being wrong and lying about it. In 1984, Jean LERAY told me about suffering because one understands more than others,[19] and later I found in a book by Clifford TRUESDELL,[20] which he offered me, a quote of PLANCK,[21] who also described this difficulty: "A new scientific

(Université Pierre et Marie Curie), Paris, France. I knew him under the French form of his first name, Henri, but he now uses his second name, Évariste.

[16] Peter David LAX, Hungarian-born mathematician, born in 1926. He received the Wolf Prize in 1987, for his outstanding contributions to many areas of analysis and applied mathematics, jointly with Kiyoshi ITO. He received the Abel Prize in 2005 for his groundbreaking contributions to the theory and application of partial differential equations and to the computation of their solutions. He works at NYU (New York University), New York, NY.

[17] János (John) VON NEUMANN, Hungarian-born mathematician, 1903–1957. He worked in Berlin, in Hamburg, Germany, and at IAS (Institute for Advanced Study), Princeton, NJ.

[18] I read that VON NEUMANN wrote in a letter in 1935 that he did not believe anymore in the mathematical framework that he devised for quantum mechanics. As he did not make this point known to all, he did not think of changing the way how one looks at physics, and he bears the responsibility that the silly rules of quantum mechanics transformed into dogmas!

[19] Jean LERAY, French mathematician, 1906–1998. He received the Wolf Prize in 1979, for pioneering work on the development and application of topological methods to the study of differential equations, jointly with André WEIL. He worked in Nancy, France, in a prisoner of war camp in Austria (1940–1945), and in Paris, France; he held a chair (théorie des équations différentielles et fonctionnelles, 1947–1978) at Collège de France, Paris.

[20] Clifford Ambrose TRUESDELL III, American mathematician, 1919–2000. He worked at Indiana University, Bloomington, IN, and at Johns Hopkins University, Baltimore, MD.

[21] Max Karl Ernst Ludwig PLANCK, German physicist, 1858–1947. He received the Nobel Prize in Physics in 1918, in recognition of the services he rendered to the advancement of physics by his discovery of energy quanta. He worked in Kiel and in Berlin, Germany. There is a Max Planck Society for the Advancement of the Sciences, which promotes research in many institutes, mostly in Germany (I spent my sabbatical year 1997–1998 at the Max Planck Institute for Mathematics in the Sciences in Leipzig, Germany).

truth does not triumph by convincing its opponents and making them see the light, but rather because its opponents finally die, and a new generation grows up that is familiar with it." For many years I wondered why so many mathematicians pretended to work on problems of mechanics, and said things known to be false by anyone who studied a little; I was not very good at communicating, and I stayed silent, but I felt that many around me showed a mixture of incompetence and intellectually dishonest behavior.

Why pretend that the world is described by ordinary differential equations, as if one did not study partial differential equations? For example, why use the term mechanics for designating classical mechanics, which is an eighteenth century point of view based on ordinary differential equations, and not continuum mechanics, which is a nineteenth century point of view based on partial differential equations, or ignore the twentieth century point of view that goes *beyond partial differential equations*, as I explained during the last 30 years?

Why be interested in studying the asymptotic behavior of equations without saying that so many known effects were neglected in the models used that their time of validity is known to be quite limited?

Why pretend that physical systems minimize their potential energy, as if one did not know the first principle of thermodynamics, that energy is conserved (when one counts all its various forms)? Why ignore the second principle of thermodynamics, despite its defects? Why not say that thermodynamics is not about dynamics but about equilibria, and that equations of state derived from equilibrium might well create havoc if one pretends that they are valid all the time? Why not discuss the defects of the Boltzmann equation,[22] and observe that it was obtained by postulating an irreversible behavior, and thus cannot help one understand how irreversibility occurs?

Why not observe that the rules of quantum mechanics could only be invented by people unaware of partial differential equations, and unable to distinguish between the point of view of NEWTON,[23] where there are forces acting at a distance, and the point of view of H. POINCARÉ,[24] which EINSTEIN did not seem to understand,[25] where there are none? Why not

[22] Ludwig BOLTZMANN, Austrian physicist, 1844–1906. He worked in Graz and Vienna, Austria, in Leipzig, Germany, and then again in Vienna.

