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In and Out of Equilibrium 3: Celebrating Vladas Sidoravicius

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Vlado Sidoravičiaus (1963–2019) atminimui.
Jis įkvėpė mus savo draugiškumu,
kūrybiškumu ir meile matematikai ir
gyvenimui.

In memory of Vladas Sidoravicius
(1963–2019).
He inspired us with his friendliness,
creativity, and love for mathematics and for
life.

Vladas Sidoravicius (courtesy of NYU-Shanghai; credit: Junbo Chen)

Preface

Vladas was born in Vilnius, Lithuania, on August 23, 1963, and did his undergraduate studies in Mathematics from 1982 to 1985 at Vilnius University. There, in 1986, he received a Master's degree with Honors under the supervision of Vygantas Paulauskas. While in Vilnius, his early career and interest for doing research in Mathematics benefited greatly from the mentorship of Donatas Surgailis. Pursuing research from the very beginning, Vladas moved to Moscow for the next 4 years and was awarded a Ph.D. from Moscow State University, under the supervision of Vadim Malyshev, with a dissertation on the convergence of the stochastic quantization method. At the VIth Vilnius Conference on Probability and Statistics, in 1990, Vladas gave what was probably his first presentation in an important international conference. In the meantime, he also had several other collaborators in Vilnius, already showing the vitality and initiative that were some of his characteristics. Vladas loved music and all expressions of fine art and always made clear that those years in Moscow offered him an extremely enriching experience in this aspect as well as for his mathematical development.

In 1991, Vladas had a postdoc experience at the University of Heidelberg, which he then continued for more than 1 year at the Université Paris Dauphine, working with the team of Claude Kipnis. This brought to his attention the existence of a probability research group in Brazil, which led to his arrival at IMPA, Rio de Janeiro, in February of 1993, where he held a position until 2015. While at IMPA, Vladas served as advisor to several PhD students, supervised a number of postdocs, and organized many meetings as well as remarkable conferences and schools. He always focused on offering challenging and stimulating opportunities to young researchers. As pointed out to us by Marco Isopi, this emphasis on supporting young scientists was something that Vladas and other postdocs of Claude Kipnis vowed to emulate following Kipnis' early death at age 43. Vladas made an immense contribution to the development of Probability in South America, particularly in Brazil.

A very important development in his scientific career began in 1995, when he made the first of many visits to Cornell University. It was the beginning of an extremely fruitful interaction with Harry Kesten, a towering figure in Probability Theory for six decades, who passed away shortly before Vladas. Not only did they write many joint papers, including a seminal work where a shape theorem without subadditivity was proven, but they also became very close friends. One anecdotal story has to do with the efforts made by Vladas to find a copy of Kesten's book *Percolation Theory for Mathematicians*, which was out of print. He searched by all possible methods until the day arrived when he somehow managed, with his usual soft and charmingly convincing attitude, to have one colleague make him a gift of his personal copy. That was a priceless gift, providing huge joy to Vladas.

Vladas' friendship with Harry Kesten extended into an approach to the Dutch stochastic community that resulted in a double appointment as researcher at the Centrum Wiskunde & Informatica (CWI) and visiting professor at Leiden University. During his tenure (2007–2011), Vladas developed an intensive research activity with the leaders of the main Dutch groups in probability, gave courses and seminars, and acted both as consultant and conference organizer at Eurandom. Vladas was a vocal supporter of this last institute, which he considered a model deserving emulation.

In 2015, Vladas became NYU Global Network Professor and was appointed Deputy Director of the Mathematical Institute at NYU-Shanghai. He quickly understood the immense potential of this new institution and invested in it all his energy and his capital of scientific networking. His enthusiasm and dedication helped to construct a remarkable Institute characterized by a continuous flow of distinguished visitors and an intense scientific activity. A particular achievement was the semester he organized on mathematical physics supported by the Chinese Science Foundation, which attracted most of the leading scientists in the field. He was there, in Shanghai, planning the next scientific visits, dreaming on building "the Eurandom of Asia" when his life came to an end.

