Collected Papers

1955–1966

Volume I

Bertram Kostant

Springer

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Kostant at family event, June 2002

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Springer

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ISBN 978-0-387-09582-0 e-ISBN 978-0-387-09583-7
DOI 10.1007/b94535
Springer Dordrecht Heidelberg London New York

Library of Congress Control Number: 2009926994

Mathematics Subject Classification (2000): 01A75, 32-06, 14-06

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Bertram Kostant is to be counted as one of the remarkable mathematicians of the second half of the twentieth century through his fundamental and varied contributions to many aspects of Lie theory, a subject which itself pervades almost the whole of mathematics. His work is marked by a rare simplicity and characteristic elegance, making it eminently readable, wonderfully enjoyable, and easily understandable even to a true novice. It is often through his work that one may understand many sophisticated developments of modern mathematics.

The mathematical work of Bertram Kostant has spanned well over fifty years in which he has published at an almost constant rhythm over 100 papers of which more than a few have become a cornerstone of rich and fruitful theories. Here, in briefly listing some of his most important contributions and their eventual ramifications we can give only a small glimpse into the depth and nature of his legacy.

In one of his earliest works Kostant introduced a partition function describing weight multiplicities. Significantly, it provided a tool which allowed Jantzen to compute certain fundamental determinants concerning weight spaces in Verma modules in the much more difficult parabolic case.

Kostant studied extensively the principal three-dimensional subalgebra of a semisimple Lie algebra, extracting from its adjoint action the Betti numbers of the corresponding Lie group. In addition, he thereby obtained a remarkable linearization of the fundamental invariants. Geometrically this realizes an affine slice to the regular coadjoint orbits, a result subsequently broadened notably by Luna, Slowody, and Premet, and now of greatly renewed interest.

Kostant followed this work by a further fundamental paper describing harmonic polynomials in the symmetric algebra. This gave a separation of variables theorem and a rather precise description of the nilpotent cone. These results have been key building blocks in many important results in representation theory, including the theory of primitive ideals.

Kostant discovered a remarkable connection between the regular nilpotent element and the Coxeter element in the Weyl group. This was subsequently brought to a classification of nilpotent orbits by Carter and Elkington and then elevated to a map from nilpotent orbits to conjugacy classes in the Weyl group by Kazhdan and Lusztig.

Kostant himself has returned to the many beautiful themes which may be developed from these early papers. They include an extensive study of Whittaker modules, the mathematical underpinnings of the Toda lattice which he generalized well beyond the dreams of physicists, and quantum cohomology.

Building upon Bott's geometric results, Kostant computed the Lie algebra cohomology of certain nilpotent Lie algebras arising as nilradicals of parabolic subalgebras, studying at the same time their relationship to the cohomology of the flag variety and its generalizations. This recovered the Bernstein–Gelfand–Gelfand resolution of simple finite-dimensional modules admitting far-reaching generalizations to Kac–Moody and later to Borcherds algebras. Here he inadvertently rediscovered the Amitsur–Levitski identity, so important in algebras with polynomial identity, giving it a far simpler proof and placing the result in the more general context of semisimple Lie algebras.

Kostant was one of the first to realize that a construction of Kirillov to classify unitary representations of nilpotent Lie groups implied that any coadjoint orbit admitted the structure of a symplectic variety, that is, the phase space of classical mechanics. He quickly seized upon the passage to quantum mechanics, developed by physicists, as a process of "geometric quantization." The Kostant line bundle detects which symplectic manifolds are quantizable and this result is fundamental in Hamiltonian geometry. When applied to the coadjoint orbits, quantization is designed to produce most unitary representations of the corresponding real Lie group. With Auslander he classified the unitary representations for real class 1 simply connected solvable Lie groups, laying the groundwork for the more complete theory of Pukanszky. The general case, though even more resistant, has come under the intensive study of many mathematicians, including notably Duflo, Rossmann, Vergne and Vogan. In more algebraic terms geometric quantization dramatically laid open the path to the classification of primitive ideals initiated by Dixmier and Gabriel. Further deep manifestations of geometric quantization are exemplified by the Duflo isomorphism, the Kashiwara–Vergne conjecture, and the Drinfeld associator themselves having been unified and extended following notably the powerful techniques of Kontsevitch inspired by Feynman diagrams and knot theory.

Stepping outside the purely algebraic framework Kostant discovered a far-reaching generalization of the Golden–Thompson rule in a convexity theorem subsequently generalized by Atiyah–Bott and Guillemin–Sternberg. Going beyond zero characteristic, he introduced the "Kostant form" on the enveloping algebra which has played a major role in modular representation theory. Moreover its generalization for quantum groups and the subsequent development of the quantum Frobenius map due to Lusztig allowed Littelmann to complete the Lakshmibai–Seshadri programme of describing standard monomial bases.

