Developments in Mathematics

Teresa W. Haynes
Stephen T. Hedetniemi
Michael A. Henning *Editors*

Structures of Domination in Graphs

Springer

Developments in Mathematics

Volume 66

Aims and Scope

The Developments in Mathematics (DEVM) book series is devoted to publishing well-written monographs within the broad spectrum of pure and applied mathematics. Ideally, each book should be self-contained and fairly comprehensive in treating a particular subject. Topics in the forefront of mathematical research that present new results and/or a unique and engaging approach with a potential relationship to other fields are most welcome. High-quality edited volumes conveying current state-of-the-art research will occasionally also be considered for publication. The DEVM series appeals to a variety of audiences including researchers, postdocs, and advanced graduate students.

More information about this series at http://www.springer.com/series/5834

Teresa W. Haynes • Stephen T. Hedetniemi
Michael A. Henning
Editors

Structures of Domination in Graphs

Springer

Editors
Teresa W. Haynes
Department of Mathematics and Statistics
East Tennessee State University
Johnson City, TN, USA

Department of Mathematics
and Applied Mathematics
University of Johannesburg
Johannesburg, South Africa

Michael A. Henning
Department of Mathematics
and Applied Mathematics
University of Johannesburg
Johannesburg, South Africa

Stephen T. Hedetniemi
School of Computing
Clemson University
Clemson, SC, USA

ISSN 1389-2177 ISSN 2197-795X (electronic)
Developments in Mathematics
ISBN 978-3-030-58891-5 ISBN 978-3-030-58892-2 (eBook)
https://doi.org/10.1007/978-3-030-58892-2

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

While concepts related to domination in graphs can be traced back to the mid-1800s in connection to various chessboard problems, domination was first defined as a graph theoretical concept in 1958. Domination in graphs has experienced rapid growth from its introduction, resulting in over 1200 papers published on domination in graphs by the late 1990s.

Noting the need for a comprehensive survey of the literature on domination in graphs, in 1998 Haynes, Hedetniemi, and Slater published the first two books on domination, *Fundamentals of Domination in Graphs* and *Domination in Graphs: Advanced Topics*. We refer to these books as Books I and II.

The explosive growth of this field since 1998 has continued, and today more than 4,000 papers have been published on domination in graphs, and the material in Books I and II is now more than 20 years old. Thus, the authors feel it is time for an update on the developments in domination theory since 1998. We also want to give a comprehensive treatment of only the major topics in domination. This coverage of domination, including the major results and updates, will be in the form of three books, which we call Books III, IV, and V.

Book III, *Domination in Graphs: Core Concepts*, is written by the authors and concentrates, as the title suggests, on the three main types of domination in graphs: domination, independent domination, and total domination. It contains major results on these basic domination numbers, including proofs of selected results that illustrate many of the proof techniques used in domination theory.

For the companion books, Books IV and V, we invited leading researchers in domination to contribute chapters.

Book IV concentrates on the most-studied types of domination that are not covered in Book III. Although well over 70 types of domination have been defined, Book IV focuses on those that have received the most attention in the literature, and contains chapters on paired domination, connected domination, restrained domination, multiple domination, distance domination, dominating functions, fractional dominating parameters, Roman domination, rainbow domination, locating-domination, eternal and secure domination, global domination, stratified domination, and power domination.

The present volume, Book V, is divided into three parts. The first part focuses on several domination-related concepts: broadcast domination, alliances, domatic numbers, dominator colorings, irredundance in graphs, private neighbor concepts, game domination, varieties of Roman domination, and spectral graph theory. The second part covers domination in (i) hypergraphs, (ii) chessboards, and (iii) digraphs and tournaments. The third part focuses on the development of algorithms and complexity of (i) signed, minus, and majority domination, (ii) power domination, and (iii) alliances in graphs. The third part also includes a chapter on self-stabilizing domination algorithms.

The authors of the chapters in Book V provide a survey of known results with a sampling of proof techniques in their areas of expertise. To avoid excessive repetition of definitions and notation, Chapter 1 provides a glossary of commonly used terms.

This book is intended as a reference resource for researchers and is written to reach the following audiences: first, established researchers in the field of domination who want an updated, comprehensive coverage of domination theory; second, researchers in graph theory who wish to become acquainted with newer topics in domination, along with major developments in the field and some of the proof techniques used; and third, graduate students with interests in graph theory, who might find the theory and many real-world applications of domination of interest for master's and doctoral theses topics. We also believe that Book V provides a good focus for use in a seminar on either domination theory or domination algorithms and complexity, including the new algorithm paradigm of self-stabilizing domination algorithms.

We wish to thank the authors who contributed chapters to this book as well as the reviewers of the chapters.

Johnson City, TN, USA — Teresa W. Haynes
Clemson, SC, USA — Stephen T. Hedetniemi
Johannesburg, South Africa — Michael A. Henning

Contents

Part II Domination in Selected Graph Families

Part III Algorithms and Complexity

Glossary of Common Terms

Teresa W. Haynes, Stephen T. Hedetniemi, and Michael A. Henning

1 Introduction

It is difficult to say when the study of domination in graphs began, but for the sake of this glossary let us say that it began in 1962 with the publication of Oystein Ore's book *Theory of Graphs* [15]. In *Chapter 13 Dominating Sets, Covering Sets and Independent Sets* of [15], we see for the first time the name *dominating set*, defined as follows: "A subset D of V is a *dominating set* for G when every vertex not in D is the endpoint of some edge from a vertex in D." Ore then defines the *domination number*, denoted $\delta(G)$, of a graph G, as "the smallest number of vertices in any minimal dominating set." So, at this point, and for the first time, domination has a "name" and a "number."

Of course, prior to this Claude Berge [3], in his book *Theory of Graphs and its Applications*, which was first published in France in 1958 by Dunod, Paris,

T. W. Haynes (✉)
Department of Mathematics and Statistics, East Tennessee State University, Johnson City, TN, USA

Department of Mathematics and Applied Mathematics, University of Johannesburg, Johannesburg, South Africa
e-mail: haynes@etsu.edu

S. T. Hedetniemi
School of Computing, Clemson University, Clemson, SC, USA
e-mail: hedet@clemson.edu

M. A. Henning
Department of Mathematics and Applied Mathematics, University of Johannesburg, Johannesburg, South Africa
e-mail: mahenning@uj.ac.za

T. W. Haynes et al. (eds.), *Structures of Domination in Graphs*, Developments in Mathematics 66, https://doi.org/10.1007/978-3-030-58892-2_1

had previously defined the same concept, but had, in *Chapter 4 The Fundamental Numbers of the Theory of Graphs* of [3], given it the name "the coefficient of external stability."

Before Berge, Dénes König, in his 1936 book *Theorie der Endlichen und Unendlichen Graphen* [13], had defined essentially the same concept, but in *VII Kapitel, Basisproblem für gerichtete Graphen*, König gave it the name "punktbasis," which we would today say is an independent dominating set.

And even before König, in the books by Dudeney in 1908 [8] and W. W. Rouse Ball in 1905 [2], one can find the concepts of domination, independent domination, and total domination discussed in connection with various chessboard problems. And it was Ball who, in turn, credited such people as W. Ahrens in 1910 [1], C. F. de Jaenisch in 1862 [7], Franz Nauck in 1850 [14], and Max Bezzel in 1848 [4] for their contributions to these types of chessboard problems involving dominating sets of chess pieces.

But it was Ore who gave the name *domination* and this name took root. Not long thereafter, Cockayne and Hedetniemi [6] gave the notation $\gamma(G)$ for the domination number of a graph, and this also took root and is the notation adopted here.

Since the subsequent chapters in this book will deal with domination parameters, there will be much overlap in the terminology and notation used. One purpose of this chapter is to present definitions common to many of the chapters in order to prevent terms being defined repeatedly and to avoid other redundancy. Also, since graph theory terminology and notation sometimes vary, in this glossary we clarify the terminology that will be adopted in subsequent chapters.

We proceed as follows. In Section 2.1, we present basic graph theory definitions. We discuss common types of graphs in Section 2.2. Some fundamental graph constructions are given in Section 2.3. In Section 3.1 and Section 3.2, we present parameters related to connectivity and distance in graphs, respectively. The covering, packing, independence, and matching numbers are defined in Section 3.3. Finally in Section 3.4, we define selected domination-type parameters that will occur frequently throughout the book.