[23] Sir Isaac NEWTON, English mathematician, 1643–1727. He worked in Cambridge, England, holding the Lucasian chair (1669–1701). There is an Isaac Newton Institute for Mathematical Sciences in Cambridge, England.

[24] Jules Henri POINCARÉ, French mathematician, 1854–1912. He worked in Paris, France. There is an Institut Henri Poincaré (IHP), dedicated to mathematics and theoretical physics, part of UPMC (Université Pierre et Marie Curie), Paris.

[25] Albert EINSTEIN, German-born physicist, 1879–1955. He received the Nobel Prize in Physics in 1921, for his services to theoretical physics, and especially for his discovery of the law of the photoelectric effect. He worked in Bern, in Zürich, Switzerland, in Prague, now capital of the Czech Republic, at ETH (Eidgenössische Technische Hochschule), Zürich, Switzerland, in Berlin, Germany, and at IAS (Institute for

observe that those who pretend to see things traveling faster than the speed of light c play with equations whose best derivation is to let c tend to ∞ in a more realistic physical description? Why not observe that the difficulties between waves and particles disappear when one understands that there are only waves satisfying partial differential equations, and that as there are no particles the question of understanding where they are is meaningless?

Is it not becoming obvious that one needs to go *beyond partial differential equations*, which I explained for about 30 years, so why are there so many people who keep thinking in terms of ordinary differential equations?

I was only aware of a few of these questions in the mid 1970s, when I developed my new approach to continuum mechanics which mixed ideas from homogenization and from *compensated compactness*, first described in my Peccot lectures at the beginning of 1977,[26] at Collège de France, in Paris. For what concerned questions of physics, the situation was more delicate, because I could not believe the classical presentations, usually obscured by an excessive amount of probabilities, so often used for masking the fact that one does not know much yet about the phenomena that one pretends to study. In 1977, I understood why the second principle of thermodynamics needed improvement, by embedding the question into a more general homogenization problem, and in 1980, I understood that the appearance of nonlocal effects by homogenization is probably behind the strange rules of spontaneous absorption and emission which physicists invented, and the key for understanding turbulence, but in the summer of 1982 I still did not see how to extend my ideas to quantum mechanics and statistical mechanics, and it was due to the help of Robert DAUTRAY,[27] that I could improve my understanding of physics. On one hand, he offered me a position at CEA (Commissariat à l'Énergie Atomique), so that I could leave Université Paris Sud, Orsay, France; on the other hand, I benefited from his advice about what to read, and this helped me understand how a part of physics could be described in the same spirit as my previous research programme, and this is how I understood about *H-measures* and their variants [105]. I am very grateful to Robert DAUTRAY for that, because *physics is a very difficult subject for a mathematically oriented mind*, as physicists' statements usually lack precision, and by following the advice of a very competent person, one learns that there

Advanced Study), Princeton, NJ. The Max Planck Institute for Gravitational Physics in Potsdam, Germany, is named after him, the Albert Einstein Institute.

[26] Claude Antoine PECCOT, French child prodigy, 1856–1876.

[27] Ignace Robert DAUTRAY (KOUCHELEVITZ), French physicist, born in 1928. It is thanks to him that I worked at CEA (Commissariat à l'Énergie Atomique) from 1982 to 1987, and that my understanding of physics improved.

are a few important questions which need to be understood in a better way, and one gains an invaluable amount of time in not having to identify these questions by oneself. After that, one observes that most mathematicians who think that they understand physics are only playing one of the many games which physicists invented, without a good physicist telling them that they should not build too much on a game which is not good physics at all;[28] even some games with a long life, like quantum mechanics or statistical physics, survive mostly because they were transformed into dogmas, which makes them difficult to discard, but their defects are too obvious to be ignored.[29]

A few years ago, I heard a talk by a physicist which showed how difficult it is for a mathematician to assess the value of what physicists say; this one, who put a lot of humor in his presentation, chose a suggestive title, "before the big-bang," and at the end I asked him a question, mentioning that temperature is an equilibrium concept, and wondering if he thought that in the first few milliseconds just after the big-bang (which he believed in), matter was in equilibrium at temperatures of a few million (or billion) degrees, and he answered yes!

