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Walter Dittrich

Reassessing Riemann's Paper

On the Number of Primes Less Than a Given Magnitude

Second Edition

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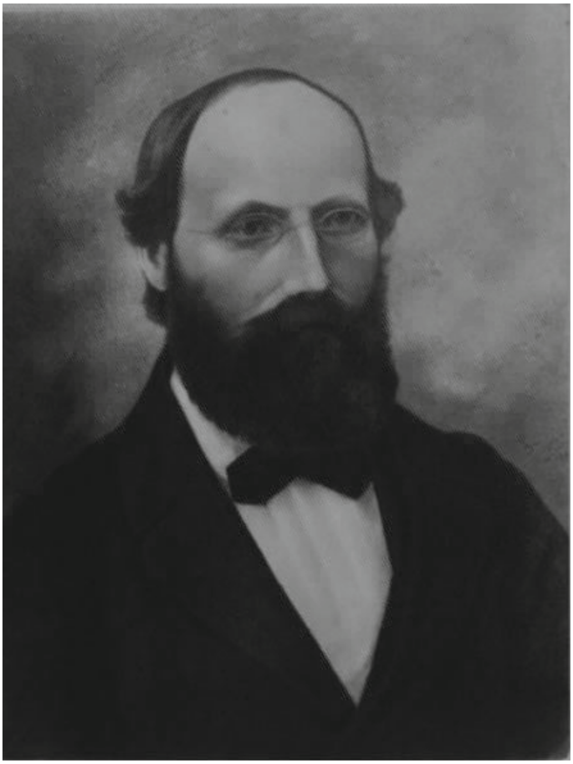
ISSN 2211-4564 ISSN 2211-4572 (electronic)
SpringerBriefs in History of Science and Technology
ISBN 978-3-030-61048-7 ISBN 978-3-030-61049-4 (eBook)
https://doi.org/10.1007/978-3-030-61049-4

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland



Georg Friedrich Bernhard Riemann (1826 – 1866)

Preface to the Second Edition

In the first edition, I concentrated mainly on the historical development of the Riemann zeta function and its application in mathematics. In this second edition, I have added three new chapters in order to underline the importance of the zeta-function regularization (ZFR) for physics, with the intent to emphasize Riemann's zeta function as a powerful tool to regularize the otherwise infinite quantities that occur in many problems in quantum mechanics and quantum field theory.

The calculations of the partition function of the Bose oscillator in Chap. 9 and of the Fermi oscillator in Chap. 10 are not only challenging but also represent historically the first use of ZFR by Gibbons and Hawking in the seventies.

Shortly thereafter, I proved the usefulness of ZFR in quantum electrodynamics (QED) (Chap. 11). Equation (11.33) represents the result. The path I took is described in great detail, where I show the computation of the one-loop effective action in spinor QED using Riemann's ZFR.

Finally, I thought it would be helpful to the reader to summarize the results of the Euler-Riemann equation and include many findings and graphic representations.

I have also re-written Appendix A.2, improving on the formulation in the first edition.

Tübingen, Germany Walter Dittrich

Preface to the First Edition

This book is devoted to one of the members of the Göttingen triumvirate, Gauß, Dirichlet, and Riemann. It is the latter to whom I wish to pay tribute, and especially to his world-famous article of 1859, which he presented in person at the Berlin Academy upon his election as a corresponding member. His article entitled, "Über die Anzahl der Primzahlen unter einer gegebenen Größe" ("On the Number of Primes Less Than a Given Magnitude"), revolutionized mathematics worldwide. Included in the present book is a detailed analysis of Riemann's article, including such novel concepts as analytical continuation in the complex plane; the product formula for entire functions; and, last but not least, a detailed study of the zeros of the so-called Riemann zeta function and its close relation to determining the number of primes up to a given magnitude, i.e., an explicit formula for the prime number counting function.

Tübingen, Germany Walter Dittrich

Contents

Chapter 1
Towards Euler's Product Formula and Riemann's Extension of the Zeta Function

There is a very close connection between the sums of the reciprocals of the integers raised to a variable power that Euler wrote down in 1737, the now-called zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots , \qquad s > 1 \tag{1.1}$$

and the primes—which, as integers, are the very signature of discontinuity. Euler considered s to be a real integer variable with $s > 1$ to insure convergence of the sum. Multiplying the definition of $\zeta(s)$ by $1/2^s$ we obtain

$$\frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots \tag{1.2}$$

and subtracting this from $\zeta(s)$ we get

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s}$$

$$\text{or} \qquad \left(1 - \frac{1}{2^s}\right)\zeta(s) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \frac{1}{11^s} + \cdots . \tag{1.3}$$

Hence all the multiples of the prime $n = 2$ disappeared from the original sum of the defined $\zeta(s)$. In short, we found

$$\left(1 - \frac{1}{2^s}\right)\zeta(s) = \sum_{\substack{n=1 \\ n \neq 2k}}^{\infty} \frac{1}{n^s} . \tag{1.4}$$

W. Dittrich, *Reassessing Riemann's Paper*,
SpringerBriefs in History of Science and Technology,
https://doi.org/10.1007/978-3-030-61049-4_1

Next, we multiply this last result by $1/3^s$ to obtain

$$\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\zeta(s)=\sum_{\substack{n=1\\ n\neq 2k}}^{\infty}\frac{1}{(3n)^s}=1+\frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\cdots \tag{1.5}$$

and so, subtracting this from $(1-1/2^s)\zeta(s)$, we have

Leonhard Euler (1707 – 1783);
Drawing by C.F. Gauß