For more details and terminology, the reader is referred to the two books *Fundamentals of Domination in Graphs* [10] and *Domination in Graphs, Advanced Topics* [11], written and edited by Haynes, Hedetniemi, and Slater and the book *Total Domination in Graphs* by Henning and Yeo [12]. An annotated glossary, from which many of the definitions in this chapter are taken, was produced by Gera, Haynes, Hedetniemi, and Henning in 2018 [9].

2 Basic Terminology

In this section, we give basic definitions, common types of graphs, and fundamental graph constructions.

2.1 Basic Graph Theory Definitions

Before we proceed with our glossary of parameters, we need to define a few basic terms, which are used in the definitions in the following subsections. For an integer $k \geq 1$, we use the standard notation $[k] = \{1, \ldots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \ldots, k\}$.

A (finite, undirected) *graph* $G = (V, E)$ consists of a finite nonempty set of *vertices* $V = V(G)$ together with a set $E = E(G)$ of unordered pairs of distinct vertices called *edges*. Each edge $e = \{u, v\}$ in E is denoted with any of e, uv, vu, and $\{u, v\}$. We say that a graph G has *order* $n = |V|$ and *size* $m = |E|$.

Two vertices u and v in G are *adjacent* if they are joined by an edge e, that is, u and v are adjacent if $e = uv \in E(G)$. In this case, we say that each of u and v is *incident* with the edge e. Further, we say that the edge e *joins* the vertices u and v. Two edges are *adjacent* if they have a vertex in common. Two vertices in a graph G are *independent* if they are not adjacent. A set of pairwise independent vertices in G is an *independent set* of G. Similarly, two edges are *independent* if they are not adjacent.

A *neighbor* of a vertex v in G is a vertex u that is adjacent to v. The *open neighborhood* of a vertex v in G is the set of neighbors of v, denoted $N_G(v)$. Thus, $N_G(v) = \{u \in V : uv \in E\}$. The *closed neighborhood of* v is the set $N_G[v] = \{v\} \cup N_G(v)$. For a set of vertices $S \subseteq V$, the *open neighborhood* of S is the set $N_G(S) = \bigcup_{v \in S} N_G(v)$ and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we omit it in the above expressions. For example, we write $N(v)$, $N[v]$, $N(S)$, and $N[S]$ rather than $N_G(v)$, $N_G[v]$, $N_G(S)$, and $N_G[S]$, respectively.

For a set of vertices $S \subseteq V$ and a vertex v belonging to the set S, the *S-private neighborhood* of v is defined by $\text{pn}[v, S] = \{w \in V : N_G[w] \cap S = \{v\}\}$, while its *open S-private neighborhood* is defined by $\text{pn}(v, S) = \{w \in V : N_G(w) \cap S = \{v\}\}$. As remarked in [12], the sets $\text{pn}[v, S] \setminus S$ and $\text{pn}(v, S) \setminus S$ are equivalent and we define the *S-external private neighborhood* of v to be this set, abbreviated $\text{epn}[v, S]$ or $\text{epn}(v, S)$. The *S-internal private neighborhood* of v is defined by $\text{ipn}[v, S] = \text{pn}[v, S] \cap S$ and its *open S-internal private neighborhood* is defined by $\text{ipn}(v, S) = \text{pn}(v, S) \cap S$. We define an *S-external private neighbor* of v to be a vertex in $\text{epn}(v, S)$ and an *S-internal private neighbor* of v to be a vertex in $\text{ipn}(v, S)$.

The *degree* $d_G(v)$ of a vertex v is the number of neighbors v has in G, that is, $d_G(v) = |N_G(v)|$. Again if the graph G is clear from the context, we use $d(v)$ rather than $d_G(v)$. We remark that some books use $\deg(v)$ and $\deg v$ to denote the degree of v. We leave it to the authors to choose which of these notations to adopt in their chapters. For a subset of vertices $S \subseteq V$, the *degree of* v *in* S, denoted $d_S(v)$, is the number of vertices in S adjacent to the vertex v; that is, $d_S(v) = |N_G(v) \cap S|$. In particular, if $S = V$, then $d_S(v) = d_G(v)$. The *degree sequence* of a graph G with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ is the sequence $d_1, d_2, \ldots, d_n$, where $d_i = d(v_i)$ for $i \in [n]$. Often the degree sequence, $d_1, d_2, \ldots, d_n$ is written in non-increasing order, and so $d_1 \geq d_2 \geq \cdots \geq d_n$.

An *isolated vertex* is a vertex of degree 0 in G. A graph is *isolate-free* if it does not contain an isolated vertex. The minimum degree among the vertices of G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. A *leaf* is a vertex of degree 1, while its neighbor is a *support vertex*. A *strong support vertex* is a (support) vertex with at least two leaf neighbors.

For subsets X and Y of vertices of G, we denote the set of edges that join a vertex of X and a vertex of Y in G by $[X, Y]$.

Two graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $\phi\colon V(G) \to V(H)$ such that two vertices u and v are adjacent in G if and only if the two vertices $\phi(u)$ and $\phi(v)$ are adjacent in H. A *parameter* of a graph G is a numerical value (usually a non-negative integer) that can be associated with a graph such that whenever two graphs are isomorphic, they have the same associated parameter value.

By a *partition* of the vertex set V of a graph G, we mean a family $\pi = \{V_1, V_2, \ldots, V_k\}$ of nonempty pairwise disjoint sets whose union equals V, that is, for all $1 \le i < j \le k$, $V_i \cap V_j = \varnothing$ and

$$\bigcup_{i=1}^{k} V_i = V.$$

For such a partition π, we will say that π has *order* k.

A *walk* in a graph G from a vertex u to a vertex v is a finite, alternating sequence of vertices and edges, starting with the vertex u and ending with the vertex v, in which each edge of the sequence joins the vertex that precedes it in the sequence to the vertex that follows it in the sequence. A *trail* is a walk containing no repeated edges, and a *path* is a walk containing no repeated vertices. We will mainly be concerned with paths. A path between two vertices u and v is called a (u, v)*-path* or a u*-*v *path* or a u, v*-path* in the literature. The *length* of a walk equals the number of edges in the walk. A graph G is *connected* if for any two vertices u and v in G, there is a (u, v)-path.

A *cycle* is a path in which the first and last vertices are the same and all other vertices are distinct. A *chord* of a cycle C is an edge between two nonconsecutive vertices of C.

The *distance* $d(u, v)$ between two vertices u and v, in a connected graph G, equals the minimum length of a (u, v)-path in G. A shortest, or minimum length, path between two vertices u and v is called a (u, v)*-geodesic*; a v*-geodesic* is any shortest path from v to another vertex; a *geodesic* is any shortest path in a graph. The *diameter* of G is the maximum length of a geodesic in G.

A graph $G' = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' of a graph G is called a *spanning subgraph* of G if $V' = V$. If $G = (V, E)$ and $S \subseteq V$, then the *subgraph of* G *induced by* S is the graph $G[S]$, whose vertex set is S and whose edges are all the edges in E both of whose vertices are in S.

Let F be an arbitrary graph. A graph G is said to be F*-free* if G does not contain F as an induced subgraph.

If $G=(V, E)$ and $S \subseteq V$, the subgraph obtained from G by deleting all vertices in S and all edges incident with one or two vertices in S is denoted by $G-S$; that is, $G-S=G[V \setminus S]$. If $S=\{v\}$, we simply denote $G-\{v\}$ by $G-v$. The *contraction* of an edge $e=xy$ in a graph G is the graph obtained from G by deleting the vertices x and y and all edges incident to x or y and adding a new vertex and edges joining this new vertex to all vertices that were adjacent to x or y in G.

A *component* of a graph is a maximal connected subgraph. An *odd (even) component* is a component of odd (even) order. Let $\mathrm{oc}(G)$ equal the number of odd components of G. A vertex $v \in V$ is a *cutvertex* if the graph $G-v$ has more components than G. An edge $e=uv$ is a *bridge* if the graph $G-e$ obtained by deleting e from G has more components than G.

2.2 Common Types of Graphs

A graph of order $n=1$ is called a *trivial graph*, while a graph with at least two vertices is called a *nontrivial graph*. A graph of size $m=0$ is an *empty graph*, while a graph with at least one edge is a *nonempty graph*. Recall that a *connected graph* is a graph for which there is a path between every pair of its vertices.