I read an article by POISSON from 1807,[30] where he pointed out that the speed of sound could not have been computed before by using the available data about compressibility of air, because the usual relation where the pressure is proportional to the density of mass gives an incorrect value for the speed of sound, and maybe NEWTON already knew this discrepancy; instead, POISSON used a law $p = c\,\varrho^\gamma$, which was proposed by LAPLACE,[31] probably for heuristic reasons. One explains now that *the propagation of a wave is too fast a phenomenon for heat to flow* so that the process is adiabatic (isentropic). This was a source of error for a few mathematicians, starting with RIEMANN,[32] who worked too much with the equations of isentropic gases, as if adiabatic changes were the rule, but it seems to me that some physicists are as deluded as some mathematicians if they believe that matter reaches instantaneously its equilibrium at a temperature of million

[28] There is a parable about that, which talks about building a house on the sand.

[29] I first learned about religions because my father was a Protestant minister, but after rejecting the idea of God for intuitive reasons when I was 12 or 13, I became interested in religions in order to make up my mind about GOD (see fn. 34, p. 10). It was only much later, after fighting against vote-rigging in Orsay, and observing the powerful allies of my political opponents and their methods of destruction, that I realized how much one can learn from the mistakes of the past concerning religions, as it helps understand how some of the actual chaos in science was generated.

[30] Siméon Denis POISSON, French mathematician, 1781–1840. He worked in Paris, France.

[31] Pierre-Simon LAPLACE, French mathematician, 1749–1827. He was made count in 1806 by Napoléon I, and marquis in 1817 by Louis XVIII. He worked in Paris, France.

[32] Georg Friedrich Bernhard RIEMANN, German mathematician, 1826–1866. He worked at Georg-August-Universität, Göttingen, Germany.

degrees, whatever this means.[33] One should remember that physicists misled the "scientific" community and the funding agencies for almost 50 years by claiming that they were going to control fusion, but now they estimate that they might succeed in the second part of the twenty first century; of course, they are quite careful not to say explicitly that one important reason is that *one must still discover the properties of matter at temperatures of a few million degrees*!

It was not so difficult for me to discover what is wrong with a few laws believed by physicists, and I think that being educated as a Calvinist and losing my faith in God by the age of 13 helped form my character in a useful way for science,[34] in that I cannot lie and I cannot accept any dogma without criticizing it, and will preferably tear it to pieces and wonder why some people believe it. However, becoming a mathematician implies that one must know the hypotheses and postulates that one makes: having postulated that God does not exist, I needed to check that particular dogma of mine.

From a mathematical point of view, it is impossible to decide if the world was created or has existed forever: in his course at École Polytechnique, in 1965–1966, Laurent SCHWARTZ pointed out that one cannot decide if the universe that we live in is an orientable manifold or not,[35] as orientability is a global property and our information on the universe is local, and the same argument shows that *one cannot decide if the universe was created or not.* Many western pseudo-scientists were so brainwashed by the creation theory in the Bible, that they think that they must reject it by adhering to the big-bang theory, without realizing that both the creationist approach and the big-bang theory are flawed, and as one cannot decide if the universe we live in was created or not, why not imagine that there could be quite a number of universes, some having always existed and some being created in a finite past, where the same particular event occurs, like that of a French mathematician preparing his fourth book, on homogenization.[36]

From a mathematical point of view, it is impossible to decide if one or many gods exist without giving mathematical definitions of divine beings and proving their properties, but again western pseudo-scientists were so brainwashed by the Bible, that they think that they must oppose those who believe that God exists and that the Bible redactors were inspired by God,

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[33] I was told recently that this physicist does not believe that matter was in equilibrium, so that either he did not understand my question, or he answered it in the spirit of his talk, as a joke.

[34] I use God to refer to the deity venerated by Jews, Christians, and Moslems, whom I believe to be just a literary character created in the seventh century BCE. I use GOD as a conjecture for a notion too transcendent to be perceived by ordinary beings, like the one Ramakrishna seemed to refer to, with a name that I do not recall [57].

[35] Obviously, Laurent SCHWARTZ postulated that the universe is a manifold!

[36] There is no obvious reason why the book should be finished, or that the finished books should be the same in all the realizations of that event.