In a veritable "tour de force" Kostant calculated certain fundamental determinants showing them to have linear factors from which their zeros could be determined. This led to a criterion for unitarity which has so far gone unsurpassed. Subsequently, many other such factorizations were achieved, particularly that of Shapovalov and Janzten noted above; but also of Gorelik–Lanzmann which determined exactly when a Verma module annihilator for a reductive super Lie algebra was generated by its intersection with the centre.

Kostant was one of the first to firmly lay the foundations of supermanifolds and super Lie groups. He also studied super Lie algebras providing a fundamental structure theorem for their enveloping algebras. He initiated the study of the tensor product of an infinite-dimensional representation with finite-dimensional ones, leading in the hands of Jantzen to the translation principle. The latter, combined with Kostant's description of the nilpotent cone, led to the Beilinson–Bernstein equivalence of categories which heralded a new enlightenment in Verma module and Harish-Chandra theory. Kostant described (in sl(4)) a non-trivial example of a unitary highest weight module, the theory of which was subsequently completed by Enright, Howe and Wallach (and independently by Jakobsen). In the theory of characteristics developed by Guillemin, Quillen and Sternberg, a result of Kostant proves in the finite-dimensional case the involutivity of characteristics ultimately resolved in full generality by Gabber.

Among some of his important collaborative works, we mention his paper with Hochschild and Rosenberg on the differential forms on regular affine algebras, his paper with Rallis on separation of variables for symmetric spaces, his work with Kumar on the cohomology and K-theory of flag varieties associated to Kac–Moody groups, his determination with R. Brylinski of invariant symplectic structures and a uniform construction of minimal representations, and his work with Wallach on Gelfand–Zeitlin theory from the point of view of the construction of maximal Poisson commutative subalgebras particularly by "shift of argument" coming from classical mechanics.

Even in the fifth and sixth decades of his career, Kostant has continued to produce results of astonishing beauty and significance. We cite here his work on the Toda lattice and the quantum cohomology of the flag variety, his re-examination of the Clifford algebra deformation of "wedge n" with its beautiful realization of the module whose highest weight is the sum of the fundamental weights, his introduction of the cubic Dirac operator, his generalization of the Borel–Bott–Weil theorem with its connection to Euler number multiplets of representations, his work on the set of abelian ideals in the nilradical of a Borel (wonderfully classified by certain elements in the affine Weyl group by Dale Peterson), his longstanding love affair with the icosahedron, his work with Wallach mentioned above and most recently an attempt to pin down the generators of the centralizer of a maximal compact subalgebra.

After receiving his Ph.D. from the University of Chicago in 1954, Bertram Kostant began his academic career as an Assistant Professor in 1956 at the University of California, Berkeley rose to full professor in 1962 and shortly after moved to MIT. He taught at MIT until retiring in 1993.

Kostant was elected to the National Academy of Sciences U.S.A. in 1978 and has received many other honours and prizes, including election as a Sackler Institute Fellow at Tel-Aviv University in 1982, a medal from the Collège de France in 1983, the Steele prize of the American Mathematical Society in 1990, and several honorary doctorates.

Kostant maintains his physical fitness after the strains of University life by a weight lifting regime well beyond the capabilities of many younger colleagues. This may be one secret to his scientific longevity.

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April, 2008 Editors

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Acknowledgments

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Reprinted from the
TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY
Vol. 80, No. 2, pp. 528-542
November, 1955

HOLONOMY AND THE LIE ALGEBRA OF INFINITESIMAL MOTIONS OF A RIEMANNIAN MANIFOLD

BY
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Introduction. .1. Let M be a differentiable manifold of class C^∞. All tensor fields discussed below are assumed to be of class C^∞. Let X be a vector field on M. If X vanishes at a point $o \in M$ then X induces, in a natural way, an endomorphism a_X of the tangent space V_o at o. In fact if $y \in V_o$ and Y is any vector field whose value at o is y, then define $a_X y = [X, Y]_o$. It is not hard to see that $[X, Y]_o$ does not depend on Y so long as the value of Y at o is y.

Now assume M has an affine connection or as we shall do in this paper, assume M is Riemannian and that it possesses the corresponding affine connection. One may now associate to X an endomorphism a_X of V_o for any point $o \in M$ which agrees with the above definition and which heuristically indicates how X "winds around o" by defining for $v \in V_o$, $a_X v = -\nabla_v X$, where ∇_v is the symbol of covariant differentiation with respect to v.

X is called an infinitesmal motion if the Lie derivative of the metric tensor with respect to X is equal to zero. This is equivalent to the statement that a_X is skew-symmetric (as an endomorphism of V_o where the latter is provided with the inner product generated by the metric tensor) for all $o \in M$.

§1 contains definitions and formulae which will be used in the remaining sections.