A *k-regular graph* is a graph in which every vertex has degree k for some $k \geq 0$. A *regular graph* is a graph that is k-regular. A 3-regular graph is also called a *cubic graph*.

A graph of order n that is itself a cycle is denoted by C_n, and a graph that is itself a path is denoted by P_n. Note that a cycle is a 2-regular graph.

A *forest* is an *acyclic* graph, that is, a graph with no cycles. A *tree* is a connected acyclic graph. Equivalently, a tree is a connected graph having size one less than its order. Hence, if T is a tree of order n and size m, then T is connected and $m=n-1$. Note that every component of a forest is a tree, and a forest in which every component is a path is called a *linear forest*.

If G is a vertex disjoint union of k copies of a graph F, we write $G=kF$.

A *complete graph* is a graph in which every two vertices are adjacent. A complete graph of order n is denoted by K_n. A *triangle* is a subgraph isomorphic to K_3 or C_3, since $K_3 \cong C_3$.

A graph G is *bipartite* if its vertex set can be partitioned into two independent sets X and Y. The sets X and Y are called the *partite sets* of G. A *complete bipartite graph*, denoted $K_{r,s}$, is a bipartite graph with partite sets X and Y, where $|X|=r$, $|Y|=s$, and every vertex in X is adjacent to every vertex in Y. The graph $K_{r,s}$ has order $r+s$, size rs, $\delta(K_{r,s}) = \min\{r, s\}$ and $\Delta(K_{r,s}) = \max\{r, s\}$.

A *star* is a nontrivial tree with at most one vertex that is not a leaf. Thus, a star is a complete bipartite graph $K_{1,k}$ for some $k \geq 1$. A *claw* is an induced copy of the graph $K_{1,3}$. Thus, a *claw-free graph* is a $K_{1,3}$-free graph.

For $r, s \geq 1$, a *double star* $S(r, s)$ is a tree with exactly two (adjacent) vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors. Equivalently, a *double star* is a tree having diameter equal to 3.

A *diamond* is an induced copy of the graph $K_4 - e$, which is obtained from a copy of the complete graph of order 4 by deleting any edge e.

A graph G can be *embedded* on a surface S if its vertices can be placed on S and all of its edges can be drawn between the vertices on S in such a way that no two edges intersect. A graph G is *planar* if it can be embedded in the plane; a *plane graph* is a graph that has been embedded in the plane.

A *rooted tree* T is a tree having a *distinguished vertex* labeled r, called the *root*. Let T be a rooted tree with root r. For each vertex v, let $P(v)$ be the unique (r, v)-path in T. The *parent* of a vertex v is its neighbor on $P(v)$, while the other neighbors of v are called its *children*. The set of children of v is denoted by $C(v)$. Note that the root r is the only vertex of T with no parent. A *descendant* of v is any vertex $u \neq v$ such that $P(u)$ contains v, while an *ancestor* of v is a vertex $u \neq v$ that belongs to $P(v)$ in T. In particular, every child of v is a descendant of v, while the parent of v is an ancestor of v. A *grandchild* of v is a descendant of v at distance 2 from v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v, denoted T_v, is the subtree of T induced by $D[v]$. The *depth* of a vertex v in T equals $d(r, v)$, and the *height* of v, denoted $\mathrm{ht}(v)$, is the maximum distance from v to a descendant of v. Thus, $\mathrm{ht}(v) = \max\{d(v, w) : w \text{ is a descendant of } v\}$.

For classes of graphs not defined here, we refer the reader to the definitive encyclopedia on graph classes, *Graph Classes: A Survey* [5] by Brandstädt, Le, and Spinrad.

2.3 Graph Constructions

Given a graph $G = (V, E)$, the *complement* of G is the graph $\overline{G} = (V, \overline{E})$, where $uv \in \overline{E}$ if and only if $uv \notin E$. Thus, the complement $\overline{G}$ of G is formed by taking the vertex set of G and joining two vertices by an edge whenever they are not joined in G.

By a *graph product* $G \otimes H$ on graphs G and H, we mean a graph whose vertex set is the Cartesian product of the vertex sets of G and H (that is, $V(G \otimes H) = V(G) \times V(H)$) and whose edge set is determined entirely by the adjacency relations of G and H. Exactly how it is determined depends on what kind of graph product we are considering.

The *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

The *direct product* (also known as the *cross product*, *tensor product*, *categorical product*, and *conjunction*) $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G \times H$ if and only if $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$.

Given a graph $G = (V, E)$ and an edge $uv \in E$, the *subdivision* of edge uv consists of (i) deleting the edge uv from E, (ii) adding a new vertex w to V, and (iii) adding the new edges uw and wv to E. In this case, we say that the edge uv has been *subdivided*. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing every edge of G exactly once.

Given a graph $G = (V, E)$, the *line graph* $L(G) = (E, E(L(G)))$ is the graph whose vertices correspond 1-to-1 with the edges in E, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G have a vertex in common, that is, if and only if the corresponding two edges are adjacent.

The *corona* $G \circ K_1$ of a graph G, also denoted $\text{cor}(G)$ in the literature, is the graph obtained from G by adding, for each vertex $v \in V$, a new vertex v' and the edge vv'. The edge vv' is called a *pendant edge*. The *k-corona* $G \circ P_k$ of G is the graph of order $(k+1)|V(G)|$ obtained from G by attaching a path of length k to each vertex of G so that the resulting paths are vertex-disjoint. In particular, the 2*-corona* $G \circ P_2$ of G is the graph of order $3|V(G)|$ obtained from G by attaching a path of length 2 to each vertex of G so that the resulting paths are vertex-disjoint. The *generalized corona* $G \circ H$ is the graph obtained by adding a copy of H for each vertex v of G and joining v to every vertex of H. Thus, a generalized corona $G \circ H$, where $H = K_1$, is the ordinary corona $G \circ K_1$. We note that whether $G \circ P_k$ is intended to denote a k-corona or a generalized corona will be clear from context or specifically stated by the author.

3 Graph Parameters

In this section, we present common graph parameters that may appear in this book.

3.1 Connectivity and Subgraph Numbers

In this subsection, we present parameters related to connectivity in graphs.

(a) *blocks* $\text{bl}(G)$, number of blocks in G. A *block* of a graph G is a maximal nonseparable subgraph of G, that is, a maximal subgraph having no cutvertices.
(b) *bridges* $\text{br}(G)$, number of bridges in G.
(c) *circumference* $\text{cir}(G)$, maximum length or order of a cycle in G.
(d) *clique number* $\omega(G)$, maximum order of a complete subgraph of G.
(e) *components* $c(G)$, number of maximal connected subgraphs of G.
(f) A *vertex cut* of a connected graph G is a subset S of the vertex set of G with the property that $G - S$ is disconnected (has more than one component). A vertex cut S is a *k-vertex cut* if $|S| = k$.
(g) *vertex connectivity* $\kappa(G)$, minimum cardinality of a vertex cut of G if G is not the complete graph, and $\kappa(K_n) = n - 1$. A graph G is *k-vertex-connected* (or *k-connected*) if $\kappa(G) \geq k$ for some integer $k \geq 0$. Thus, $\kappa(G)$ is the smallest number of vertices whose deletion from G produces a disconnected graph or the trivial graph K_1. A nontrivial graph has connectivity 0 if and only if it is disconnected.

(h) An *edge cut* of a nontrivial connected graph G is a nonempty subset F of the edge set of G with the property that $G - F$ is disconnected (has more than one component). Thus, the deletion of an edge cut from the connected graph G results in a disconnected graph. An edge cut F is a *k-edge cut* if $|F| = k$.
(i) *edge connectivity* $\lambda(G)$, minimum cardinality of an edge cut of G if G is nontrivial, while $\lambda(K_1) = 0$. A graph G is *k-edge-connected* if $\lambda(G) \geq k$ for some integer $k \geq 0$. Thus, $\lambda(G)$ is the smallest number of edges whose deletion from G produces a disconnected graph or the trivial graph K_1. Hence, $\lambda(G) = 0$ if and only if G is disconnected or trivial.
(j) *girth* of G, denoted *girth(G)* or $g(G)$ in the literature, the length of a shortest cycle in G.