Besides his great talent and creativity, Vladas had an unlimited enthusiasm for his work. He truly enjoyed it and would not be stopped by ordinary difficulties: he would make huge efforts to attend a conference or meeting that he considered important, working full day in between two long flights; he would put in full energy while organizing events and making sure that everyone felt as comfortable as possible. At IMPA, students and collaborators always knew the clock drawn on his blackboard as his daily agenda. This was always full but also always open to find some extra time. Everyone could see his immense energy and his passionate enthusiasm for the profession.

This volume contains a collection of papers by many of his collaborators and on a variety of topics in probability and statistical physics that reflects Vladas' main research interests. Among them are two projects in collaboration with him, in preparation at the time of his death.

The idea of preparing this volume grew during the XXIII Brazilian School of Probability that took place at the end of July 2019, in USP-São Carlos, also dedicated to him. We thank the scientific and organizing committees as well as all the speakers of the School for their full support.

After we wrote to Vladas' many collaborators, we received great support and excellent cooperation from them and many others who helped with this project, including series editors, authors, and anonymous referees. Most of the review

process took place during a period when everyone was affected by the pandemic of Covid-19, with extra time needed for online teaching activities, but our referees were extremely generous with their help. Our sincere thanks also goes to a group of Vladas' close friends from Lithuania, for their valuable and inspiring feedback.

We acknowledge the important role played by NYU-Shanghai in the latter portion of Vladas' career and its cooperation in the preparation of this volume. The 1-day memorial event held in Shanghai, on October 22, 2019, when several of us came together to remember him, was also a source of inspiration.

As a consequence of his enormous enthusiasm and dedication to the probability community, besides organizing wonderful meetings, Vladas edited many special volumes, mostly associated to schools or conferences in probability and mathematical physics. The list includes proceedings of two editions of the Brazilian School of Probability, of which he was one of the initiators, and that he titled *In and Out of Equilibrium*. As a way to honor him and at the same time reflecting well the content of the scientific papers, we keep the title for this memorial volume.

Anyone who had the opportunity of being close to Vladas, in the profession or outside, knows his huge energy and joy for life. He also took great care of his mother, Galina, who survives him. No matter where in the world he was located, he would call her almost daily to make sure she was well and well-provided for. We all remember him in constant Celebration of Life. We miss his joyful laugh but have powerful reasons to celebrate his life and his achievements.

Rio de Janeiro, Brazil — Maria Eulália Vares
Shanghai, China — Roberto Fernández
São Paulo, Brazil — Luiz Renato Fontes
New York, NY, USA — Charles M. Newman
August 2020

With Harry Kesten (Ithaca, 2003)

Vladas, 2nd from right in top row at Vilnius University, in 1984 (courtesy of Arvydas Strumskis)

Vladas at leisure in the early 90s (courtesy of Renata Sidoraviciene)

Publications of Vladas Sidoravicius

The editors believe that this list of publications was complete at the time when this volume was prepared, but since there are a number of ongoing projects that Vladas was involved in, it is likely that there will be some future publications that include Vladas as a coauthor.

Research Articles

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[1] We have left the misspelling of Vladas' last name because that was the way it appeared in the published version of the paper.

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Vladas at NYU-Shanghai in 2016 (courtesy of NYU-Shanghai; credit: Junbo Chen)

Contents

Existence and Coexistence in First-Passage Percolation

Daniel Ahlberg

Abstract We consider first-passage percolation with i.i.d. non-negative weights coming from some continuous distribution under a moment condition. We review recent results in the study of geodesics in first-passage percolation and study their implications for the multi-type Richardson model. In two dimensions this establishes a dual relation between the existence of infinite geodesics and coexistence among competing types. The argument amounts to making precise the heuristic that infinite geodesics can be thought of as 'highways to infinity'. We explain the limitations of the current techniques by presenting a partial result in dimensions $d > 2$.