In §2 we consider $\mathfrak{g}$, the space of all infinitesimal motions on M. $\mathfrak{g}$ is a finite dimensional Lie algebra under the usual bracket operation for vector fields. Each element $X \in \mathfrak{g}$ is uniquely determined over M by the value x of X and a_X at any single point $o \in M$. These known facts and a statement giving $[X, Y]_o$ and $a_{[X,Y]}$ for $X, Y \in \mathfrak{g}$ (and consequently giving the structure of $\mathfrak{g}$) in terms of x, y, a_X, a_Y and the curvature tensor at o, generalizing that given by E. Cartan when M is symmetric, are contained in Theorem 2.3 of this section.

In §3 we consider $\mathfrak{s}_o$ the Lie algebra of the restricted homogeneous holonomy group at $o \in M$. ($\mathfrak{s}_o$ is a Lie algebra of skew-symmetric endomorphisms of V_o.) We are interested in the question as to when $a_X \in \mathfrak{s}_o$ for $X \in \mathfrak{g}$. As Euclidean space clearly illustrates, this is not true in general. When $x = 0$, Lichnerowicz [12] has shown that it is true if $\mathfrak{s}_o$ acts irreducibly on V_o (M is thereby called irreducible as this is independent of o) and the Ricci tensor does not vanish. It is true under these conditions whether or not $x = 0$. It is the main result of this section that this is true ($a_X \in \mathfrak{s}_o$) whenever M is compact.

Received by the editors March 20, 1955.

B. Kostant, *Collected Papers*,
DOI 10.1007/b94535_1,

In §4 we consider $\mathfrak{h}_o$ the Lie algebra of skew-symmetric endomorphisms of V_o generated by all a_X for $X\in\mathfrak{g}$. One knows that in general $\mathfrak{h}_o\subseteq n(\mathfrak{s}_o)$ where $n(\mathfrak{s}_o)$ is the normalizer of $\mathfrak{s}_o$ in the Lie algebra, $\mathfrak{a}_o$, of all skew-symmetric endomorphisms of V_o. In this section we take up the reverse question of §3, namely, when does $\mathfrak{s}_o\subseteq\mathfrak{h}_o$. We show that this holds whenever $\mathfrak{g}$ is sufficiently big, i.e. when $\mathfrak{g}$ is transitive in the sense that for any vector $v\in V_o$ at any point $o\in M$ there exists $X\in\mathfrak{g}$ such that $x=v$. This is of course true whenever M is any open submanifold of a homogeneous space. It is then the principal result of this section that $\mathfrak{s}_o=\mathfrak{h}_o$ when M is a compact homogeneous space.

1. **Preliminaries.** .1. Let M be a Riemannian manifold of class C^∞. By this we mean that not only is M a differentiable manifold of class C^∞ but that the metric tensor is likewise of class C^∞. Similarly, without any mention to the contrary all tensor fields considered in this paper are assumed to be defined on all of M and are of class C^∞.

We adopt the following notations:

(α) o is an arbitrary point of M,

(β) V_o is the tangent space at o,

(γ) X, Y, Z are vector fields on M, and unless specified otherwise x, y, z are, respectively, the values of these vector fields at o,

(δ) S is an arbitrary tensor field and s its value at o,

(ϵ) u, v, are arbitrary vectors.

.2. For any $o\in M$ we define for the present $V_o^1=V_o$ and $V_o^{-1}=V_o^*$ (the dual of V_o) and where γ is the k-tuple $(\gamma_1,\gamma_2,\cdots,\gamma_k)$, $\gamma_i=\pm 1$, $i=1,2,\cdots,n$, define $V_o^\gamma=V_o^{\gamma_1}\otimes\cdots\otimes V_o^{\gamma_k}$. Then the direct sum

$$\mathfrak{T}(V_o)=\sum_\gamma V_o^\gamma$$

is of course the mixed tensor algebra at $o\in M$.

If $\mathfrak{T}$ is the algebra of all mixed tensor fields on M let us consider any linear transformation D_o from $\mathfrak{T}$ into $\mathfrak{T}(V_o)$ which satisfies

(1) $D_o(S_1S_2)=s_1D_o(S_2)+D_o(S_1)s_2$,

(2) D_o preserves tensor type,

(3) $D_o(S^C)=D_o(S)^C$

(where C designates any specific contraction). We shall call such a linear transformation a $\mathfrak{T}$-differentiation at $o\in M$.

An example of a $\mathfrak{T}$-differentiation at o, with which we shall be concerned, is given by covariant differentiation ∇_v of $\mathfrak{T}$ by a vector $v\in V_o$.