3.2 Distance Numbers

This subsection contains the definitions of parameters, which are defined in terms of the distances $d(u, v)$ between vertices u and v in a graph.

(a) *eccentricity* $\text{ecc}(v) = \max\{d(v, w) : w \in V(G)\}$.
(b) *diameter* $\text{diam}(G)$, maximum distance among all pairs of vertices of G. Equivalently, the diameter of G is the maximum length of a geodesic in G. Thus, the diameter of G is the maximum eccentricity taken over all vertices of G. Two vertices u and v in G for which $d(u, v) = \text{diam}(G)$ are called *antipodal* or *peripheral vertices* of G. A *diametral path* in G is a geodesic whose length equals the diameter of G.
(c) The *periphery* of a graph G is the subgraph of G induced by its peripheral vertices.
(d) *radius* $\text{rad}(G) = \min\{\text{ecc}(v) : v \in V(G)\}$.
(e) The *center* of a graph G, denoted $C(G)$, is the subgraph of G induced by the vertices in G whose eccentricity equals the radius of G. A vertex $v \in C(G)$ is called a *central vertex* of G.

3.3 Covering, Packing, Independence, and Matching Numbers

As previously defined, a set S is *independent* if no two vertices of S are adjacent.

A set M of edges is called a *matching* if no two edges of M are adjacent, and a matching of maximum cardinality is a *maximum matching*. Given a matching M, we denote by $V[M]$ the set of vertices in G incident with an edge in M. A matching M of G is a *perfect matching* if $V[M] = V(G)$. Thus, if G has a perfect matching M, then G has even order $n = 2k$ for some $k \geq 1$ and $|M| = k$.

A vertex and an edge are said to *cover* each other in a graph G if they are incident in G. A *vertex cover* in G is a set of vertices that covers all the edges of G, while an *edge cover* in G is a set of edges that covers all the vertices of G. Thus, a vertex cover in G is a set of vertices that contains at least one vertex of every edge in G.

A subset S of vertices in G is a *packing* if the closed neighborhoods of vertices in S are pairwise disjoint. Equivalently, S is a packing in G if for every $u, v \in S$, $d(u, v) > 2$. Thus, if S is a packing in G, then $|N_G[v] \cap S| \leq 1$ for every vertex $v \in V(G)$. A packing is also called a 2-*packing* in the literature. More generally, for $k \geq 2$, a set S is a *k-packing* in G if for $u, v \in S$, $d(u, v) > k$.

A subset S of vertices in G is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint. Thus, if S is an open packing in G, then $|N_G(v) \cap S| \leq 1$ for every vertex $v \in V(G)$.

All of the parameters in this subsection have to do with sets that are independent or cover other sets. These include some of the most basic of all parameters in graph theory.

(a) *vertex independence numbers* $i(G)$ and $\alpha(G)$, minimum and maximum cardinalities of a maximal independent set in G. The lower vertex independence number, $i(G)$, is also called the *independent domination number* of G, while the upper vertex independence number, $\alpha(G)$, is also called the *independence number* of G. (While the notation $i(G)$ is fairly standard for the independent domination number, we remark that the independence number is also denoted by $\beta_0(G)$ in the literature.)

(b) *vertex covering numbers* $\beta(G)$ and $\beta^+(G)$, minimum and maximum cardinalities of a minimal vertex cover in G. (We remark that the vertex covering number is also denoted by $\tau(G)$ or by $\alpha(G)$ in the literature.)

(c) *edge covering numbers* $\beta'(G)$ and $\beta'^+(G)$, minimum and maximum cardinalities of a minimal edge cover in G.

(d) *k-packing numbers* $\rho_k(G)$, maximum cardinality of a k-packing in G for $k \geq 2$. When $k = 2$, the k-packing number $\rho_k(G)$ is called the *packing number* of G, denoted by $\rho(G)$. Thus, $\rho(G)$ is the maximum cardinality of a packing in G.

(e) *open packing numbers* $\rho^o(G)$, maximum cardinality of an open packing in G.

(f) *matching numbers* $\alpha'^-(G)$ and $\alpha'(G)$, minimum and maximum cardinalities of a maximal matching in G. The upper matching number, $\alpha'(G)$, is also called the *matching number* of G. Recall that a *perfect matching* is a matching in which every vertex is incident with an edge of the matching. Thus, if a graph G of order n has a perfect matching, then $\alpha'(G) = \frac{1}{2}n$. It should be noted that by a well-known theorem of Gallai, that if G is a graph of order n with no isolated vertices, then $\alpha(G) + \beta(G) = n = \alpha'(G) + \beta'(G)$. (We remark that the matching number is also denoted by $\beta_1(G)$ in the literature.)

3.4 Domination Numbers

A *dominating set* in a graph $G = (V, E)$ is a set S of vertices of G such that every vertex in $\overline{S} = V \setminus S$ has a neighbor in S. Thus, if S is a dominating set of G, then

$N_G[S] = V$ and every vertex in $\overline{S}$ is therefore adjacent to at least one vertex in S. For subsets X and Y of vertices of G, if $Y \subseteq N_G[X]$, then the set X *dominates* the set Y in G. In particular, if X dominates $V(G)$, then X is a dominating set of G.

The many variations of dominating sets in a graph G are based on (i) conditions that are placed on the subgraph $G[S]$ induced by a dominating set S, (ii) conditions that are placed on the vertices in $\overline{S}$, or (iii) conditions that are placed on the edges between vertices in S and vertices in $\overline{S}$. We mention only the major domination numbers here.

A *total dominating set*, abbreviated TD-set, in a graph G with no isolated vertex is a set S of vertices of G such that every vertex in V is adjacent to at least one vertex in S. Thus, a subset $S \subseteq V$ is a TD-set in G if $N_G(S) = V$. Every graph without isolated vertices has a TD-set, since $S = V$ is such a set. If X and Y are subsets of vertices in G, then the set X *totally dominates* the set Y in G if $Y \subseteq N_G(X)$. In particular, if X totally dominates $V(G)$, then X is a TD-set in G.

A *paired dominating set*, abbreviated PD-set, of G is a set S of vertices of G such that every vertex is adjacent to some vertex in S and the subgraph $G[S]$ induced by S contains a perfect matching M. Two vertices joined by an edge of M are said to be *paired* and are also called *partners* in S.

A *connected dominating set*, abbreviated CD-set, in a graph G is a dominating set S of vertices of G such that $G[S]$ is connected.

(a) *domination numbers* $\gamma(G)$ and $\Gamma(G)$, minimum and maximum cardinalities of a minimal dominating set in G. The parameters $\gamma(G)$ and $\Gamma(G)$ are referred to as the *domination number* and *upper domination number* of G, respectively. A dominating set of G of cardinality $\gamma(G)$ is called a *γ-set* of G, while a minimal dominating set of cardinality $\Gamma(G)$ is called a Γ-set of G.

(b) *independent domination number* $i(G)$, minimum cardinality of a dominating set in G that is also independent. An independent dominating set of G of cardinality $i(G)$ is called an i-set of G. We note that the maximum order of a minimal independent dominating set equals the vertex independence number $\alpha(G)$.

(c) *total domination numbers* $\gamma_t(G)$ and $\Gamma_t(G)$, minimum and maximum cardinalities of a minimal total dominating set of G. The parameters $\gamma_t(G)$ and $\Gamma_t(G)$ are referred to as the *total domination number* and *upper total domination number* of G, respectively. A TD-set of G of cardinality $\gamma_t(G)$ is called a *γ_t-set* of G, while a minimal TD-set of cardinality $\Gamma_t(G)$ is called a Γ_t-set of G.

(d) *paired domination numbers* $\gamma_{\rm pr}(G)$ and $\Gamma_{\rm pr}(G)$, minimum and maximum cardinalities of a minimal PD-set of G. The parameters $\gamma_{\rm pr}(G)$ and $\Gamma_{\rm pr}(G)$ are referred to as the *paired domination number* and *upper paired domination number* of G, respectively. A PD-set of G of cardinality $\gamma_{\rm pr}(G)$ is called a *$\gamma_{\rm pr}$-set* of G, while a minimal PD-set of cardinality $\Gamma_{\rm pr}(G)$ is called a $\Gamma_{\rm pr}$-set of G.