Keywords First-passage percolation · Competing growth · Geodesics · Busemann functions

1 Introduction

In first-passage percolation the edges of the $\mathbb{Z}^d$ nearest neighbour lattice, for some $d \geq 2$, are equipped with non-negative i.i.d. random weights ω_e, inducing a random metric T on $\mathbb{Z}^2$ as follows: For $x, y \in \mathbb{Z}^d$, let

$$T(x, y) := \inf\Big\{ \textstyle\sum_{e\in\pi} \omega_e : \pi \text{ is a self-avoiding path from } x \text{ to } y\Big\}. \tag{1}$$

Since its introduction in the 1960s, by Hammersley and Welsh [18], a vast body of literature has been generated seeking to understand the large scale behaviour of distances, balls and geodesics in this random metric space. The state of the art has been summarized in various volumes over the years, including [4, 21, 23, 32]. We will here address questions related to geodesics, and shall for this reason make

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the common assumption that the edge weights are sampled from a continuous distribution. Since many of the results we shall rely on require a moment condition for their conclusions to hold, we shall assume in what follows that $\mathbb{E}[Y^d] < \infty$, where Y denotes the minimum weight among the $2d$ edges connected to the origin.

In the 1960s, the study of first-passage percolation led to the development of an ergodic theory for subadditive ergodic sequences, culminating with the ergodic theorem due to Kingman [24]. As a consequence thereof, one obtains the existence of a norm $\mu : \mathbb{R}^d \to [0, \infty)$, simply referred to as the *time constant*, such that for every $z \in \mathbb{Z}^d$, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} T(0, nz) = \mu(z).$$

Richardson [30], and later work of Cox and Durrett [9], extended the above *radial* convergence to *simultaneous* convergence in all directions. Their results show that the ball $\{z \in \mathbb{Z}^d : T(0, z) \leq t\}$ in the metric T once rescaled by $1/t$ approaches the unit ball in the norm μ. The unit ball in μ, henceforth denoted by Ball $:= \{x \in \mathbb{R}^d : \mu(x) \leq 1\}$, is therefore commonly referred to as the *asymptotic shape*, and known to be compact and convex with non-empty interior. In addition, the shape retains the symmetries of $\mathbb{Z}^d$. However, little else is known regarding the properties of the shape in general. This, we shall see, is a major obstacle for our understanding of several other features of the model.

Although questions regarding geodesics were considered in the early work of Hammersley and Welsh, it took until the mid 1990s before Newman [28] together with co-authors [25, 26, 29] initiated a systematic study of the geometry of geodesics in first-passage percolation. Under the assumption of continuous weights there is almost surely a unique path attaining the minimum in (1); we shall denote this path geo(x, y) and refer to it as the *geodesic* between x and y. The graph consisting of all edges on geo$(0, y)$ for some $y \in \mathbb{Z}^d$ is a tree spanning the lattice. Understanding the properties of this object, such as the number of topological ends, leads one to the study of *infinite* geodesics, i.e. infinite paths of which every finite segment is a geodesic. We shall write $\mathscr{T}_0$ for the collection of infinite geodesics starting at the origin. A simple compactness argument shows that the cardinality $|\mathscr{T}_0|$ of $\mathscr{T}_0$ is always at least one.[1] In two dimensions, Newman [28] predicted that $|\mathscr{T}_0| = \infty$ almost surely, and proved this under an additional assumption of uniform curvature of the asymptotic shape, which remains unverified to this day.

As a means to make rigorous progress on Newman's prediction, Häggström and Pemantle [17] introduced a model for competing growth on $\mathbb{Z}^d$, for $d \geq 2$, known as the *two-type Richardson model*. In this model, two sites x and y are initially coloured red and blue respectively. As time evolves an uncoloured site turns red

[1]Consider the sequence of finite geodesics between the origin and $n\mathbf{e}_1$, where $\mathbf{e}_1$ denotes the first coordinate vector. Since the number of edges that connect to the origin is finite, one of them must be traversed for infinitely many n. Repeating the argument results in an infinite path which by construction is a geodesic.

at rate 1 times the number of red neighbours, and blue at rate λ times the number or blue neighbours. A central question of interest is for which values of λ there is positive probability for both colours to coexist, in the sense that they both are responsible for the colouring of infinitely many sites.