Where f is a function $\nabla_v f$ is just vf, i.e. differentiation of f by the vector v. However, any $\mathfrak{T}$-differentiation at o defines a differentiation, at $o\in M$, of the space of functions of class C^∞ on M and hence there exists a vector $v\in V_o$ such that $D_o-\nabla_v$ vanishes on all such functions. If we then apply $D_o-\nabla_v$ to a vector field X we see easily that $(D_o-\nabla_v)X$ depends only on x so that $D_o-\nabla_v$ defines an endomorphism a on V_o. Similarly, it defines an endomorphism on V_o^* which by (3) is simply $-a^*$ (where a^* designates the transpose of a on V_o^*). Moreover, letting $a_1=a$ and $a_{-1}=-a^*$ for the present, it follows that $D_o-\nabla_v$ applied to any tensor field S of type $\gamma=(\gamma_1, \gamma_2, \cdots, \gamma_k)$ is equal to

$$a_\gamma = \sum_{i=1}^{k} 1 \otimes 1 \otimes \cdots \otimes a_{\gamma_i} \otimes \cdots \otimes 1 \quad (a_{\gamma_i} \text{ the } i\text{th term, } k = \text{number of terms})$$

applied to s. Conversely, given any endomorphism a on V_o we may consider the endomorphism a_γ of V_o defined in just this way. Furthermore if S is any tensor field decomposing into the sum $S=\sum_\gamma S^\gamma$ according to type and we define $\nu(a)S=\sum_\gamma a_\gamma s^\gamma$ we see that $\nu(a)$ is just a $\mathfrak{T}$-differentiation at o. Thus we may associate to every endomorphism a of V_o a $\mathfrak{T}$-differentiation $\nu(a)$ at o. In the future we shall drop the symbol ν in $\nu(a)$ and simply understand that any endomorphism a of V by an extension of its domain may be applied to any tensor field in the manner discussed above. It follows then that any $\mathfrak{T}$-differentiation D_o at $o\in M$ may be uniquely written as

$$D_o = \nabla_v + a$$

for some vector $v\in V_o$ and some endomorphism a of V_o and that every such sum is a $\mathfrak{T}$-differentiation. We see then that a $\mathfrak{T}$-differentiation is determined once we know how it acts on functions and vector fields.

.3. If X is a vector field we shall let ∇_X be the derivation of $\mathfrak{T}$ defined by setting $\nabla_X S$ equal to $\nabla_x S$ at any point $o\in M$.

Its action on functions already given, the definition of covariant differentiation on M is uniquely determined in that it satisfies for any two vector fields X and Y the equations

$$v(X, Y) = (\nabla_v X, Y) + (X, \nabla_v Y), \tag{1.3.1}$$

$$[X, Y] = \nabla_X Y - \nabla_Y X \tag{1.3.2}$$

where (,) designates the inner product in the tangent space at any point given by the metric tensor at the point and $[X, Y]$ is just the Poisson bracket of the vector fields X and Y.

(1.3.1) expresses the fact that the metric tensor has covariant derivative zero and (1.3.2) the fact that the torsion tensor is zero.

.4. Now if X and Y are any two vector fields, then it is easy to see that

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

defines, at each point $o\in M$, a $\mathfrak{T}$-differentiation. Moreover this $\mathfrak{T}$-differentiation vanishes on functions and thus is given by an endomorphism of V_o. Depending only on x and y this endomorphism, $t(x, y)$, is well known to be the contraction of the curvature tensor, t, at o on the last two indices by x and y. Letting T be the curvature tensor field and contracting in the same way with X and Y we thus have

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = T(X, Y) \tag{1.4.1}$$

as a field of tangent space endomorphisms.

Besides being skew-symmetric in u and v the endomorphism $t(u, v)$ satisfies the first Bianchi identity

$$t(u, v)w + t(v, w)u + t(w, u)v = 0. \tag{1.4.2}$$

.5. We shall call an endomorphism a of V_o symmetric (resp. skew-symmetric) if a satisfies $(au, v) = (u, av)$ (resp. $(au, v) = -(u, av)$) for all $u, v \in V_o$, i.e. if the second order covariant tensor obtained from a by lowering its upper index is symmetric (resp. skew-symmetric).

It is well known that $t(u, v)$ is skew-symmetric.

The final relation involving the curvature tensor which we shall make use of in this paper is the second Bianchi identity,

$$[\nabla_u T](v, w) + [\nabla_v T](w, u) + [\nabla_w T](u, v) = 0. \tag{1.5.1}$$

The contraction again being on the last two indices.

The above identities are proved, for example, in [13].

2. **A structure theorem.** .1. Let X be any vector field. We may associate to X a field of tangent space endomorphisms A_X, i.e. a tensor field of one upper index and one lower index, in the following way: For $v \in V_o$ we define the value a_X of A_X at o by

$$a_X v = - \nabla_v X. \tag{2.1.1}$$

Consider the derivation $\nabla_X + A_X$ of $\mathfrak{T}$. Clearly

$$(\nabla_X + A_X)f = Xf$$

where f is a function.

$$\begin{aligned}(\nabla_X + A_X)Y &= \nabla_X Y + A_X Y \\ &= \nabla_X Y - \nabla_Y X = [X, Y]\end{aligned}$$

by (1.3.2).