(e) *connected domination numbers* $\gamma_c(G)$ and $\Gamma_c(G)$, minimum and maximum cardinalities of a minimal CD-set of G. The parameters $\gamma_c(G)$ and $\Gamma_c(G)$ are referred to as the *connected domination number* and *upper connected domination number* of G, respectively. A CD-set of G of cardinality $\gamma_c(G)$ is called a *γ_c-set* of G, while a minimal CD-set of cardinality $\Gamma_c(G)$ is called a Γ_c-set of G.

References

1. W. Ahrens, *Mathematische Unterhaltungen und Spiele* (Druck und Verlag von B. G. Teubner, Berlin, 1910), pp. 311–312
2. W.W.R. Ball, *Mathematical Recreations and Essays*, 4th edn. (Macmillan, London, 1905)
3. C. Berge, *The Theory of Graphs and its Applications* (Methuen, London, 1962)
4. M. Bezzel, Schachfreund. Berliner Schachzeitung **3**, 363 (1848)
5. A. Brandstädt, V.B. Le, J.P. Spinrad, *Graph Classes: A Survey*. SIAM Monographs on Discrete Mathematics and Applications (SIAM, Philadelphia, 1999)
6. E.J. Cockayne, S.T. Hedetniemi, Towards a theory of domination in graphs. Networks **7**, 247–261 (1977)
7. C.F. de Jaenisch, *Applications de l'Analyse Mathematique au Jeu des Echecs* (Petrograd, Moscow, 1862)
8. H.E. Dudeney, *The Canterbury Puzzles and Other Curious Problems* (E. P. Dutton and Company, New York, 1908)
9. R. Gera, T.W. Haynes, S.T. Hedetniemi, M.A. Henning, An annotated glossary of graph theory parameters with conjectures, in *Graph Theory, Favorite Conjectures and Open Problems, Volume 2*, ed. by R. Gera, T.W. Haynes, S.T. Hedetniemi (Springer, Berlin, 2018), pp. 177–281
10. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, *Fundamentals of Domination in Graphs* (Marcel Dekker, New York, 1998)
11. T.W. Haynes, S.T. Hedetniemi, P.J. Slater, eds., *Domination in Graphs, Advanced Topics* (Marcel Dekker, New York, 1998)
12. M.A. Henning, A. Yeo, *Total Domination in Graphs*. Springer Monographs in Mathematics (Springer, New York, 2013). ISBN: 978-1-4614-6524-9 (Print) 978-1-4614-6525-6 (eBook)
13. D. König, *Theorie der Endlichen und Unendlichen Graphen*. Akademische Verlagsgesellschaft M. B. H., Leipzig, 1936 (Chelsea, New York, 1950)
14. F. Nauck, Briefwechsel mit allen für alle. Illustrirte Zeitung **15**, 182 (1850)
15. O. Ore, *Theory of Graphs*. American Mathematical Society Colloquium Publications, vol. 38 (American Mathematical Society, Providence, 1962)
16. A.M. Yaglom, I.M. Yaglom, *Challenging Mathematical Problems with Elementary Solutions, Combinatorial Analysis and Probability Theory*, vol. I (Holden-Day, San Francisco, 1964)

Part I
Related Parameters

Broadcast Domination in Graphs

Michael A. Henning, Gary MacGillivray, and Feiran Yang

AMS Subject Classification: 05C65, 05C69

1 Introduction

The concept of broadcast domination was birthed by combining the concepts of distance and domination in graphs and applying them to modeling the problem of positioning broadcasting radio transmitters, where each transmitter may have a different effective radiated power. To formally define broadcast domination, we

The research of the author Michael A. Henning supported in part by the University of Johannesburg. The research of the author Gary MacGillivray supported by the Natural Sciences and Engineering Research Council of Canada.

M. A. Henning
Department of Mathematics and Applied Mathematics, University of Johannesburg, Johannesburg, South Africa
e-mail: mahenning@uj.ac.za

G. MacGillivray (✉)
Department of Mathematics and Statistics, University of Victoria, V8W 2Y2, P.O. Box 1700 STN CSC, Victoria, BC, Canada
e-mail: gmacgill@math.uvic.ca

F. Yang
Department of Mathematics and Applied Mathematics, University of Johannesburg, Auckland Park, 2006 South Africa

Department of Mathematics and Statistics, University of Victoria, V8W 2Y2, P.O. Box 1700 STN CSC, Victoria, BC, Canada
e-mail: fyang@uvic.ca

T. W. Haynes et al. (eds.), *Structures of Domination in Graphs*, Developments in Mathematics 66, https://doi.org/10.1007/978-3-030-58892-2_2

recall the fundamental concepts of distance and domination in graph theory. The *distance* between two vertices u and v in a graph G, denoted by $d_G(u, v)$, or simply $d(u, v)$ if the graph G is clear from context, is the length of a shortest (u, v)-path in G. The *eccentricity* $\mathrm{ecc}_G(v)$ of a vertex v in G is the maximum distance of a vertex from v in G. The maximum eccentricity among the vertices of G is the *diameter* of G, denoted by $\mathrm{diam}(G)$, while the minimum eccentricity among the vertices of G is the *radius* of G, denoted by $\mathrm{rad}(G)$. A *central vertex* of G is a vertex whose eccentricity equals $\mathrm{rad}(G)$. A tree is either *central* or *bicentral*, depending on whether it has one or two central vertices. A *diametrical path* in G is a shortest path whose length is equal to $\mathrm{diam}(G)$. We note that the two vertices at the end of a diametrical path have maximum eccentricity in G.

A *dominating set* in a graph G is a set S of vertices of G such that every vertex outside S is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G.

A *neighbor* of a vertex v in G is a vertex adjacent to v. The *open neighborhood* of a vertex v in G, denoted by $N_G(v)$, is the set of all neighbors of v in G, while the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. We denote the *degree* of a vertex v in G by $d_G(v) = |N_G(v)|$. The minimum and maximum degrees among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

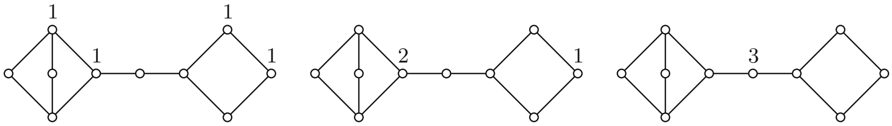
For an integer $k \geq 1$, the *closed k-neighborhood* of v in G, denoted by $N_k[v;G]$, is the set of all vertices within distance k from v, that is, $N_k[v;G] = \{u : d_G(u, v) \leq k\}$. The *open k-neighborhood* of v, denoted by $N_k(v;G)$, is the set of all vertices different from v and at distance at most k from v in G, that is, $N_k(v;G) = N_k[v;G] \setminus \{v\}$.

If the graph G is clear from context, we omit the subscript G. For example, we simply write $N(v)$, $N[v]$, $N_k(v)$, and $N_k[v]$ rather than $N_G(v)$, $N_G[v]$, $N_k(v;G)$, and $N_k[v;G]$, respectively. When $k = 1$, the set $N_k[v] = N[v]$ and the set $N_k(v) = N(v)$. In what follows, for an integer $k \geq 1$, we use the standard notation $[k] = \{1, \dots, k\}$ and $[k]_0 = [k] \cup \{0\} = \{0, 1, \dots, k\}$.

For a graph $G = (V, E)$ with a vertex set V and an edge set E, a function $f : V \to \{0, 1, 2, \dots, \mathrm{diam}(G)\}$ is called a *broadcast* on G. For each vertex v in G, the value $f(v)$ is called the *strength* (or the *weight*) of the broadcast from v. For each vertex $u \in V$, if there exists a vertex v in G (possibly, $u = v$) such that $f(v) > 0$ and $d(u, v) \leq f(v)$, then f is called a *dominating broadcast* on G. A vertex v with $f(v) > 0$ can be thought of as the site from which the broadcast is transmitted with strength $f(v)$, and such a vertex is called an *f-broadcast vertex* or simply a *broadcast vertex* if the function f is clear from context. The *ball of radius r around v* is defined as $N_r[v] = \{u \in V : d(u, v) \leq r\}$. Thus, the ball $N_{f(v)}[v]$ is the set of vertices that *hear* the broadcast from v. Vertices u with $f(u) = 0$ do not broadcast. For $X \subseteq V$, we define

$$f(X) = \sum_{v \in X} f(v).$$

The *cost* of the dominating broadcast f is the quantity $f(V)$, which is the sum of the strengths of the broadcasts over all vertices in G. The minimum cost of a dominating broadcast is the *broadcast domination number* of G, denoted by $\gamma_b(G)$.