There is an intimate relation between the existence of infinite geodesics and coexistence in the Richardson model that we shall pay special interest in. In the case of equal strength competitors ($\lambda = 1$), one way to construct the two-type Richardson model is to equip the edges of the $\mathbb{Z}^d$ lattice with independent exponential weights, thus exhibiting a direct connection to first-passage percolation. The set of sites eventually coloured red in the two-type Richardson model is then equivalent to the set of sites closer to x than y in the first-passage metric. That is, an analogous way to phrase the question of coexistence is whether there are infinitely many points closer to x than y as well as infinitely many points closer to y than x in the first-passage metric. As before, a compactness argument will show that on the event of coexistence there are disjoint infinite geodesics g and g' that respectively originate from x and y. Häggström and Pemantle [17] showed that, for $d = 2$, coexistence of the two types occurs with positive probability, and deduced as a corollary that

$$\mathbb{P}(|\mathscr{T}_0| \geq 2) > 0.$$

Their results were later extended to higher dimensions and more general edge weight distributions in parallel by Garet and Marchand [13] and Hoffman [19]. In a later paper, Hoffman [20] showed that in two dimensions coexistence of four different types has positive probability, and that $\mathbb{P}(|\mathscr{T}_0| \geq 4) > 0$. The best currently known general lower bound on the number of geodesics is a strengthening of Hoffman's result due to Damron and Hanson [10], showing that

$$\mathbb{P}(|\mathscr{T}_0| \geq 4) = 1.$$

In this paper we shall take a closer look at the relation between existence of infinite geodesics and coexistence in competing first-passage percolation. We saw above that on the event of coexistence of various types, a compactness argument gives the existence of equally many infinite geodesics. It is furthermore conceivable that it is possible to locally modify the edge weight in such a way that these geodesics are re-routed through the origin. Conversely, interpreting infinite geodesics as 'highways to infinity', along which the different types should be able to escape their competitors, it seems that the existence of a given number of geodesics should accommodate an equal number of surviving types. These heuristic arguments suggest a duality between existence and coexistence, and it is this dual relation we shall make precise.

Given sites $x_1, x_2, \ldots, x_k$ in $\mathbb{Z}^d$, we let $\mathrm{Coex}(x_1, x_2, \ldots, x_k)$ denote the event that for every $i = 1, 2, \ldots, k$ there are infinitely many sites $z \in \mathbb{Z}^d$ for which the distance $T(x_j, z)$ is minimized by $j = i$. (The continuous weight distribution assures that there are almost surely no ties.) In two dimensions the duality between existence and coexistence that we prove takes the form:

$$\exists\, x_1, x_2, \ldots, x_k \text{ such that } \mathbb{P}\big(\text{Coex}(x_1, x_2, \ldots, x_k)\big) > 0 \quad \Leftrightarrow \quad \mathbb{P}(|\mathscr{T}_0| \geq k) > 0. \tag{2}$$

Turning the above heuristic into a proof is more demanding that it may seem. In order to derive the relation in (2) we shall rely on the recently developed ergodic theory for infinite geodesics. This theory has its origins in the work of Hoffman [19, 20], and was developed further by Damron and Hanson [10, 11], before it reached its current status in work of Ahlberg and Hoffman [1]. The full force of this theory is currently restricted to two dimensions, which prevents us from obtaining an analogue to (2) in higher dimensions. In higher dimensions we deduce a partial result based on results of Damron and Hanson [10] and Nakajima [27].

1.1 The Dual Relation

Before we state our results formally, we remind the reader that Y denotes the minimum weight among the $2d$ edges connected to the origin. We recall (from [9]) that $\mathbb{E}[Y^d] < \infty$ is both necessary and sufficient in order for the shape theorem to hold in dimension $d \geq 2$.

Theorem 1 *Consider first-passage percolation on $\mathbb{Z}^2$ with continuous edge weights satisfying $\mathbb{E}[Y^2] < \infty$. For any $k \geq 1$, including $k = \infty$, and $\varepsilon > 0$ we have:*

(i) If $\mathbb{P}\big(\text{Coex}(x_1, \ldots, x_k)\big) > 0$ for some $x_1, \ldots, x_k$ in $\mathbb{Z}^2$, then $\mathbb{P}(|\mathscr{T}_0| \geq k) = 1$.
(ii) If $\mathbb{P}(|\mathscr{T}_0| \geq k) > 0$, then $\mathbb{P}\big(\text{Coex}(x_1, \ldots, x_k)\big) > 1 - \varepsilon$ for some $x_1, \ldots, x_k$ in $\mathbb{Z}^2$.