However if L_X is the derivation of $\mathfrak{T}$ corresponding to Lie differentiation with respect to X, then at each point L_X defines a $\mathfrak{T}$-differentiation. Moreover,

$$L_X f = Xf$$

for f a function and

$$L_X Y = [X, Y]$$

for Y a vector field. Thus by §1.2

$$L_X = \nabla_X + A_X. \tag{2.1.2}$$

It is well known that for X and Y

$$[L_X, L_Y] = L_{[X,Y]}. \tag{2.1.3}$$

In fact this follows easily since $[L_X, L_Y]-L_{[X,Y]}$ defines a $\mathfrak{T}$-differentiation at each point which vanishes on functions and by the Jacobi identity vanishes on vector fields and hence is zero.

The relation (2.1.3) enables us to give an expression for $A_{[X,Y]}$ in terms of A_X and A_Y. Indeed by (2.1.2) and (2.1.3)

$$[\nabla_X + A_X, \nabla_Y + A_Y] = \nabla_{[X,Y]} + A_{[X,Y]}$$

or

$$[\nabla_X, \nabla_Y] + [\nabla_X, A_Y] - [\nabla_Y, A_X] + [A_X, A_Y] = \nabla_{[X,Y]} + A_{[X,Y]}.$$

But

$$[\nabla_X, A_Y] = \nabla_X(A_Y),$$
$$[\nabla_Y, A_X] = \nabla_Y(A_X)$$

and since

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = T(X, Y)$$

we have

$$T(X, Y) + \nabla_X(A_Y) - \nabla_Y(A_X) + [A_X, A_Y] = A_{[X,Y]}. \tag{2.1.4}$$

.2. A vector field X is called an infinitesimal motion or Killing vector if the Lie derivative of the metric tensor with respect to X is equal to 0. This is equivalent to saying that a_X is skew-symmetric at all points $o \in M$, or as we shall say A_X is skew-symmetric.

Since it is trivial to show that if A_X is skew-symmetric then $\nabla_v(A_X)$ is likewise skew-symmetric it follows from (2.1.4) that if X and Y are infinitesimal motions then so is $[X, Y]$. Thus if $\mathfrak{g}$ is the space of all infinitesimal motions on M, $\mathfrak{g}$ becomes a Lie algebra under the Poisson bracket. $\mathfrak{g}$, as is well known, and which will follow later, is finite dimensional; however, finite dimensionality is not understood here to be a requirement for the definition of a Lie algebra.

It is a known fact (see [13, p. 45]) that if X is a vector field then the endomorphism

$$\nabla_v(A_X) - t(x, v)$$

of V_o is the value at o of the Lie derivative, by X, of the affine connection contracted by $v \in V_o$. Since if $X \in \mathfrak{g}$ the Lie derivative of the affine connection is zero it follows of course that for $X \in \mathfrak{g}$ the above expression vanishes.

For completeness we shall give a proof of this which does not involve a computation of the Lie derivative of the affine connection. We state it in the following form.

LEMMA 2.2. *Let X be a vector field. Then at any point $o \in M$*, $\nabla_v(A_X)$

$=t(x, v)$ *for all* $v \in V_o$ *if and only if* $\nabla_v(A_X)$ *is skew-symmetric for all* $v \in V_o$.

Proof. It is of course trivial that if the equality holds then $\nabla_v(A_X)$ is skew-symmetric for all $v \in V_o$.

Let X be arbitrary. We first show that the vector

$$\nabla_y(A_X)z - t(x, y)z$$

as a bilinear function of y and z is symmetric in y and z. In fact embed y and z in the respective vector fields Y and Z.

Now

$$\begin{aligned}\nabla_Y(A_X)Z &= [\nabla_Y, A_X]Z \\ &= \nabla_Y A_X Z - A_X \nabla_Y Z = - \nabla_Y \nabla_Z X - A_X \nabla_Y Z.\end{aligned}$$

Then

$$\begin{aligned}\nabla_Y(A_X)Z - \nabla_Z(A_X)Y &= [\nabla_Z, \nabla_Y]X + A_X[Z, Y] \quad \text{(by (1.3.2))} \\ &= ([\nabla_Z, \nabla_Y] - \nabla_{[Z,Y]})X \\ &= T(Z, Y)X \quad \text{(by (1.4.1))}.\end{aligned}$$