(a) f_1 (b) f_2 (c) f_3

Fig. 1 Three broadcast dominating functions of a graph G

An *optimal broadcast* is a broadcast with cost equal to $\gamma_b(G)$. For the graph G shown in Figure 1, three broadcast dominating functions are illustrated in Figure 1(a), 1(b) and 1(c). The cost of f_1, f_2, and f_3 is 4, 3, and 3, respectively. For this graph G, we have $\gamma_b(G) = 3$ and both f_2 and f_3 are optimal broadcasts.

Broadcast domination in graphs was first introduced and studied in 2001 by Erwin [21, 22]. Erwin observed that if a dominating broadcast f satisfies $f(v) \in \{0, 1\}$ for all $v \in V$, then f is the characteristic function of a dominating set and hence has cost at most $\gamma(G)$. Furthermore, he observed that a broadcast $f : V \to \{0, 1, \ldots, \text{diam}(G)\}$ that assigns the strength rad(G) to a central vertex of a connected graph G and the strength 0 to all remaining vertices of G has cost $f(V) = \text{rad}(G)$. If $G = K_1$, then $\gamma_b(G) = 1 = \gamma(G)$, while rad$(G) = 0$. Hence, we assume that $G \neq K_1$ and therefore has order at least 2. Thus, the broadcast domination number of a graph G is at most its domination number and at most its radius. We state this formally as follows.

Observation 1. ([21, 22]) *If G is a connected graph of order at least* 2*, then*

$$\gamma_b(G) \leq \min\{\gamma(G), \text{rad}(G)\}.$$

Graphs for which the broadcast domination number is equal to the radius are called *radial*. In view of Observation 1, we can replace diam(G) by rad(G) in the definition of a dominating broadcast in a graph G. Erwin [21, 22] showed that if the domination number or the radius of a graph is at most 3, then the broadcast domination number is determined.

Proposition 2. ([21, 22]) *If G is a connected graph of order at least* 2 *and* $k = \min\{\gamma(G), \text{rad}(G)\}$ *where* $k \in [3]$*, then* $\gamma_b(G) = k$.

In 2006, Dunbar, Erwin, Haynes, Hedetniemi, and Hedetniemi [20] defined a key concept called efficient broadcast. A dominating broadcast is *efficient* if no vertex hears a broadcast from two different vertices. If f is not an efficient dominating broadcast in a graph $G = (V, E)$, then there exists a vertex v such that $d(v, x) \leq f(x)$ and $d(v, y) \leq f(y)$, where x and y are broadcasting vertices in G. In this case, we can reassign the value 0 to both x and y, assign the value $f(w) + f(x) + f(y)$ to a vertex w that is within distance $f(y)$ from x and also within distance $f(x)$ from y, and leave the value of all other vertices unchanged under f. The cost of the new broadcast is equal to the cost of the original broadcast. This process can be repeated until an efficient broadcast is found. This yields the following result.

Theorem 3. ([20]) *Every graph G has an optimal dominating broadcast that is efficient.*

As first observed by Herke [31], the broadcast domination number of a connected graph is equal to the minimum broadcast domination number among its spanning trees.

Observation 4. ([31]) *If G is a connected graph, then*

$$\gamma_b(G) = \min\{\gamma_b(T) \mid T \text{ is a spanning tree of } G\}.$$

2 The Dual of Broadcast Domination

Graph theoretic minimization (respectively, maximization) problems expressed as linear programming problems have dual maximization (respectively, minimization) problems. Much of the early work on linear programming duality problems for domination type parameters is done by Slater. A survey of these results can be found in the 1998 survey paper of Slater [44]. The dual concept of coverings and packings is also well studied in graph theory. For a survey on the combinatorics underlying set packing and set covering problems, we refer the reader to the 2001 monograph by Cornuéjols [17].

In this section, we discuss the dual (in the sense of linear programming) of broadcast domination, namely *multipacking*. The term multipacking was first introduced in the Master's thesis of Teshima [47] in 2012. Here, broadcast domination was considered as a linear programming problem, and the linear programming dual was used to give the definition of a multipacking. A *multipacking* is a set $S \subseteq V$ in a graph $G = (V, E)$ such that for every vertex $v \in V$ and for every integer $r \geq 1$, the ball of radius r around v contains at most r vertices of S, that is, there are at most r vertices in S at distance at most r from v in G. We note that in this definition of a multipacking, we may restrict our attention to $r \in [\text{diam}(G)]$. By our earlier observations, we can in fact restrict the integer r to belong to the set $[\text{rad}(G)]$. The *multipacking number* of G is the maximum cardinality of a multipacking of G and is denoted by $\text{mp}(G)$. We define next the multipacking number in terms of the dual of the linear programming problem for broadcast domination.

Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$. The definition of $\gamma_b(G)$ leads to a 0–1 integer program, which we now describe. For each vertex v_i and integer $k \in [\text{rad}(G)]$, let x_{ik} be an indicator variable giving the truth value of the statement "the strength of the broadcast f at vertex v_i equals k," that is,

$$x_{ik} = \begin{cases} 1 \text{ if } f(v_i) = k \\ 0 \text{ otherwise.} \end{cases}$$

The formulation of the primal integer program for broadcast domination is given by

Broadcast Domination γ_b

Minimize $\sum_{k=1}^{\mathrm{rad}(G)} \sum_{i=1}^{n} kx_{ik}$,

subject to

$\sum_{d(v_i,v_j)\leq k} x_{ik} \geq 1$ for all vertices v_i and v_j,

$x_{ik} \in \{0, 1\}$ for each vertex v_i and integer $k \in [\mathrm{rad}(G)]$.

Multipacking Number mp(G)

Maximize $\sum_{k=1}^{n} y_j$,

subject to

$\sum_{d(v_i,v_j)\leq k} y_j \leq k$ for all vertices v_i and v_j and integer $k \in [\mathrm{rad}(G)]$,

$y_k \in \{0, 1\}$ for each $k \in [n]$.

Since broadcast domination and multipacking are dual problems, we have the following observation.

Observation 5. *For every graph G, we have* $\mathrm{mp}(G) \leq \gamma_b(G)$.

The graph G shown in Figure 2 satisfies $\mathrm{mp}(G) = 3$, where the darkened vertices form a multipacking of maximum cardinality in G. As observed earlier, $\gamma_b(G) = 3$, and so for this example, we have $\mathrm{mp}(G) = \gamma_b(G)$.

In 2014, Hartnell and Mynhardt [26] provided the following lower bound on the multipacking number of a graph.

Theorem 6. ([26]) *If G is a connected graph, then* $\mathrm{mp}(G) \geq \left\lceil \frac{1}{3}(\mathrm{diam}(G) + 1) \right\rceil$.

Proof.. Let $P: v_0, v_1, \ldots, v_d$ be a diametrical path of G, where $d = \mathrm{diam}(G)$. Let $V_i = \{v \in V : d(v, v_0) = i\}$ for all $i \in [d]$, and let $M = \{v_i : i \equiv 0 \pmod 3\}$. We note that $|M| = \lceil \frac{1}{3}(d + 1) \rceil$. By our choice of the set M, every vertex $v \in V(P)$ satisfies $|N_r[v] \cap M| \leq r$ for all integers $r \geq 1$. We now consider an arbitrary vertex $w \in V$. We note that $w \in V_j$ for some $j \in [d]_0$. Since $v_j \in V_j$ and $M \subseteq V(P)$, we note that $N_r[w] \cap M \subseteq N_r[v_j] \cap M$, implying that $|N_r[w] \cap M| \leq r$ for all integers $r \geq 1$. Since $w \in V$ is arbitrary, this implies that the set M is a multipacking in G. Thus, $\mathrm{mp}(G) \geq |M| = \lceil \frac{1}{3}(d + 1) \rceil = \lceil \frac{1}{3}(\mathrm{diam}(G) + 1) \rceil$. □

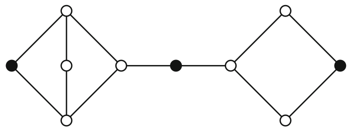


Fig. 2 A graph G with $\mathrm{mp}(G) = 3$

As an immediate consequence of Observation 5 and Theorem 6, we have the following lower bound on the broadcast domination number first observed by Erwin [21, 22].