In dimensions higher than two we shall establish parts of the above dual relation, and recall next some basic geometric concepts in order to state this result precisely. A hyperplane in the d-dimensional Euclidean space divides $\mathbb{R}^d$ into two open half-spaces. A *supporting hyperplane* to a convex set $S \subset \mathbb{R}^d$ is a hyperplane that contains some boundary point of S and contains all interior points of S in one of the two half-spaces associated to the hyperplane. It is well-known that for every boundary point of a convex set S there exists a supporting hyperplane that contains that point. A supporting hyperplane to S is called a *tangent hyperplane* if it is the unique supporting hyperplane containing some boundary point of S. Finally, we define the number of *sides* of a compact convex set S as the number of (distinct) tangent hyperplanes to S. Hence, the number of sides is finite if and only if S is a (finite) convex polygon ($d = 2$) or convex polytope ($d \geq 3$). A deeper account on convex analysis can be found in [31].

Theorem 2 *Consider first-passage percolation on $\mathbb{Z}^d$, for $d \geq 2$, with continuous edge weights. For any $k \geq 1$, including $k = \infty$, and $\varepsilon > 0$ we have*

(i) If $\mathbb{E}[\exp(\alpha\omega_e)] < \infty$ and $\mathbb{P}\big(\mathrm{Coex}(x_1, \dots, x_k)\big) > 0$ for some $\alpha > 0$ and $x_1, \dots, x_k$ in $\mathbb{Z}^d$, then $\mathbb{P}(|\mathscr{T}_0| \geq k) = 1$.
(ii) If $\mathbb{E}[Y^d] < \infty$ and Ball has at least k sides, then $\mathbb{P}\big(\mathrm{Coex}(x_1, \dots, x_k)\big) > 1 - \varepsilon$ for some $x_1, \dots, x_k$ in $\mathbb{Z}^d$.

In Sect. 2 we shall review the recent development in the study of infinite geodesics that will be essential for the deduction, in Sect. 3, of the announced dual result. Finally, in Sect. 4, we prove the partial result in higher dimensions.

1.2 A Mention of Our Methods

One aspect of the connection between existence and coexistence is an easy observation, and was hinted at already above. Namely, if $\mathrm{Geos}(x_1, x_2, \dots, x_k)$ denotes the event that there exist k pairwise disjoint infinite geodesics, each originating from one of the points $x_1, x_2, \dots, x_k$, then

$$\mathrm{Coex}(x_1, x_2, \dots, x_k) \subseteq \mathrm{Geos}(x_1, x_2, \dots, x_k). \tag{3}$$

To see this, let V_i denote the set of sites closer to x_i than to any other x_j, for $j \neq i$, in the first-passage metric. (Note that $T(x, y) \neq T(z, y)$ for all $x, y, z \in \mathbb{Z}^2$ almost surely, due to the assumptions of continuous weights.[2]) On the event $\mathrm{Coex}(x_1, x_2, \dots, x_k)$ each set V_i is infinite, and for each i a compactness argument gives the existence of an infinite path contained in V_i, which by construction is a geodesic. Since $V_1, V_2, \dots, V_k$ are pairwise disjoint, due to uniqueness of geodesics, so are the resulting infinite geodesics.

Let $\mathscr{N}$ denote the maximal number of pairwise disjoint infinite geodesics. Since $\mathscr{N}$ is invariant with respect to translations (and measurable) it follows from the ergodic theorem that $\mathscr{N}$ is almost surely constant. Hence, positive probability for coexistence of k types implies the almost sure existence of k pairwise disjoint geodesics. That $|\mathscr{T}_0| \leq \mathscr{N}$ is trivial, given the tree structure of $\mathscr{T}_0$. The inequality is in fact an equality, which was established by different means in [1, 27]. Together with (3), this resolves the first part of Theorems 1 and 2.