But by (1.4.2)

$$t(z, y)x = t(x, y)z - t(x, z)y.$$

Thus we have the symmetry in y and z expressed in the relation

$$\nabla_y(A_x)z - t(x, y)z = \nabla_z(A_x)y - t(x, z)y.$$

Let

$$\phi(v, u, w) = (\nabla_v(A_X)u, w) - (t(x, v)u, w).$$

Then ϕ is symmetric in u and v. If we now make the assumption that $\nabla_v(A_X)$ is skew-symmetric for all $v \in V_o$, then $\phi(v, u, w)$ is skew-symmetric in u and w. But any trilinear form symmetric in two variables and skew-symmetric in two others is necessarily identically zero. Thus

$$\nabla_v(A_X) = t(x, v). \qquad \text{Q.E.D.}$$

.3. If we consider the question as to how much information is yielded about $X \in \mathfrak{g}$ by knowledge of x and a_X, we are led, heuristically at any rate, to thinking of X decomposing into a sum of an infinitesimal translation at o in the direction of x and an infinitesimal rotation about o given in the tangent space of o by a_X. Indeed if we consider for the moment the case where M is Euclidean n-space with the usual metric, and we let o be the origin, we know that $\mathfrak{g}$ decomposes into the direct sum $\mathfrak{g} = \mathfrak{a} + V$ where $\mathfrak{a}$ is the subalgebra of $\mathfrak{g}$ composed of all elements in $\mathfrak{g}$ vanishing at o and V is the commutative subalgebra of all infinitesimal translations. Now if we let $\mathfrak{a}_o$ be the Lie algebra of all skew-symmetric endomorphisms of V_0 and we define $\mathfrak{g}_o = \mathfrak{a}_o + V_o$ and introduce a bracket operation into $\mathfrak{g}_o$ by setting

$$[a_1, a_2] = a_1a_2 - a_2a_1 \in \mathfrak{a}_o, [a, v] = a(v) \in V_o,$$
$$[v, a] = -a(v) \in V_o, [v_1, v_2] = 0$$

for $a, a_1, a_2 \in \mathfrak{a}_o$; $v, v_1, v_2 \in V_o$, then $\mathfrak{g}_o$ is a Lie algebra and $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_o$ under the mapping which sends X into $a_X + x$. This mapping of course sends $\mathfrak{a}$ into $\mathfrak{a}_o$ and V into V_o.

More generally it was shown by E. Cartan that if M is a simply connected Riemannian symmetric space and o is any point of M, then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}_o = \mathfrak{k}_o + V_o$ where $\mathfrak{k}_o$ is the subalgebra of all skew-symmetric endomorphisms a of V_o which leave invariant the curvature tensor at $o \in M$ (infinitesimally, i.e., $aT=0$). The bracket relation in $\mathfrak{g}_o$ is introduced in the same way as before with respect to $[\mathfrak{k}_o, \mathfrak{k}_o]$, $[\mathfrak{k}_o, V_o]$, $[V_o, \mathfrak{k}_o]$, but for $[V_o, V_o]$ we now have more generally $[V_o, V_o] \subseteq \mathfrak{k}_o$ where

$$[v_1, v_2] = t(v_2, v_1) \qquad \text{for } v_1, v_2 \in V_o.$$

The isomorphism between $\mathfrak{g}$ and $\mathfrak{g}_o$ is again established by mapping X into $a_X + x$.

We return now to the case where M is perfectly general. Let $\mathfrak{a}_o$ be all skew-symmetric endomorphisms of V_o. Let $\hat{\mathfrak{g}}_o = \mathfrak{a}_o + V_o$. We introduce into $\hat{\mathfrak{g}}_o$ a bracket operation where

$$[a_1, a_2] = a_1a_2 - a_2a_1, [a, v] = a(v),$$
$$[v, a] = -a(v), [v_1, v_2] = t(v_2, v_1)$$

for $a, a_1, a_2 \in \mathfrak{a}_o$ and $v, v_1, v_2 \in V_o$. $\hat{\mathfrak{g}}_o$ is not in general a Lie algebra under this bracket (the Jacobi identity fails in the general case) but of course subalgebras of $\hat{\mathfrak{g}}_o$ may be Lie algebras. We have seen that in the symmetric case $\mathfrak{g}$ is isomorphic to a subalgebra $\mathfrak{g}_o$ of $\hat{\mathfrak{g}}_o$ under the mapping $X \rightarrow a_X + x$. The following theorem states that this is true in general.

THEOREM 2.3. *Let $o \in M$. Let $\hat{\mathfrak{g}}_o = \mathfrak{a}_o + V_o$ be the algebra defined in §2.3. Let $\mathfrak{g}$ be the Lie algebra of infinitesimal motions on M. Let $\theta_o: \mathfrak{g} \rightarrow \hat{\mathfrak{g}}_o$ be the mapping defined by*

$$\theta_o(X) = X_o = a_X + x$$

and $\mathfrak{g}_o = \theta_o(\mathfrak{g})$; then θ_o is an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}_o$.