Corollary 7. ([21, 22]) *If G is a connected graph, then* $\gamma_b(G) \geq \left\lceil \frac{1}{3}(\text{diam}(G) + 1) \right\rceil$.

We note that if G is a path P_n on $n \geq 2$ vertices, then $\gamma(G) = \lceil \frac{1}{3}n \rceil = \lceil \frac{1}{3}(\text{diam}(G) + 1) \rceil$. Hence, by Observations 1 and 5 and Theorem 6, we have that the lower bound of Theorem 6 is tight. Furthermore, we have the following result.

Proposition 8. *For every integer $n \geq 2$,*

$$\text{mp}(P_n) = \gamma_b(P_n) = \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil.$$

By Observation 4, for $n \geq 3$, we have $\gamma_b(C_n) = \gamma_b(P_n)$, and so by Proposition 8, $\gamma_b(C_n) = \lceil \frac{n}{3} \rceil$. However, $\text{mp}(C_n) = \lfloor \frac{n}{3} \rfloor$ for all $n \geq 3$. Thus, for cycles, we have the following result.

Proposition 9. ([47]) *For every integer $n \geq 3$,* $\text{mp}(C_n) = \gamma_b(C_n)$ *if and only if* $n \equiv 0$ (*mod* 3)*. For n* (*mod* 3) $\in \{1, 2\}$*, we have* $mp(C_n) = \gamma_b(C_n) - 1$.

By Theorem 6, if G is a connected graph, then $3\text{mp}(G) \geq \text{diam}(G) + 1$. By definition, $\text{diam}(G) \geq \text{rad}(G)$. By Observation 1, $\text{rad}(G) \geq \gamma_b(G)$. Hence, $3\text{mp}(G) \geq \text{diam}(G) + 1 \geq \text{rad}(G) + 1 \geq \gamma_b(G) + 1$, or, equivalently, $\gamma_b(G) \leq 3\text{mp}(G) - 1$. Hence, as a consequence of our earlier results, we have the following upper bound on the broadcast domination number in terms of its multipacking number.

Corollary 10. ([26]) *If G is a connected graph, then* $\gamma_b(G) \leq 3\text{mp}(G) - 1$.

If the multipacking number of a graph G is at least 2, then Hartnell and Mynhardt [26] improved the upper bound in Corollary 10 slightly.

Theorem 11. ([26]) *If G is a connected graph with* $\text{mp}(G) \geq 2$*, then* $\gamma_b(G) \leq 3\text{mp}(G) - 2$.

As a consequence of Corollary 10, we have the following upper bound on the ratio $\gamma_b(G)/\text{mp}(G)$.

Corollary 12. ([26]) *If G is a connected graph, then* $\dfrac{\gamma_b(G)}{\text{mp}(G)} < 3$.

In 2012, Teshima [47] proved that the graph G shown in Figure 3 satisfies $\gamma_b(G) = 4$ and $\text{mp}(G) = 2$. Assigning a strength 2 to each of the two vertices of degree 2 in G as illustrated in Figure 3, and a strength of 0 to the remaining vertices of G produces an optimal broadcast of G. An example of a multipacking of maximum cardinality in G is given by the set of two darkened vertices of G illustrated in Figure 3. This example serves to show the existence of a graph G for which the ratio $\gamma_b(G)/\text{mp}(G) = 2$.

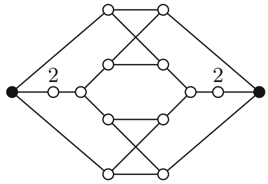


Fig. 3 A graph G with $\gamma_b(G) = 4$ and $\mathrm{mp}(G) = 2$

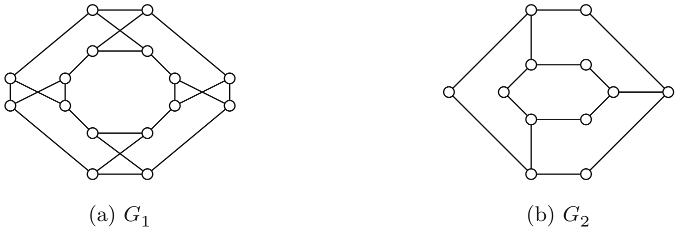


Fig. 4 Two graphs satisfying $\gamma_b(G) = 4$ and $\mathrm{mp}(G) = 2$

To date, no graph G has been found satisfying $\gamma_b(G)/\mathrm{mp}(G) > 2$. Beaudou, Brewster, and Foucaud [4] posed the following conjecture.

Conjecture 1. ([4]) *If G is a connected graph, then* $\gamma_b(G) \le 2\mathrm{mp}(G)$.

There are a few known examples of connected graphs G which achieve the conjectured bound, that is, $\gamma_b(G) = 2\mathrm{mp}(G)$. For example, if G is a cycle C_4 or C_5, then $\gamma_b(G) = 2$ and $\mathrm{mp}(G) = 1$, and so $\gamma_b(G) = 2\mathrm{mp}(G)$. As observed earlier, if G is the graph shown in Figure 3, then $\gamma_b(G) = 4$ and $\mathrm{mp}(G) = 2$, and so $\gamma_b(G) = 2\mathrm{mp}(G)$. Two additional examples of graphs G with $\gamma_b(G) = 4$ and $\mathrm{mp}(G) = 2$ are the graphs $G = G_1$ and $G = G_2$ shown in Figure 4(a) and 4(b), respectively. Graph G_1 is attributed to C. R. Dougherty in [4, Figure 3(c)] as private communication, while graph G_2 is given in [4].

In 2014, Hartnell and Mynhardt [26] gave a construction of a graph G_k such that $\gamma_b(G_k) - \mathrm{mp}(G_k) = k$ for any given integer $k \ge 1$, showing that the difference $\gamma_b - \mathrm{mp}$ can be arbitrarily large. In order to explain their construction, let H be the graph obtained from three vertex-disjoint copies F_1, F_2, *and* F_3 of $K_{2,4}$ as follows. Let u_i be a vertex of degree 2 in F_i for $i \in [2]$, and let v_1 and v_2 be two vertices of degree 2 in F_3. Let H be obtained from the disjoint union of F_1, F_2, *and* F_3 by joining v_i to u_i for $i \in [2]$. Let x be a vertex of degree 2 in F_1 different from u_1, and let y be a vertex of degree 2 in F_2 different from u_2. The graph H is illustrated in Figure 5.

Let M be a multipacking of maximum cardinality in H. Each induced subgraph F_i of H contains at most one vertex of M, implying that $\mathrm{mp}(H) = |M| \le 3$. By Theorem 6, $\mathrm{mp}(H) \ge \left\lceil \frac{1}{3}(\mathrm{diam}(H) + 1) \right\rceil = \left\lceil \frac{1}{3}(8 + 1) \right\rceil = 3$. Consequently, $\mathrm{mp}(H) = 3$. An example of a multipacking of maximum cardinality in H is given by the set

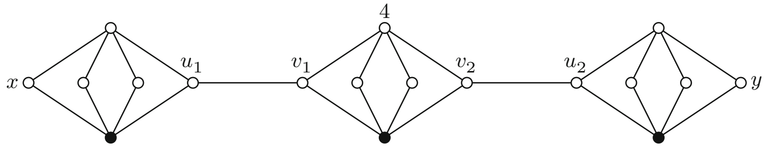


Fig. 5 A graph H with $\gamma_b(H) = 4$ and $\mathrm{mp}(H) = 3$

of three darkened vertices of H illustrated in Figure 5. By Observation 5, we have $\gamma_b(H) \geq \mathrm{mp}(H) = 3$. If $\gamma_b(H) = 3$, then since $\mathrm{rad}(H) = 4$, every optimal broadcast in H must contain at least two broadcast vertices (of positive strength), one of which therefore has strength 1 and the other strength 2. But then at least one of the vertices x and y hears no broadcast, a contradiction. Hence, $\gamma_b(H) \geq 4$. Since $\mathrm{rad}(H) = 4$ and $\gamma(H) = 6$, by Observation 1, we have $\gamma_b(H) \leq 4$. Consequently, $\gamma_b(H) = 4$.