Above it was suggested that infinite geodesics should, at least heuristically, be thought of as 'highways to infinity' along which the different types may escape the competition. The concept of Busemann functions, and their properties, will be central in order to make this heuristic precise. These functions have their origin in the work of Herbert Busemann [7] on metric spaces. In first-passage percolation, Busemann-related limits first appeared in the work of Newman [28] as a means to

[2]This will be referred to as having *unique passage times*.

describe the microscopic structure of the boundary (or surface) of a growing ball $\{z \in \mathbb{Z}^2 : T(0, z) \leq t\}$ in the first-passage metric. Later work of Hoffman [19, 20] developed a method to describe asymptotic properties of geodesics via the study of Busemann functions. Hoffman's approach has since become indispensable in the study of various models for spatial growth, including first-passage percolation [1, 10, 11], the corner growth model [15, 16] and random polymers [2, 14]. In a tangential direction, Bakhtin et al. [5] used Busemann functions to construct stationary space-time solutions to the one-dimensional Burgers equation, inspired by earlier work of Cator and Pimentel [8].

Finally, we remark that (for $d = 2$) it is widely believed that the asymptotic shape is not a polygon, in which case it follows from [20] that both $\mathbb{P}(|\mathscr{T}_0| = \infty) = 1$ and for every $k \geq 1$ there are $x_1, x_2, \ldots, x_k$ such that $\mathbb{P}(\mathrm{Coex}(x_1, x_2, \ldots, x_k)) > 0$. The latter was extended to infinite coexistence by Damron and Hochman [12]. Thus, proving that the asymptotic shape is non-polygonal would make our main theorem obsolete. However, understanding the asymptotic shape is a notoriously hard problem, which is the reason an approach sidestepping Newman's curvature assumption has been developed in the first place.

2 Geodesics and Busemann Functions

In this section we review the recent developments in the study of infinite geodesics in first-passage percolation. We shall focus on the two-dimensional setting, and remark on higher dimensions only at the end. We make no claim in providing a complete account of previous work, and instead prefer to focus on the results that will be of significance for the purposes of this paper. A more complete description of these results, save those reported in the more recent studies [1, 27], can be found in [4].

2.1 Geodesics in Newman's Contribution to the 1994 ICM Proceedings

The study of geodesics in first-passage percolation was pioneered by Newman and co-authors [25, 26, 28, 29] in the mid 1990s. Their work gave rise to a precise set of predictions for the structure of infinite geodesics. In order to describe these predictions we shall need some notation. First, we say that an infinite geodesic $g = (v_1, v_2, \ldots)$ has *asymptotic direction* θ, in the unit circle $S^1 := \{x \in \mathbb{R}^2 : |x| = 1\}$, if the limit $\lim_{k\to\infty} v_k/|v_k|$ exists and equals θ. Second, two infinite geodesics g and g' are said to *coalesce* if their symmetrical difference $g \Delta g'$ is finite. The predictions originating from the work of Newman and his collaborators can be summarized as, under mild conditions on the weight distribution, the following should hold:

(a) with probability one, every infinite geodesic has an asymptotic direction;
(b) for every direction θ, there is an almost surely unique geodesic in $\mathscr{T}_0$ with direction θ;
(c) for every direction θ, any two geodesics with direction θ coalesce almost surely.

In particular, these statements would imply that $|\mathscr{T}_0| = \infty$ almost surely.

Licea and Newman [25, 28] proved conditional versions of these statements under an additional curvature assumption of the asymptotic shape. While this assumption seems plausible for a large family of edge weight distributions, there is no known example for which it has been verified. Rigorous proofs of the corresponding statements for a rotation invariant first-passage-like model, where the asymptotic shape is known to be a Euclidean disc, has been obtained by Howard and Newman [22]. Since proving properties like strict convexity and differentiability of the boundary of the asymptotic shape in standard first-passage percolation appears to be a major challenge, later work has focused on obtaining results without assumptions on the shape.