Proof. Implicit in the statement of the theorem is that, first of all, θ_o is one-one. This is equivalent to the statement that an infinitesimal motion X is uniquely determined over all of M by the value of A_X and X at any one point. To see this, let M and let $r \rightarrow o(r)$ be a differentiable curve, $1 \leqq r \leqq 2$, connecting o_1 with o_2. Then if $a_X(r)$ and $x(r)$ are respectively the values of A_X and X respectively at $o(r)$, and $v(r)$ is the tangent vector to the curve at $o(r)$, then it follows from (2.1.1) and Lemma 2.2 that the pair $x(r)$ and $a_X(r)$ satisfy the linear differential equations

(2.3.1) $$\nabla_{v(r)}x(r) = -a_X(r)v(r),$$

(2.3.2) $$\nabla_{v(r)}a_X(r) = t(x(r), v(r))$$

so that a knowledge of them at any one point of the curve determines their value at any other.

To show θ_o is a homomorphism, let $X, Y \in \mathfrak{g}$ and let $X_o = \theta_o X$, $Y_o = \theta_o Y$. Now by (2.1.4)

$$T(X, Y) + \nabla_X(A_Y) - \nabla_Y(A_X) + [A_X, A_Y] = A_{[X,Y]}.$$

But since X and Y are infinitesimal motions we have by Lemma 2.2 that

$$\nabla_X(A_Y) = T(Y, X),$$
$$\nabla_Y(A_X) = T(X, Y).$$

Thus

(2.3.3) $$A_{[X,Y]} = T(Y, X) + [A_X, A_Y].$$

Moreover by (1.3.2)

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

or by (2.1.1)

$$[X, Y] = A_X Y - A_Y X.$$

Hence, by evaluating $[X, Y]$ and $A_{[X,Y]}$ at o we see that

$$\theta_o[X, Y] = [X_o, Y_o].$$ Q.E.D.

Note. With respect to the curve $r \to o(r)$ mentioned in the proof of Theorem 2.3 and by means of the linear differential equations (2.3.1) and (2.3.2) we obtain, by solving these equations, a linear one-one mapping from $\hat{\mathfrak{g}}_{o_1}$ to $\hat{\mathfrak{g}}_{o_2}$. If now $o_1 = o_2 = o$ it is clear that the elements of $\mathfrak{g}_o$ are pointwise fixed. In fact it is not hard to see that $\mathfrak{g}_o$ is characterized as the set of those elements in $\hat{\mathfrak{g}}_o$ which remain invariant under all mappings of $\hat{\mathfrak{g}}_o$ into itself obtained in this way for all closed differentiable curves passing through o.

For any $X_o \in \hat{\mathfrak{g}}_o$, $X_o = a + x$, we define l_{X_o} to be the $\mathfrak{T}$-differentiation $\nabla_x + a$ at o. The question as to whether the Jacobi identity is satisfied for three elements X_o, Y_o and $Z_o \in \hat{\mathfrak{g}}_o$ is, by using the second Bianchi identity (1.5.1), seen to be equivalent to the question as to whether

$$(l_{X_o}T)(y, z) + (l_{Y_o}T)(z, x) + (l_{Z_o}T)(x, y)$$

vanishes. (The contraction is again on the last two indices.) In case $X_o = \theta_o X$ for some vector field X, then l_{X_o} is by (2.1.2) just the Lie derivative L_X evaluated at o. If therefore $X \in \mathfrak{g}$ it follows that $l_{X_o}T = 0$ since the curvature tensor is invariant under motions.

3. **Holonomy and infinitesimal motions.** .1. Let ψ_o be the homogeneous holonomy group at $o \in M$. ψ_o is a group of orthogonal endomorphisms of the

tangent space V_o, orthogonal with respect to the inner product in V_o. We recall, by definition, that an orthogonal endomorphism α belongs to ψ_o if the endomorphism, $\alpha: V_o \rightarrow V_o$, is induced by transporting (by parallelism) V_o completely around a closed piecewise differentiable curve on M. It returns to its initial position only after being subjected to the rotation α.

The restricted homogeneous holonomy group σ_o at o is a normal subgroup of ψ_o and is obtained in the same way as ψ_o except that the curves are restricted to be homotopic to zero. It is a known result [3] that σ_o as a group of orthogonal endomorphisms is compact and connected and is in fact the connected component of the identity in ψ_o.

.2. Let $\mathfrak{s}_o$ be the Lie algebra of σ_o. $\mathfrak{s}_o$ is of course a subalgebra of $\mathfrak{a}_o$. Consider, in $\mathfrak{a}_o$, the inner product,

$$B(a_1, a_2) = \text{trace } a_1 a_2. \tag{3.2.1}$$

B is clearly negative definite. Let $\mathfrak{u}_o$ be the orthocomplement of $\mathfrak{s}_o$ in $\mathfrak{a}_o$ with respect to B.