We now return to the general construction given by Hartnell and Mynhardt [26]. For $k = 1$, let $G_k = H$. For $k \geq 2$, let $H_1, H_2, \ldots, H_k$ be k vertex-disjoint copies of the graph H, where x_i and y_i are the vertices in H_i named x and y in H. Let G_k be constructed from the disjoint union of the graphs $H_1, H_2, \ldots, H_k$ by adding the edges $y_i x_{i+1}$ for $i \in [k-1]$. As shown in [26], $\gamma_b(G_k) = 4k$ and $\mathrm{mp}(G_k) = 3k$. This yields the following result.

Theorem 13. ([26]) *For every integer $k \geq 1$, there exists a connected graph G_k such that $\gamma_b(G_k) = 4k$ and $\mathrm{mp}(G_k) = 3k$. Thus, the following hold in the graph G_k.*

(a) $\gamma_b(G_k) - \mathrm{mp}(G_k) = k$.
(b) $\gamma_b(G_k)/\mathrm{mp}(G_k) = \frac{4}{3}$.

Recall that in Theorem 11, if G is a connected graph with $\mathrm{mp}(G) \geq 2$, then $\gamma_b(G) \leq 3\mathrm{mp}(G) - 2$. Hartnell and Mynhardt [26] asked whether the factor 3 in this bound can be improved. In 2019, Beaudou, Brewster, and Foucaud [4] answered their question in the affirmative, resulting in a significant improvement of this upper bound on the broadcast domination number in terms of its multipacking number.

Theorem 14. ([4]) *If G is a connected graph, then $\gamma_b(G) \leq 2\mathrm{mp}(G) + 3$.*

Hartnell and Mynhardt [26] were the first to observe that Conjecture 1 is true when $\mathrm{mp}(G) \leq 2$. The conjecture is shown in [4] to hold for all graphs with multipacking number at most 4.

Theorem 15. ([4]) *If G is a connected graph and $\mathrm{mp}(G) \leq 4$, then $\gamma_b(G) \leq 2\mathrm{mp}(G)$.*

By Observation 5, for every graph G, we have $\mathrm{mp}(G) \leq \gamma_b(G)$. In 2017, Mynhardt and Teshima [47] proved that equality holds here for the class of trees, thereby extending a classic result due to Meir and Moon [37] that the domination number equals the 2-packing number for trees.

Theorem 16. ([47]) *For every tree T, we have $\gamma_b(T) = \mathrm{mp}(T)$.*

For any integer programming problem, a natural variation of the problem can be obtained by considering the LP relaxation. Since broadcast domination

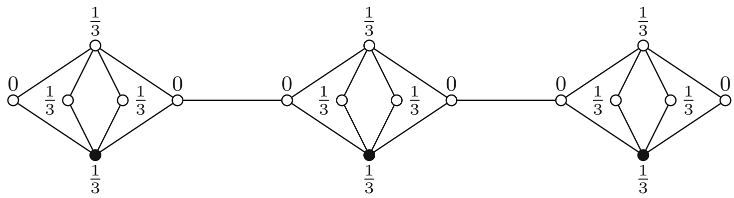


Fig. 6 A graph H with $\mathrm{mp}_f(H) = 4$ and $\mathrm{mp}(H) = 3$

and multipacking can be regarded as integer programming problems, Brewster, Mynhardt, and Teshima [11] used this idea to study fractional broadcast domination and fractional multipacking. Here, the broadcast strength of a vertex can be a fraction, and a vertex can be considered to be fractionally in a multipacking. For example, we can assign 1/3 strength to all vertices in C_4, for a total cost of 4/3, resulting in a fractional dominating broadcast where each vertex hears a total strength at least one. On the other hand, we can pack 1/3 for each vertex in C_4 and it will give a multipacking of size 4/3. We denote the fractional broadcast domination number as $\gamma_{b,f}(G)$ and the fractional multipacking number as $\mathrm{mp}_f(G)$. The duality theorem of linear programming yields the result below.

Theorem 17. ([11]) *If G is a connected graph, then*

$$\mathrm{mp}(G) \leq \mathrm{mp}_f(G) = \gamma_{b,f}(G) \leq \gamma_b(G).$$

The difference $\mathrm{mp}_f(G) - \mathrm{mp}(G)$ can be arbitrarily large. The graph H shown in Figure 5 has fractional multipacking number at least 4 since we can pack 1/3 on the degree 2 and 4 vertices with the exception of x and y, which are not packed. The resulting fractional multipacking is shown in Figure 6. Thus, $\mathrm{mp}_f(H) \geq 4$. As observed earlier, $\gamma_b(G) = 4$, implying by Theorem 17 that $\mathrm{mp}_f(H) \leq 4$. Consequently, $\mathrm{mp}_f(H) = 4$.

Using the previous construction G_k given by Hartnell and Mynhardt [26], we can readily deduce the following result.

Theorem 18. *For every integer $k \geq 1$, there exists a connected graph G_k such that $\mathrm{mp}_f(G_k) = 4k$ and $\mathrm{mp}(G_k) = 3k$.*

3 Broadcast Domination in Trees

Broadcasts in trees have a special structure, which was exploited in the thesis by Herke [31] in 2007 and in the papers by Herke and Mynhardt [32] in 2009 and Cockayne, Herke, and Mynhardt [16] in 2011. In order to determine the broadcast domination number of a tree, the above authors introduced the concept of a *shadow tree* of a tree, defined as follows.

Suppose $P: v_0v_1v_2\ldots v_d$ is a diametrical path in a tree T. The shadow tree is constructed in two steps. First, consider the forest $F = T - E(P)$ obtained from T by deleting all edges on the path P. For each vertex v_k of P, let Q_k be a longest path in F emanating from v_k. Let Q_k start at v_k and end at the vertex b_k (possibly, $v_k = b_k$). We note, for example, that Q_0 is the trivial path consisting of the vertex $v_0 = b_0$, and Q_d is the trivial path consisting of the vertex $v_d = b_k$. For example, consider the tree T in Figure 7, where the vertices of the diametrical path $P: v_0v_1v_2\ldots v_d$ and the vertices b_1, b_4, and b_5 are labeled as shown. We note that in this example, $v_i = b_i$ for $i \in \{0, 2, 3, 6, 7\}$.

In the first step of the construction of a shadow tree, we reduce the tree T to the subtree, T_{reduced}, of T induced by the vertices belonging to the set

$$V(P) \cup \left(\bigcup_{k=1}^{d-1} V(Q_k)\right).$$

For the tree T in Figure 7, the resulting reduced tree T_{reduced} is shown in Figure 8.

In the second step of the construction of a shadow tree, if $d(v_k, b_k) \geq d(v_k, b_i)$ for some $k \in [d]$ and $i \in [d] \setminus \{k\}$, then we remove the vertices in $V(Q_i) \setminus \{v_i\}$ from the tree T_{reduced}. We repeat this process until no such indices k and i exist. The resulting tree is a *shadow tree* of T, denoted by T_{shadow}. The *shadow* of vertex b_k is the set of vertices $\{v \in V(T_{\text{shadow}}) : d(v_k, b_k) \geq d(v_k, v)\}$. In our example, in the

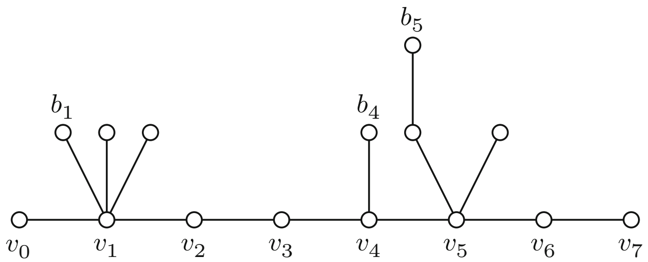


Fig. 7 A tree T

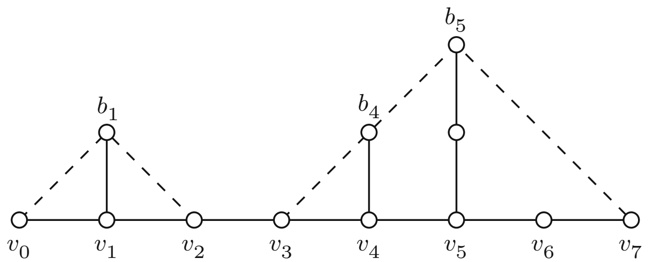


Fig. 8 A reduced tree T_{reduced} of T