2.2 Busemann Functions

Limits reminiscent of Busemann functions first appeared in the first-passage literature in the work of Newman [28], as a means of describing the microscopic structure of the boundary of a growing ball in the first passage metric. The method for describing properties of geodesics via Busemann functions developed in later work of Hoffman [19, 20].

Given an infinite geodesic $g = (v_1, v_2, \ldots)$ in $\mathscr{T}_0$ we define the *Busemann function* $B_g : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$ of g as the limit

$$B_g(x, y) := \lim_{k\to\infty} \big[T(x, v_k) - T(y, v_k)\big]. \tag{4}$$

As observed by Hoffman [19], with probability one the limit in (4) exists for every $g \in \mathscr{T}_0$ and all $x, y \in \mathbb{Z}^2$, and satisfies the following properties:

- $B_g(x, y) = B_g(x, z) + B_g(z, y)$ for all $x, y, z \in \mathbb{Z}^2$;
- $|B_g(x, y)| \le T(x, y)$;
- $B_g(x, y) = T(x, y)$ for all $x, y \in g$ such that $x \in \mathrm{geo}(0, y)$.

In [19] Hoffman used Busemann functions to establish that there are at least two disjoint infinite geodesics almost surely. In [20] he used Busemann functions to associate certain infinite geodesics with sides (tangent lines) of the asymptotic shape. The approach involving Busemann functions in order to study infinite geodesics was later developed further in work by Damron and Hanson [10, 11] and Ahlberg and Hoffman [1]. Studying Busemann functions of geodesics, as opposed to the geodesics themselves, has allowed these authors to establish rigorous versions of Newman's predictions regarding the structure of geodesics. Describing parts of these results in detail will be essential in order to understand the duality between existence of geodesics and coexistence in competing first-passage percolation.

2.3 Linearity of Busemann Functions

We shall call a linear functional $\rho : \mathbb{R}^2 \to \mathbb{R}$ *supporting* if the line $\{x \in \mathbb{R}^2 : \rho(x) = 1\}$ is a supporting line to ∂Ball through some point, and *tangent* if $\{x \in \mathbb{R}^2 : \rho(x) = 1\}$ is the unique supporting line (i.e. the tangent line) through some point of ∂Ball. Given a supporting functional ρ and a geodesic $g \in \mathscr{T}_0$ we say that the Busemann function of g is *asymptotically linear* to ρ if

$$\limsup_{|y|\to\infty} \frac{1}{|y|}\big|B_g(0, y) - \rho(y)\big| = 0. \tag{5}$$

Asymptotic linearity of Busemann functions is closely related to asymptotic directions of geodesics in the sense that (5), together with the third of the properties of Busemann functions exhibited by Hoffman, provides information on the direction of $g = (v_1, v_2, \ldots)$: The set of limit points of the sequence $(v_k/|v_k|)_{k\geq 1}$ is contained in the arc $\{x \in S^1 : \mu(x) = \rho(x)\}$, corresponding to a point or a flat edge of ∂Ball.

Building on the work of Hoffman [20], Damron and Hanson [10] showed that for every tangent line of the asymptotic shape there exists a geodesic whose Busemann function is described by the corresponding linear functional. In a simplified form their result reads as follows:

Theorem 3 *For every tangent functional $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ there exists, almost surely, a geodesic in $\mathscr{T}_0$ whose Busemann function is asymptotically linear to ρ.*

While the work of Damron and Hanson proves *existence* of geodesics with linear Busemann functions, later work of Ahlberg and Hoffman [1] has established that *every* geodesic has a linear Busemann function, and that the associated linear functionals are *unique*. We summarize these results in the next couple of theorems.

Theorem 4 *With probability one, for every geodesic $g \in \mathscr{T}_0$ there exists a supporting functional $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that the Busemann function of g is asymptotically linear to ρ.*

To address uniqueness, note that the set of supporting functionals is naturally parametrized by the direction of their gradients. Due to convexity of the shape, these functionals stand in 1-1 correspondence with the unit circle S^1. We shall from now on identify the set of supporting functionals with S^1.

Theorem 5 *There exists a closed (deterministic) set $\mathscr{C} \subseteq S^1$ such that, with probability one, the (random) set of supporting functionals ρ for which there exists*