Let $\mathfrak{k}^o$ be the set of $X \in \mathfrak{g}$ such that x vanishes and let $\mathfrak{k}_o = \theta_o \mathfrak{k}^o$. $\mathfrak{k}^o$ is a subalgebra of $\mathfrak{g}$; in fact $\mathfrak{k}_o$ is a subalgebra of $\mathfrak{a}_o$. If K^o is the group of all motions of M which keep o fixed then its Lie algebra, represented as vector fields on M, is a subalgebra of $\mathfrak{k}^o$ so that it is natural to call $\mathfrak{k}^o$ the isotropy algebra at o and $\mathfrak{k}_o$ the linear isotropy algebra at o.

It is a simple fact that if $a \in \mathfrak{k}_o$ then a lies in the normalizer of $\mathfrak{s}_o$. It is not much harder to show this is true for any a_X where $X \in \mathfrak{g}$. A more general statement is derived as an easy consequence of the next lemma; it is more general in that it deals with ψ_o instead of σ_o or $\mathfrak{s}_o$.

For any $X \in \mathfrak{g}$ we consider the decomposition of a_X,

$$a_X = b_X + e_X, \tag{3.2.2}$$

where $b_X \in \mathfrak{s}_o$ and $e_X \in \mathfrak{u}_o$. Let E_X (resp. B_X) be the field of tangent space endomorphisms which at any point o takes the value e_X (resp. b_X).

Lemma 3.2. *The field E_X is covariant constant.*

Proof. We use in the proof two basic facts about holonomy; (1) if $o(r)$, $r_1 \leqq r \leqq r_2$, is a differentiable curve then $\mathfrak{s}_{o(r_1)}$ as a subspace of $\mathfrak{a}_{o(r_1)}$ is carried into $\mathfrak{s}_{o(r_2)}$ under parallel transport along the curve, and (2) at any point o and for any $u, v \in V_o$, $t(u, v) \in \mathfrak{s}_o$, see, for example, [1].

But from (1) it follows that the orthocomplement $\mathfrak{u}_{o(r_1)}$ of $\mathfrak{s}_{o(r_1)}$ is carried into $\mathfrak{u}_{o(r_2)}$ under parallel transport along the curve. Thus to prove the lemma it is sufficient to show the covariant derivative of A_X by any vector $v \in V_o$ lies in $\mathfrak{s}_o$. Now by Lemma 2.2,

$$\nabla_v(A_X) = t(x, v) \in \mathfrak{s}_o.$$

Thus we have shown $\nabla_v(E_X) = 0$. Q.E.D.

.3. We are interested in the question as to when $a_X \in \mathfrak{g}_o$, i.e. when $e_X = 0$, or equivalently, $E_X = 0$.

It follows from a result of Lichnerowicz [12] that this is the case when the Ricci tensor is not zero and $\mathfrak{g}_o$ acts irreducibly on V_o. In fact that author uses his result to prove that $a_X \in \mathfrak{g}_o$ when $X \in \mathfrak{k}^o$ under those assumptions. For completeness we repeat his argument (slightly modified) here. Assume $E_X \neq 0$. Since $\mathfrak{g}_o$ acts irreducibly it follows that M is pseudo-Kahlerian. The result in [12] alluded to above is that if M is pseudo-Kahlerian and the Ricci tensor is not zero then $\mathfrak{g}_o$ has a nonzero center. But if $c \in \mathfrak{g}_o$ is in the center then c, e_X, and the identity endomorphism generate a three-dimensional commutative algebra in the centralizer of $\mathfrak{g}_o$. Since the latter acts irreducibly this contradicts Schur's lemma.

Of course we know that in general $a_X \notin \mathfrak{g}_o$, as Euclidean space clearly illustrates. We shall show, however, that $a_X \in \mathfrak{g}_o$ whenever M is compact.

THEOREM 3.3. *Let M be a compact Riemannian manifold, X an infinitesimal motion on M, and a_X the endomorphism of the tangent space V_o at an arbitrary point $o \in M$ defined by $a_X v = -\nabla_v X$ for any $v \in V_o$. Then $a_X \in \mathfrak{g}_o$, the Lie algebra of the restricted homogeneous holonomy group at o.*

Proof. We use the notation of §3.2. We shall show $E_X = 0$. Let Y be a vector field defined by

$$Y = E_X X.$$

That is, E_X is applied to X at every point. For any $v \in V_o$

$$\nabla_v Y = E_X \nabla_v X \qquad \text{(by Lemma 3.2).}$$

Thus

$$a_Y v = e_X a_X v$$

and consequently

$$A_Y = E_X A_X.$$

Assume for the present that M is orientable. It is just the statement of Green's theorem, for example see [13, p. 31] that if Z is any vector field on M then

$$\int \mathrm{tr}\, A_Z dv = 0$$

where dv is the volume element on M associated with the metric.

Thus

$$\int \mathrm{tr}\, A_Y dv = \int \mathrm{tr}\, [E_X^2 + E_X B_X] dv = 0.$$

But tr $E_X B_X = 0$ at every point o by definition of $\mathfrak{u}_o$. Thus